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A new fuzzy multi-objective higher order moment portfolio selection model for diversified portfolios

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HIGHLIGHTS

- A fuzzy multi-objective higher order moment portfolio selection model is proposed.
- A new entropy function based on Minkowski measure is proposed to obtain better diversification portfolios.
- A new multi-objective evolutionary algorithm is designed to solve the proposed model efficiently.
- Results show the effectiveness of the proposed portfolio mode.

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ABSTRACT

Due to the important effect of the higher order moments to portfolio returns, the aim of this paper is to make use of the third and fourth moments for fuzzy multi-objective portfolio selection model. Firstly, in order to overcome the low diversity of the obtained solution set and lead to corner solutions for the conventional higher moment portfolio selection models, a new entropy function based on Minkowski measure is proposed as a new objective function and a novel fuzzy multi-objective weighted possibilistic higher order moment portfolio model is presented. Secondly, to solve the proposed model efficiently, a new multi-objective evolutionary algorithm is designed. Thirdly, several portfolio performance evaluation techniques are used to evaluate the performance of the portfolio models. Finally, some experiments are conducted by using the data of Shanghai Stock Exchange and the results indicate the efficiency and effectiveness of the proposed model and algorithm.

1. Introduction

Traditional portfolio selection theory is derived from the mean–variance probabilistic model by Markowitz [1], which assumes an underlying normal/quadratic utility distribution for asset's returns. However, a plethora of empirical studies show that the distributions of asset returns usually tend to be of asymmetric leptokurtic and heavy-tailed features, and are not normally distributed [2–4]. This implies that the higher order moments cannot be neglected. Thus the investors should consider higher order moments in their investment decision. Some researchers have tried to study portfolio problems in a three-moment or four-moment framework. For example, Campbell et al. [5] used the mean–variance–skewness framework with the skew normal distribution, and suggested that it is important to incorporate higher order moments in portfolio selection. Adcock [6] studied the mean–variance–skewness portfolio model under the multivariate extended skew-Student

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distribution, and used the quadratic programming to solve the model. Maringer et al. [7] proposed an extension of the classical Markowitz model by taking into consideration the higher order moments and used stochastic algorithms to find the optimal portfolio. Doana et al. [8] studied the systematic skewness and systematic kurtosis of Australian stock returns in the spirit of the higher-moment asset pricing model, and presented the empirical results of examining the roles of systematic skewness and kurtosis. However, these higher order moments portfolio models come with some serious issues, and the major issues are: (a) the error in the estimation of corner solutions, and (b) the low diversity in the portfolio. The low diversity of the portfolio may result in loss while some of the invested assets experience unexpected gains [9,10]. In the last few years, Jana et al. [11] and Usta et al. [12] used the Shannon's entropy to measure the diversification, and generate a well-diversified portfolio. Additionally, Huang [13] used the Shannon's entropy as a constraint condition in the mean–variance framework, and avoided the concentrative portfolio allocation. Recently, Yu et al. [14] discussed the diversified portfolios with different entropy measures, and concluded that the models with Yager's entropy yield the higher economic value of diversification than the portfolio model with Shannon's entropy.

Besides the higher order moments, uncertainty is another important factor in portfolio model because investors may face uncertain, imprecise and vague data. Over the decades, the vast majority of the existing portfolio selection models are based on probability theory under some random states. However, if there is not enough historical data, it is more difficult to use statistical variable to describe model. It is more reasonable to use fuzzy variables [15]. So, in portfolio selection literature, many researchers extended the probabilistic portfolio model to fuzzy environment in different ways. For example, Li et al. [16] developed a fuzzy portfolio selection model with background risk, based on the definitions of the possibilistic theory and used a genetic algorithm to solve the proposed model. Sadatia et al. [17] dealt with a portfolio optimization model involving fuzzy random variables by using the possibility and necessity-based model, and proposed a two-level linear programming model to find the optimum solution. Kocadagli et al. [18] introduced a novel fuzzy portfolio selection model by means of the fuzzy goal programming techniques and presented some numerical examples of a portfolio selection problem to illustrate the effectiveness of the proposed model. More recently, Mashayekhi et al. [19] incorporated the DEA cross-efficiency into Markowitz mean–variance model and proposed a novel fuzzy portfolio model and used NSGA-II algorithm to solve the proposed model.

Also, the transaction cost and the liquidity are very important factor for investors, ignoring transaction costs and the liquidity would result in inefficient portfolios. Some researchers took into account the transaction cost or the liquidity in the portfolio selection model. For instance, Najafi et al. [20] considered the transaction costs, developed a dynamic portfolio selection model and proposed an efficient heuristic method to tackle this problem. Liu et al. [21] discussed the asset allocation in the presence of small proportional transaction costs, which objective is to keep the asset portfolio close to a target portfolio and at the same time to reduce the trading cost in doing so. Oriakhi et al. [22] considered the problem of rebalancing an existing financial portfolio, where transaction costs (fixed and/or changed in nature) have to be paid when the amount of any asset changes. Yu et al. [23] proposed a rebalancing multiple criteria portfolio model by comparing risk, return, skewness, kurtosis, and transaction cost. But they did not consider the portfolio diversification, the liquidity of portfolio and fuzzy return of assets in their model.

It can be seen from above discussion that in order to construct a reasonable portfolio selection model, it is desirable to consider the effect of the following factors: the higher order moments, uncertainty, lower diversity of portfolio, transaction cost and liquidity. However, the existing research works only considered a part of these factors, and few works considered the liquidity of portfolio.

In order to achieve the better portfolio selection and set up a reasonable portfolio selection model, in this paper, in addition to consider the first two order moments (mean and variance) for portfolio selection, we also use the third and fourth order moments (skewness and kurtosis) to design the portfolio selection model. To improve the diversity of portfolio selection and avoid the portfolio concentrating on a few assets, we present a new entropy function based on Minkowski measure, which is defined by the sum of Minkowski distance between weights of the invested assets for the portfolio selection. By using the entropy function, we can directly acquire a well-diversified portfolio. To handle the uncertainty and consider the case with not enough historical data, we introduce the fuzzy variable in the designed model. Furthermore, the transaction cost and liquidity are also considered in our model. Based on these and by making full use of the advantages of three entropies, i.e., the two most commonly used entropies (the Shannon's and Yager's entropies) and our proposed entropy, we construct three different mean–variance–skewness–kurtosis–entropy portfolio models. Moreover, in order to evaluate the performance of these portfolio selection models, we develop a new performance metric based on the Euclidean norms to evaluate the reliability of the solutions. Finally, considering these models are fuzzy multi-objective with liquidity constraints, they become more complicated. Although evolutionary algorithms are an effective way to tackle the multi-objective optimization models and there have been some such algorithms proposed for these problems (e.g. Refs. [24–26]), it is almost impossible to efficiently solve such problems by non-specific-designed algorithms. We design a new efficient multi-objective evolutionary algorithm for these models.

The rest of this paper is organized as follows. Section 2 introduces basic definitions and preliminary results related to fuzzy variables, and give the formulation of the multi-objective optimization problem. In Section 3, we develop a new entropy function based on Minkowski measure to generate a well-diversified portfolio, and present three fuzzy mean–variance–skewness–kurtosis–entropy portfolio models with different entropy functions in the multi-objective framework. Section 4 presents a detailed description of our designed multi-objective evolution algorithm. In Section 5, several portfolio performance evaluation techniques are used to evaluate the performance of these portfolio models, and give the comparison analysis among these models through some numerical examples based on the data of the Shanghai Stock Exchange Market. Finally, conclusion is drawn in Section 6.

2. Preliminaries

Some basic definitions and preliminary results will be introduced in this section.

2.1. Fuzzy numbers and notations

In this section, the basic concepts and notations are given as follows [27,28]:

Definition 1. A fuzzy number $\tilde{A}$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$ satisfies the following conditions:

(i) $\tilde{A}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$;
(ii) $\mu_{\tilde{A}}(x)$ is quasi-concave, i.e., $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$, for all $\lambda \in [0, 1]$;
(iii) $\mu_{\tilde{A}}(x)$ is upper semi-continuous, i.e., $\left\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \leq \varepsilon\right\}$ is a closed set, for all $\varepsilon \in [0, 1]$;
(iv) The closure of the set $\left\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) > 0\right\}$ is a compact set.

Definition 2. A γ-level set of $\tilde{A}$ is defined by an ordinary set $\tilde{A}_\gamma = \left\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \gamma\right\}$ for $\gamma \in (0, 1]$, and $\tilde{A}_\gamma = \mathrm{cl}\left\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq 0\right\}$ (the closure of the support of $\tilde{A}$) if $\gamma = 0$. As well known, if $\tilde{A}$ is a fuzzy number, then $\tilde{A}_\gamma = \left\{x \in \mathbb{R} | \mu_{\tilde{A}} \geq \gamma\right\} = [\underline{a}(\gamma), \overline{a}(\gamma)]$ is a compact subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$.

Definition 3. A fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number with core $[c, d]$, left width $\delta > 0$ and right width $\theta > 0$, if its membership function has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{c-x}{\delta}, & \text{if } c - \delta \leq x \leq c \\ 1, & \text{if } c \leq x \leq d \\ 1 - \dfrac{x-d}{\theta}, & \text{if } d \leq x \leq d + \theta \\ 0, & \text{if otherwise} \end{cases} \tag{1}$$

and it can be denoted by $\tilde{A} = (c, d, \delta, \theta)$.

Definition 4. Let $\tilde{A}$ be a fuzzy number with $\tilde{A}_\gamma = \left[\underline{a}(\gamma), \overline{a}(\gamma)\right]$, $\gamma \in [0, 1]$. Then the weighted possibilistic mean (WPM) and the weighted possibilistic variance (WPV) of fuzzy number $\tilde{A}$ are defined as follows, respectively.

(i) $M_f\left(\tilde{A}\right) = \int_0^1 f(\gamma) \frac{\underline{a}(\gamma) + \overline{a}(\gamma)}{2} \mathrm{d}\gamma$,
(ii) $Var_f\left(\tilde{A}\right) = \frac{1}{2} \int_0^1 f(\gamma) \left[\left(\underline{a}(\gamma) - M_f(\tilde{A})\right)^2 + \left(\overline{a}(\gamma) - M_f(\tilde{A})\right)^2\right] \mathrm{d}\gamma$.

where, $f(\gamma) = (n+1)\gamma^n$ is a weighted function such that $\int_0^1 f(\gamma)\mathrm{d}\gamma = 1$.

Definition 5. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with weighted possibilistic mean $M_f(\tilde{A})$ and $M_f(\tilde{B})$, respectively. Then the weighted possibilistic covariance between $\tilde{A}$ and $\tilde{B}$ is defined as

$$Cov_f\left(\tilde{A}, \tilde{B}\right) = \frac{1}{2} \int_0^1 f(\gamma) \left[\left(\underline{a}(\gamma) - M_f(\tilde{A})\right)\left(\underline{b}(\gamma) - M_f(\tilde{B})\right) + \left(\overline{a}(\gamma) - M_f(\tilde{A})\right)\left(\overline{b}(\gamma) - M_f(\tilde{B})\right)\right] \mathrm{d}\gamma,$$

where $f(\gamma) = (n+1)\gamma^n$ is a weighted function such that $\int_0^1 f(\gamma)\mathrm{d}\gamma = 1$.

Theorem 1 ([28]). *Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers, and let λ and μ be nonnegative numbers. Then the following conclusions can be obtained:*

(i) $M_f\left(\lambda\tilde{A} \pm \mu\tilde{B}\right) = \lambda M_f\left(\tilde{A}\right) \pm \mu M_f\left(\tilde{B}\right)$;
(ii) $Var_f\left(\lambda\tilde{A} \pm \mu\tilde{B}\right) = \lambda^2 Var_f\left(\tilde{A}\right) + \mu^2 Var_f\left(\tilde{B}\right) \pm 2\lambda\mu Cov_f\left(\tilde{A}, \tilde{B}\right)$.

From Theorem 1, we can easily deduce the following conclusion.

Theorem 2. *Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be n fuzzy numbers, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n nonnegative real number. Then*

(i) $M_f\left(\sum_{i=1}^n \lambda_i \tilde{A}_i\right) = \sum_{i=1}^n \lambda_i M_f\left(\tilde{A}_i\right)$

(ii) $Var_f\left(\sum_{i=1}^n \lambda_i \tilde{A}_i\right) = \sum_{i=1}^n \lambda_i^2 Var_f\left(\tilde{A}_i\right) \pm 2\sum_{i=1}^n \sum_{j=1}^n \lambda_i\lambda_j Cov_f(\tilde{A}_i, \tilde{A}_j)$.

According to the above definitions of weighted possibilistic moments, Theorems 1 and 2, we can get the following proposition.

Proposition 1. *Let $\tilde{r}(x) = \sum_{i=1}^n x_i \tilde{r}_i = \left(\sum_{i=1}^n x_i c_i, \sum_{i=1}^n x_i d_i, \sum_{i=1}^n x_i \delta_i, \sum_{i=1}^n x_i \theta_i\right)$ be the portfolio return. The weighted possibilistic variance of $\tilde{r}(x)$ is*

$$Var_f\left[\tilde{r}(x)\right] = \frac{\left[\sum_{i=1}^n x_i\left(\theta_i + \delta_i\right)\right]^2 + \left[\sum_{i=1}^n x_i(\theta_i - \delta_i)\right]^2}{72} + \left[\sum_{i=1}^n x_i\left(\frac{d_i - c_i}{2} + \frac{\theta_i + \delta_i}{6}\right)\right]^2. \tag{2}$$

2.2. Higher order moments of fuzzy variables

In the following subsection, we will introduce the weighted possibilistic higher order moments for the fuzzy variable.

Definition 6 ([29]). Let $\tilde{A}$ be a fuzzy number with weighted possibilistic mean $M_f\left(\tilde{A}\right)$. For any positive integer r, the f-weighted possibilistic moments (f-WPM) of order r about the weighted possibilistic mean value of $\tilde{A}$ is defined as $E_r\left(\tilde{A}\right) = \frac{1}{2}\int_0^1 f(r)\left[\left(\underline{a}(\gamma) - M_f(\tilde{A})\right)^r + \left(\overline{a}(\gamma) - M_f(\tilde{A})\right)^r\right] d\gamma$.

Definition 7 ([29]). Let $\tilde{A}$ be a fuzzy number, $f(r)$ be a weighted function, and $E_r(\tilde{A})$ with $r = 1, 2, \ldots, n$ be weighted possibilistic moments of $\tilde{A}$. Then the weighted possibilistic skewness (WPS) of fuzzy number $\tilde{A}$ is defined as follows:

$$\text{Skewness}_f\left(\tilde{A}\right) = \text{WPS}\left(\tilde{A}\right) = \frac{E_3(\tilde{A})}{\left(\sqrt{E_2(\tilde{A})}\right)^3}. \tag{3}$$

The weighted possibilistic skewness of a fuzzy number $\tilde{A}$ shows the weight of fuzzy number at the left or right side. A positive weighted possibilistic skewness specifies that fuzzy number $\tilde{A}$ is skewed to the right, that is, the tail on the right side is longer than the one on the left side. A negative weighted possibilistic skewness specifies that fuzzy number $\tilde{A}$ is skewed to the left, that is, the tail on the left side is longer than the one on the right side. A zero value indicates that fuzzy number $\tilde{A}$ with the symmetric membership function, that is, its right tail and left tail are equal.

In order to facilitate the calculation and reduce the computation complexity, we will use the third possibilistic moments as the weighted possibilistic skewness of fuzzy number $\tilde{A}$. Denote it as Skewness$(\tilde{A}) = E_3(\tilde{A})$. As well known, these two definitions of the possibilistic skewness about fuzzy number do not affect the way we study the problem.

Theorem 3 ([29]). *Let $\tilde{r}(x) = \sum_{i=1}^n x_i \tilde{r}_i = \left(\sum_{i=1}^n x_i c_i, \sum_{i=1}^n x_i d_i, \sum_{i=1}^n x_i \delta_i, \sum_{i=1}^n x_i \theta_i\right)$ be the portfolio return. The weighted possibilistic skewness of $\tilde{r}(x)$ is*

$$\begin{aligned}\text{Ske}_f\left[\tilde{r}(x)\right] = {} & \frac{19}{1080}\left[\left(\sum_{i=1}^n x_i\theta_i\right)^3 - \left(\sum_{i=1}^n x_i\delta_i\right)^3\right] + \frac{1}{24}\left[\sum_{i=1}^n x_i(d_i - c_i)\right]\left[\left(\sum_{i=1}^n x_i\theta_i\right)^2 - \left(\sum_{i=1}^n x_i\delta_i\right)^2\right] \\ & + \frac{1}{72}\left[\left(\sum_{i=1}^n x_i\delta_i\right)\left(\sum_{i=1}^n x_i\theta_i\right)^2 - \left(\sum_{i=1}^n x_i\theta_i\right)\left(\sum_{i=1}^n x_i\delta_i\right)^2\right].\end{aligned} \tag{4}$$

Definition 8. Let $\tilde{A}$ be a fuzzy number with weighted possibilistic mean $M_f\left(\tilde{A}\right)$. Then the weighted possibilistic kurtosis is defined as follows:

$$\text{Kur}_f(\tilde{A}) = \frac{E_4(\tilde{A})}{(E_2(\tilde{A}))^2} = \frac{\int_0^1 f(\gamma)\left[(\underline{a}(\gamma) - M_f(\tilde{A}))^4 + (\overline{a}(\gamma) - M_f(\tilde{A}))^4\right] d\gamma}{\int_0^1 f(\gamma)\left[\left(\underline{a}(\gamma) - M_f(\tilde{A})\right)^2 + \left(\overline{a}(\gamma) - M_f(\tilde{A})\right)^2\right] d\gamma}. \tag{5}$$

In the same way as above-discussed, we will use the fourth possibilistic moments as the weighted possibilistic kurtosis of fuzzy number $\tilde{A}$. Denote it as $\text{Kurtosis}(\tilde{A}) = E_4(\tilde{A})$.

Theorem 4 *([30]). Let $\tilde{A} = (c, d, \delta, \theta)$ be a trapezoidal fuzzy number with weighted possibilistic mean $M\left(\tilde{A}\right) = \frac{c+d}{2} + \frac{\theta-\delta}{6}$. The weighted possibilistic kurtosis of $\tilde{A}$ can be computed as:*

$$\begin{aligned} E_4\left(\tilde{A}\right) = &\frac{5}{432}\left(\delta^4+\theta^4\right) + \frac{1}{16}\left(c^4+d^4\right) + \frac{3}{8}c^2d^2 + \frac{1}{72}\theta^2\delta^2 - \frac{1}{6}cd\left(\theta^2+\delta^2+\theta\delta\right) \\ &- \frac{1}{4}cd\left(c^2+d^2-(d-c)(\theta+\delta)\right) - \frac{1}{18}\theta\delta(c-d)(\theta+\delta) - \frac{2}{45}(c-d)\left(\theta^3+\delta^3\right) \\ &+ \frac{2}{135}\theta\delta\left(\theta^2+\delta^2\right) + \frac{1}{12}\left(d^3-c^3\right)(\theta+\delta) + \frac{1}{12}(d^2+c^2)(\delta^2+\theta^2+\theta\delta). \end{aligned} \tag{6}$$

Lemma 1 *([30,31]). Let $\tilde{r}(x) = \sum_{i=1}^n x_i\tilde{r}_i = \left(\sum_{i=1}^n x_ic_i, \sum_{i=1}^n x_id_i, \sum_{i=1}^n x_i\delta_i, \sum_{i=1}^n x_i\theta_i\right)$ be the portfolio return. The weighted possibilistic kurtosis of $\tilde{r}(x)$ is*

$$\begin{aligned} \text{Kur}_f\left[\tilde{r}(x)\right] = &\frac{5}{432}\left[\left(\sum_{i=1}^n x_i\theta_i\right)^4 + \left(\sum_{i=1}^n x_i\delta_i\right)^4\right] + \frac{1}{16}\left[\left(\sum_{i=1}^n x_ic_i\right)^4 + \left(\sum_{i=1}^n x_id_i\right)^4\right] + \frac{3}{8}\left[\sum_{i=1}^n x_id_ic_i\right]^2 \\ &+ \frac{1}{72}\left[\sum_{i=1}^n x_i\theta_i\delta_i\right]^2 - \frac{1}{6}\left(\sum_{i=1}^n x_ic_id_i\right)\left[\left(\sum_{i=1}^n x_i\theta_i\right)^2 + \left(\sum_{i=1}^n x_i\delta_i\right)^2 + \sum_{i=1}^n x_i\theta_i\delta_i\right] \\ &- \frac{1}{4}\left(\sum_{i=1}^n x_ic_id_i\right)\left[\left(\sum_{i=1}^n x_ic_i\right)^2 + \left(\sum_{i=1}^n x_id_i\right)^2 + \sum_{i=1}^n x_i(d_i-c_i)(\theta_i+\delta_i)\right] \\ &- \frac{1}{18}\left(\sum_{i=1}^n x_i\theta_i\delta_i\right)\left[\sum_{i=1}^n x_i(c_i-d_i)(\theta_i+\delta_i)\right] - \frac{2}{45}\left[\sum_{i=1}^n x_i(c_i-d_i)\right] \\ &\times\left[\left(\sum_{i=1}^n x_i\theta_i\right)^3 + \left(\sum_{i=1}^n x_i\delta_i\right)^3\right] + \frac{2}{135}\left(\sum_{i=1}^n x_i\theta_i\delta_i\right)\left[\left(\sum_{i=1}^n x_i\theta_i\right)^2 + \left(\sum_{i=1}^n x_i\delta_i\right)^2\right] \\ &+ \frac{1}{12}\left[\left(\sum_{i=1}^n x_id_i\right)^3 - \left(\sum_{i=1}^n x_ic_i\right)^3\right]\left[\sum_{i=1}^n x_i(\theta_i+\delta_i)\right] \\ &+ \frac{1}{12}\left[\left(\sum_{i=1}^n x_id_i\right)^2 + \left(\sum_{i=1}^n x_ic_i\right)^2\right]\left[\left(\sum_{i=1}^n x_i\delta_i\right)^2 + \left(\sum_{i=1}^n x_i\theta_i\right)^2 + \left(\sum_{i=1}^n x_i\theta_i\delta_i\right)\right]. \end{aligned} \tag{7}$$

2.3. Multi-objective optimization problems (MOPs)

In this paper, the proposed portfolio selection problems will be modeled as multi-objective optimization problems (MOPs). Typically, a multi-objective optimization problem can be formulated as follows [32]:

$$\begin{cases} \min & y = F(x) = [f_1(x), f_2(x), \ldots, f_k(x)] \\ \text{s.t} & g_j(x) \geq 0, \quad j = 1, 2, \ldots, q \\ & h_j(x) = 0, \quad j = q+1, \ldots, m. \end{cases} \tag{8}$$

The feasible region $\mathbb{F}$ is defined as follows:

$$\mathbb{F} = \left\{x | g_j(x) \geq 0, j = 1, 2, \ldots, q; h_j(x) = 0, j = q+1, \ldots, m\right\} \subseteq S \subseteq R^n. \tag{9}$$

Definition 9 (*Pareto-Dominance*). A solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is said to dominate (Pareto-optimal) another solution $x = (x_1, x_2, \ldots, x_n)$ (denoted by $x^* \succ x$), if both the conditions mentioned below are satisfied:

$$\begin{aligned} &\text{(i) } \forall j \in \{1, 2, \ldots, k\} : f_j(x^*) \le f_j(x) \\ &\text{(ii) } \exists j \in \{1, 2, \ldots, k\} : f_j(x^*) < f_j(x). \end{aligned} \tag{10}$$

Definition 10 (*Pareto-Optimal*). A solution $x^* \in \mathbb{F}$ is said to be non-dominated (Pareto-optimal)

$$\text{Iff}: \ \neg\exists x \text{ such that } x \succ x^*.$$

Definition 11 (*Pareto-Optimal Set*). The set of all Pareto optimal solutions is defined as:

$$PS = \left\{x^* | \neg\exists x \in \mathbb{F} : x \succ x^*\right\}. \tag{11}$$

Definition 12 (*Pareto-Optimal Front*). The set of all Pareto solutions in objective space is defined as

$$\text{Pareto front and denoted by}: \ PF = \{[f_1(x), f_2(x), \ldots, f_k(x)] \,|x \in PS\}. \tag{12}$$

3. Weighted possibilistic higher order moment portfolio selection models

In this section, we first introduce the problem descriptions and notations used in the following sections. Second, we present a novel entropy function based on Minkowski metric for diversifying the allocation on various assets. Then, based on the new entropy function, we set up three fuzzy mean–variance–skewness–kurtosis–entropy portfolio models in multi-objective framework.

Let us consider a multi-objective fuzzy portfolio selection problem with n risk assets. The return rates and turnover rates of the risk assets are denoted as trapezoidal fuzzy numbers. For the notation convenience, we introduce the following notations.

x_i: Proportion of the total investment devoted to the risk asset i, $i = 1, \ldots, n$;
k_i: Rate of transaction cost on the risk asset i, $i = 1, \ldots, n$;
$\tilde{r}_i$: Fuzzy rate of return on the risk asset i, $i = 1, \ldots, n$;
$\tilde{l}_i$: Fuzzy turnover rate of the risk asset i, $i = 1, 2, \ldots, n$.

In this paper, the transaction cost is assumed to be a V-shape function, which measures the difference between a new portfolio $x = (x_1, x_2, \ldots, x_n)$ and a given portfolio $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$. The total transaction cost of the portfolio is defined by $\sum_{i=1}^n k_i \left|x_i - x_i^0\right|$.

For any risk asset, liquidity may be measured by using the turnover rate which is defined by the ratio of the average trading volume of the assets trade in the market and the trading volume of the tradable asset (i.e., shares held by the public) corresponding to the asset. It is well known that the future turnover rates of assets cannot be accurately predicted in the uncertain financial market. Therefore, the fuzzy set theory provides an effective tool to deal with this imprecision. Without loss of generality, the turnover rate of the ith asset is regarded as a trapezoidal fuzzy number $\tilde{l}_i = (c_i, d_i, \delta_i, \theta_i)$, $i = 1, 2, \ldots, n$, in this paper. Note that the turnover rate of the portfolio $x = (x_1, x_2, \ldots, x_n)$ is $\tilde{l}(x) = \sum_{i=1}^n \tilde{l}_i x_i$. In the model, the portfolio liquidity is always greater or equal to a given tolerate level of fuzzy turnover rate $\tilde{l}_0$ by the investor, that is, $\tilde{l}(x) = \sum_{i=1}^n \tilde{l}_i x_i \ge \tilde{l}_0$. According to the method for ranking fuzzy number reported in Ref. [33], the fuzzy inequality $\sum_{i=1}^n \tilde{l}_i x_i \ge \tilde{l}_0$ can be transformed into the crisp inequality $M\left(\sum_{i=1}^n \tilde{l}_i x_i\right) \ge M(\tilde{l}_0)$.

3.1. Our proposed entropy function based on Minkowski metric and its properties

Although some researchers have tried to optimize portfolios in a three-moment or four-moment framework, however, such techniques also come with serious issue and result in corner solutions and low diversity in the portfolio. In order to satisfy the requisition of decentralized investment, we develop a new entropy function based on Minkowski metric, which denotes the sum of Minkowski distances between weights of the invested asset for the portfolio selection. The following is the definition of our proposed entropy function:

$$\text{Pn}(x) = -\left(\sum_{i=1}^{n-1} |x_i - x_{i+1}|^z\right)^{\frac{1}{z}} \tag{13}$$

where x_i is proportion of the total investment devoted to the risk asset i, $i = 1, \ldots, n-1$, z is a constant and $z \geq 1$. Eq. (13) denotes the sum of Minkowski distance between weights of the invested asset. From Eq. (13), it can be seen that, the more equally the budget allocates on assets, the larger entropy function is. It is worth noticed that the entropy function can be transferred into a linear type when $z = 1$. The following is a linear type proportion entropy when $z = 1$.

$$\text{PE}(x) = -\sum_{i=1}^{n-1} |x_i - x_{i+1}| . \tag{14}$$

In order to simplify the process of computation, we will use the entropy function when $z = 1$. The linearization of the proportion entropy can directly obtain a diversified portfolio because of the linear feature of the proportion entropy. By using our proposed linear proportion entropy as an objective function to be maximized, we construct a new fuzzy mean–variance–skewness–kurtosis–entropy portfolio model. In essence, we can construct a well-diversified portfolio model, which meets the requirement of investors.

3.2. Shannon's entropy and Yager's entropy

Some researchers, for instance Jana et al. [11], Usta et al. [12] and Huang [13], used the Shannon's entropy in portfolio selection to diversify the allocation on various assets, while meeting the requirement of investors. The following is Shannon's entropy:

$$\text{SE}(x) = -\sum_{i=1}^{n} x_i \ln x_i \tag{15}$$

where x_i is the weight of risk asset i, $i = 1, \ldots, n$; n is the number of invested assets. SE (x) has the maximum value, when $x_i = \frac{1}{n}$; the larger the SE (x), the more equally the budget allocates on assets. The other extreme case occurs when $x_i = 1$ for one i, and $x_i = 0$ for the rest, then SE $(x) = 0$.

Yager's entropy is defined by [34]

$$\text{YE}(x) = -\sum_{i=1}^{n} \left| x_i - \frac{1}{n} \right| \tag{16}$$

where x_i is the weight of risk asset i, $i = 1, \ldots, n$; n is the number of invested assets. It aimed to minimize the distance between the weight of invested asset and $\frac{1}{n}$ in terms of the portfolio selection. If the $x_i = \frac{1}{n}$, the Yager's entropy has the maximum value. Therefore, the budget allocates on assets more equally when Yager's entropy is larger.

3.3. Fuzzy mean–variance–skewness–kurtosis–entropy portfolio models

In order to set up a model tackling all above mentioned issues, we maximize weighted possibilistic expected return and skewness, while minimize weighted possibilistic variance and kurtosis, simultaneously. We then can set up the following fuzzy multi-objective portfolio selection model:

$$\begin{cases} \max & f_1(x) = M_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] - \sum_{i=1}^{n} k_i \left| x_i - x_i^0 \right| \\ \min & f_2(x) = V_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max & f_3(x) = \text{Ske}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \min & f_4(x) = \text{Kur}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \text{s.t} & M\left(\sum_{i=1}^{n} \tilde{l}_i x_i\right) \geq M\left(\tilde{l}_0\right) \\ & \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \; i = 1, 2, \ldots, n \end{cases} \tag{17}$$

where the first constraint means the liquidity is greater than or equal to a given value $\tilde{l}_0$ by the investor, and the second constraint implies that all the capital will be invested to n assets and short-selling is not allowed. To further improve the diversity of portfolio and compare the effect of different entropies to the diversity of portfolio, we add our proportion entropy, Shannon's entropy, and Yager's entropy respectively to the fuzzy mean–variance–skewness–kurtosis framework and obtain the following three models.

Mode I (MVSK-PE): the fuzzy multi-objective mean–variance–skewness–kurtosis–proportion–entropy portfolio selection model.

$$\begin{cases} \max \quad f_1(x) = M_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] - \sum_{i=1}^{n} k_i \left|x_i - x_i^0\right| \\ \min \quad f_2(x) = V_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_3(x) = \mathrm{Ske}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \min \quad f_4(x) = \mathrm{Kur}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_5(x) = -\sum_{i=1}^{n-1} \left|x_i - x_{i+1}\right| \\ \text{s.t} \quad M\left(\sum_{i=1}^{n} \tilde{l}_i x_i\right) \geq M\left(\tilde{l}_0\right) \\ \qquad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \ i = 1, 2, \ldots, n. \end{cases} \tag{18}$$

Model II (MVSK-SE): the fuzzy multi-objective mean–variance–skewness–kurtosis–Shannon-entropy portfolio selection model.

$$\begin{cases} \max \quad f_1(x) = M_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] - \sum_{i=1}^{n} k_i \left|x_i - x_i^0\right| \\ \min \quad f_2(x) = V_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_3(x) = \mathrm{Ske}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \min \quad f_4(x) = \mathrm{Kur}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_5(x) = -\sum_{i=1}^{n} x_i \ln x_i \\ \text{s.t} \quad M\left(\sum_{i=1}^{n} \tilde{l}_i x_i\right) \geq M\left(\tilde{l}_0\right) \\ \qquad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \ i = 1, 2, \ldots, n. \end{cases} \tag{19}$$

Model III (MVSK-YE): the fuzzy multi-objective mean–variance–skewness–kurtosis–Yager-entropy portfolio selection model.

$$\begin{cases} \max \quad f_1(x) = M_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] - \sum_{i=1}^{n} k_i \left|x_i - x_i^0\right| \\ \min \quad f_2(x) = V_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_3(x) = \mathrm{Ske}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \min \quad f_4(x) = \mathrm{Kur}_f\left[\sum_{i=1}^{n} \tilde{r}_i x_i\right] \\ \max \quad f_5(x) = -\sum_{i=1}^{n} \left|x_i - \frac{1}{n}\right| \\ \text{s.t} \quad M\left(\sum_{i=1}^{n} \tilde{l}_i x_i\right) \geq M\left(\tilde{l}_0\right) \\ \qquad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0,\ i = 1, 2, \ldots, n. \end{cases} \tag{20}$$

4. A new multi-objective evolutionary algorithm for the models

It is well known that the convergence and the diversity are two main goals for many-objective optimization problems. However, achieving these two goals simultaneously is a difficult and challenging work for multi-objective evolutionary algorithms. In order to achieve simultaneously the two goals, a new multi-objective evolutionary algorithm based on population decomposition is proposed. The proposed algorithm is specifically designed and included the four parts, i.e. population decomposition, crossover operations, selection strategy and update strategy. Population decomposition, and update strategy are used for keeping the diversity of the obtained solutions. The crossover operations are used to search the decision space. The selection strategy has a great impact on the performance of local search and global search, which can make the obtained solutions converge to the PF.

4.1. Population decomposition and update

We use a set of uniformly distributed weight vectors to decompose the population P into M sub-populations $P^1, P^2, \ldots, P^M$ [35]. For a given set of uniformly distributed weight vectors $(\lambda^1, \lambda^2, \ldots, \lambda^M)$, and the set P of current obtained solutions, each sub-population can be defined by the following formula:

$$P^i = \left\{x | x \in P, \triangle\left(F(x), \lambda^i\right) = \min_{1\leq j\leq N}\{\triangle(F(x), \lambda^j)\}\right\} \tag{21}$$

where $\triangle\left(F(x), \lambda^i\right)$ is the angle between λ^i and $F(x) - Y$ and $Y = (Y_1, \ldots, Y_m)$ is a reference point with $Y_i = \min\{f_i(x) | x \in \Theta\}$.

In this way, some $P^i (1 \leq i \leq M)$ may contain more solutions than another P^j, and some P^i may be empty. To get an approximate uniformly distributed and well diversified solution set, it is necessary to make the following update strategy:

(a) To avoid empty sub-population. If a sub-population P^i is empty, a solution is randomly selected from its second nearest non-empty neighbor sub-population P^j (i.e., the non-empty sub-population defined by λ^j whose angle and λ^i is the second smallest among the angles between λ^i and all other weight vectors) and is put into P^i. In this way, all sub-populations are non-empty.

(b) To avoid crowded solutions in a sub-population. If solutions in a sub-population P^i are densely distributed and crowded, we keep only two solutions in it. One is the current best solution of P^i (i.e., the non-dominated solution of P^i which has the smallest angle with λ^i), and another is randomly chosen from P^i.

We could make the following comments on the decomposition method:

(i) This method uses the principle that the PFs of all these sub-populations make up the whole PF.
(ii) Even if the whole PF has a nonlinear geometric shape, the PF of each sub-population could be approached linearly, because it is just a small part of the whole PF. Thus, formula (21) makes the MOPs be simpler than before, at least in accordance with PF shapes.
(iii) This decomposition method does not need any aggregation methods. A user only needs to select a set of direction vectors.

4.2. Crossover operations and mutation operator

In our multi-objective algorithm, we design two crossover operators.

Crossover operator 1. Choose three solutions x^1, x^2, x^3, one of its offspring x^{new} can be generated as follows:

$$x_i^{new} = \begin{cases} x_i^1 + \theta\left(x_i^2 - x_i^3\right), & \text{if } f_i\left(x^2\right) < f_i\left(x^3\right) \\ x_i^1 + \theta\left(x_i^3 - x_i^2\right), & \text{otherwise} \end{cases} \tag{22}$$

where $\theta \in [0, 1]$ is a randomly generated scaling factor which controls the length of the exploration, x_i^j represents the ith component of x^j for $i = 1, 2, \ldots, n$ and $j = 1, 2, 3$.

Crossover operator 2. For the chosen three solutions x^1, x^2, x^3, if $f_i\left(x^1\right) > f_i\left(x^2\right) < f_i\left(x^3\right)$, then the second offspring $x^{new} = \left(x_1^{new}, \ldots, x_n^{new}\right)$ is generated by the following way:

$$x_i^{new} = 0.5\left[\left(x_i^1 + x_i^2\right) + \frac{\left(f_i\left(x^1\right) - f_i(x^2)\right)\left(x_i^2 - x_i^3\right)\left(x_i^3 - x_i^1\right)}{f_i\left(x^1\right)\left(x_i^2 - x_i^3\right) + f_i\left(x^2\right)\left(x_i^3 - x_i^1\right) + f_i\left(x^3\right)\left(x_i^1 - x_i^2\right)}\right]. \tag{23}$$

Mutation Operator. For an offspring x^{new} of crossover operator and a randomly chosen individual x from P, let $\delta > 0$ be a small number and ε^i be a vector whose ith element is 1 and others are all zeros. Compute

$$\varphi_i = \frac{f_i\left(x^{new} + \delta\varepsilon^i\right) - f_i(x)}{\delta} \tag{24}$$

and the offspring $O = (O_1, O_2, \ldots, O_n)$ of x^{new} is defined by

$$O_i = -\frac{\varphi_i}{2[f_i\left(x^{new}\right) - f_i\left(x\right) - \varphi_i]} \quad \text{for } i = 1, 2, \ldots, n. \tag{25}$$

4.3. Selection strategy

The selection strategy is very important for the performance of local search and global search, thus an appropriate selection strategy can improve the performance of an algorithm. In this paper, a selection strategy based on the decomposition and sub-population strategy is designed to achieve the goal. Firstly, classify all offspring of crossover and mutation into M sub-populations $P^1, P^2, \ldots, P^M$ by using the method in Section 4.1. Then, find the set Q^i of non-dominated solutions of each sub-population P^i, i.e.,

$$Q^i = \{x \in P^i | x \text{ is a nondominated solution of } P^i\}$$

and choose the required number K of individuals in Q^i whose angles with λ^i are the first K smallest ones

$$P^i = \{x \in Q^i | \text{ angle between } x \text{ and } \lambda^i \text{ is among the first } K \text{ smallest ones}\}$$

where the population size $N = M * K$.

4.4. Steps of the proposed algorithm

According to the above discussion, a new multi-objective evolutionary algorithm is proposed and the steps of the algorithm are as follows.

Algorithm 1 (*The Proposed Algorithm*).

Step 1. Initialization:
- Step 1.1. Give the crossover and mutation probabilities $p_c > 0$ and $p_m > 0$. Randomly generate M weight vectors $\lambda^1, \lambda^2, \ldots, \lambda^M$ and an initial population $x^1, x^2, \ldots, x^{M*K}$ (where K is the size of the sub-population).
- Step 1.2. Initialize reference point $Y = (Y_1, \ldots, Y_m)$.

Table 1
Sample statistics for the weekly returns on the assets (2012–2015) from historical data.

Returns	5th percentile	40th percentile	60th percentile	95th percentile
Asset 1	−0.0627	−0.0065	0.0188	0.0869
Asset 2	−0.0626	−0.0094	0.0077	0.1111
Asset 3	−0.0798	−0.0062	0.0219	0.0913
Asset 4	−0.0709	−0.0076	0.0154	0.0951
Asset 5	−0.0609	−0.0047	0.0144	0.0804
Asset 6	−0.0804	−0.0093	0.0189	0.1143
Asset 7	−0.0867	−0.0053	0.0180	0.0975
Asset 8	−0.0861	−0.0044	0.0228	0.0992
Asset 9	−0.0859	−0.0029	0.0271	0.1177
Asset 10	−0.0868	−0.0124	0.0199	0.1457
Asset 11	−0.0700	0.0063	0.0139	0.1215
Asset 12	−0.0571	−0.0060	0.01930	0.0939

Step 1.3. The initial population is firstly grouped into N sub-populations $P^1, P^2, \ldots, P^M$ by the formula (21), and there are K solutions in each sub-population.

Step 2. Crossover: Randomly choose $[K * p_c]$ individuals from each P^i to undergo crossover. For each chosen one, randomly choose one individual from each of two nearest neighbor subpopulations of P^i. In this way, three individuals denoted by x^1, x^2, x^3 will be chosen. Then use crossover operator in Section 4.2 to generate offspring.

Step 3. Mutation: Randomly choose $[K * p_m]$ individuals from each P^i to undergo crossover. For each chosen one, use the mutation operator to generate an offspring.

Step 4. Selection: Use selection strategy in Section 4.3 to generate the sub-populations of the next generation.

Step 5. Update: Update each subpopulation by update strategy in Section 4.1.

Step 6. Termination: If stopping criteria is met, then stop; otherwise, go to Step 2.

5. Numerical examples and analysis

In this section, to demonstrate the idea of our model and the effectiveness of the designed algorithm, we conduct the experiments by using our proposed multi-objective algorithm to solve the multi-objective portfolio selection models whose data are taken from the historical data of the Shanghai Stock Exchange Market.

5.1. Data processing

In the experiments, we choose 12 assets from Shanghai Stock Exchange for the candidate assets. The exchange codes of the 12 assets are 601098, 601880, 600563, 600038, 601888, 601377, 600721, 600681, 600571, 600419,600570, 600201, respectively. For convenience of description, we denote the 12 assets successively as Assets 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 in the experiments. We collect original data from the weekly closing pricing and turnover rate of these assets in three years from January 2012 to January 2015 as the sample data. Using the simple estimation method in Vercher et al. [36], we get the statistics of the historical data of 12 rates. Table 1 provides the summary statistics of the data.

For the fuzzy return rates of the models, we use the sample percentiles to approximate the core and spreads of the trapezoidal fuzzy returns on the assets. In fact, we have decided to set the core $[a_i, b_i]$ of the fuzzy return $\tilde{r}_i$ as the interval [40th, 60th] and the quantities 40th − 5th and 95th − 60th as the left α_i and right β_i spreads, respectively, where kth is the kth percentile of the sample. Then, the corresponding membership function is given by Definition 1.

$$\mu_{\tilde{r}_i}(x) = \begin{cases} 1 - \dfrac{a_i - x}{\alpha_i}, & \text{if } a_i - \alpha_i \le x \le a_i \\ 1, & \text{if } a_i \le x \le b_i \\ 1 - \dfrac{x - d}{\beta_i}, & \text{if } b_i \le x \le b_i + \beta_i \\ 0, & \text{if otherwise.} \end{cases} \tag{26}$$

In the same way, we deal with the fuzzy turnover rates of ith asset by using the historical data. Then, the parameters of the fuzzy return rate $\tilde{r}_i = (a_i, b_i, \alpha_i, \beta_i)$ and the turnover rates $\tilde{l}_i = (c_i, d_i, \delta_i, \theta_i)$ of the ith asset are shown respectively in Tables 2 and 3.

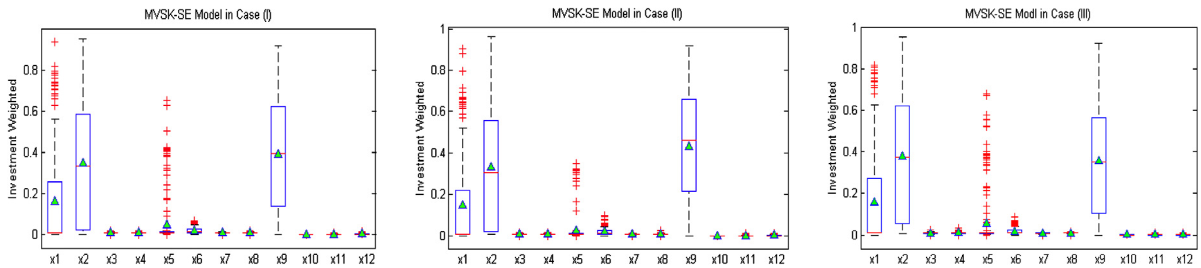
In our experiments, we assume that the transaction costs of assets are identical, i.e., $k_i = 0.003$ for all $i = 1, 2, \ldots, 12$, and the fuzzy turnover rate confidence level given by the investor is $\tilde{l}_0 = (0.0227, 0.0322, 0.0658, 0.2324)$.

Table 2
Fuzzy returns of 12 assets.

Asset i	1	2	3	4
Return rate	(−0.0065, 0.0188 0.0562 0.0681)	(−0.0094, 0.0077 0.0532 0.1034)	(−0.0062, 0.0219 0.0736 0.0694)	(−0.0076, 0.0154 0.0633 0.0797)
Asset i	5	6	7	8
Return rate	(−0.0047, 0.0144, 0.0562, 0.0660)	(−0.0093, 0.0189, 0.0711, 0.0954)	(−0.0053, 0.0180, 0.0814, 0.0795)	(−0.0044, 0.0228, 0.0817, 0.0764)
Asset i	9	10	11	12
Return rate	(−0.0029, 0.0271, 0.0830, 0.0906)	(−0.0124, 0.0199, 0.0744, 0.1258)	(0.0063, 0.0139, 0.0637, 0.1076)	(−0.0060, 0.0193, 0.0511, 0.0746)

Table 3
Fuzzy turnover rates of 12 assets.

Asset i	1	2	3	4
Turnover rate	(0.0416, 0.0662, 0.0224, 0.01315)	(0.0434, 0.0639, 0.0352, 0.2148)	(0.0526, 0.0657, 0.0290, 0.0599)	(0.0508, 0.0723, 0.0338, 0.0994)
Asset i	5	6	7	8
Turnover rate	(0.0220, 0.0278, 0.0124, 0.0571)	(0.0449, 0.0699, 0.0239, 0.1900)	(0.0723, 0.0990, 0.0481, 0.1264)	(0.0708, 0.0954, 0.0426, 0.1297)
Asset i	9	10	11	12
Turnover rate	(0.0499, 0.0820, 0.0315, 0.0965)	(0.0705, 0.0970, 0.0515, 0.2344)	(0.0299, 0.0503, 0.0194, 0.0875)	(0.0290, 0.0379, 0.0164, 0.0590)



(a) Portfolio distribution of MVSK-SE in case (I).
(b) Portfolio distribution of MVSK-SE in case (II).
(c) Portfolio distribution of MVSK-PE in case (III).

Fig. 1. Portfolio distribution of MVSK-SE model in three cases. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5.2. Parameter settings

The models are executed on a Genuine Intel CPU W35052.53 GHz and a 4.0 GB RAM personal computer with MATLAB software. The individuals are all coded as the real vectors. The parameter settings in this paper are as follows.

(1) Mutation probability is 0.05 and Crossover probability is 0.1.
(2) The population size is $N = 200$, the algorithm is run 30 times independently for each model. The algorithms stop after1000 generations.
(3) $K = 5$ and $M = 40$ ($N = K * M$).

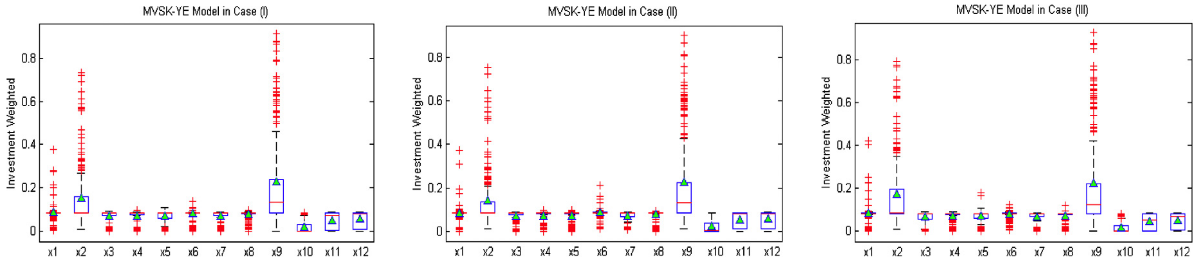
5.3. Comparisons of models MVSK-SE, MVSK-YE and MVSK-PE in decision space

In this subsection, the comparisons of the MVSK-SE, MVSK-YE and MVSK-PE portfolio models will be made through experiments based on the data of the Shanghai Stock Exchange Market.

Furthermore, in order to show the usefulness of our proposed model more clearly, we consider the following three cases: (I) returns of all assets remain unchanged, (II) returns of all assets decrease more than the sample returns up to 10% in Table 2, (III) returns of all assets increase more than the sample returns up to 10% in Table 2.

To visually compare the performance of these portfolio models, Figs. 1–3 show the box plots of average portfolio allocations that correspond to the MVSK-SE, MVSK-YE and MVSK-PE models, respectively, where x_i is the percentage of the sharing fund corresponding to the ith asset. In these Figs, the symbol '△' represents the mean of x_i, the box in the plot contains 50% of the data points from the 25th to 75th percentile, and the red line drawn across the box is the median of x_i. The whiskers are lines extending above and below each box.

Fig. 1 shows the box plots of average portfolio allocations for the MVSK-SE model in three cases. It can be seen from Fig. 1 that the MVSK-SE model prefers to a part of the portfolios. In fact, the vast majority of portfolios weights extremely concentrate on the 1th asset (x_1), 2th asset (x_2) and 9th asset (x_9), while the portfolio weights of other assets are very small, even negligible, which means the MVSK-SE model generates the low diversity of the portfolios. It may result in loss while some of the invested assets may experience unexpected gains. This is because that the Shannon's entropy measure is non-linear in portfolio selection which can result in very small positive weights for some assets in the portfolio.

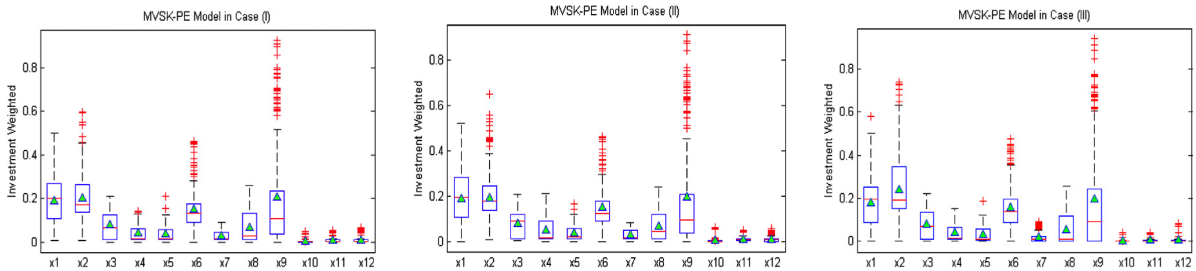


(a) Portfolio distribution of MVSK-YE in case (I).

(b) Portfolio distribution of MVSK-YE in case (II).

(c) Portfolio distribution of MVSK-YE in case (III).

Fig. 2. Portfolio distribution of MVSK-YE model in three cases. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) Portfolio distribution of MVSK-PE in case (I).

(b) Portfolio distribution of MVSK-PE in case (II).

(c) Portfolio distribution of MVSK-PE in case (III).

Fig. 3. Portfolio distribution of MVSK-PE model in three cases. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 2 shows the box plots of average portfolio allocations for the MVSK-YE model in three cases. It can be seen from Fig. 2 that the MVSK-YE model allocates almost equal the portfolio weights in each asset, except for the portfolio weights for the second (x_2) and ninth (x_9) assets which increase slightly. In essence, the MVSK-YE model allocates the budget onto assets almost equally. This is because that when $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$, the Yager's-entropy will uniquely take its maximum value. Thus, maximizing the Yager's-entropy will result in an equally-weighted portfolio. However, in the practical investment management, an investor often may not hope to distribute equally his/her wealth among all assets. Because such extreme equality of investment weights usually cannot provide good asset returns.

Fig. 3 shows the box plots of average portfolio allocations for the MVSK-PE model in three cases. Obviously, the MVSK-PE model generates a well-diversified portfolios, and not only avoids the low diversity of the portfolio but also avoids to allocate almost equal the portfolio weights to assets. This is because the linear feature of our proposed proportion entropy. The figure clearly shows that the MVSK-PE model is able to find a good set of diversified portfolios. The following subsections will give the quantitative comparison analysis of the three models.

5.4. Portfolio performance evaluation by using the hyper-volume indicator (HV) and generational distance (GD)

Because the diversity and convergence of solutions are very important for evaluating the performance of the algorithms, in order to compare the performance between the models MVSK-SE, MVSK-YE and MVSK-PE quantitatively, the generational distance (GD) [37] and hyper-volume indicator (HV) [38] are used as the performance metrics of the both diversity and convergence, where GD measures the distance of elements of the non-dominated vectors found from the ideal Pareto front, and is mathematically expressed as:

$$\begin{cases} GD\,(A,P) = \dfrac{\sqrt{\sum\limits_{v\epsilon A}(d(v,P))^2}}{|A|} \\ d\,(v,P) = \min\{\|v-z\|\;|z\epsilon P\} \end{cases} \tag{27}$$

where $|A|$ is the number of non-dominated solutions in the set A. If GD is equal to 0, all points of the known PF belong to the ideal PF. The smaller the value of GD is, the closer the solution is to the ideal PF, and the better the convergence and diversity of the set A is. The hyper-volume indicator has been widely used to evaluate the performance of the convergence, diversity and uniformity of the obtained solutions. It computes the volume of the dominated portion of the objective space relative to a reference point. Higher values of this performance indicator imply more diversity and better convergence of the obtained solutions.

Table 4
The statistical index of HV metric between MVSK-SE, MVSK-YE and MVSK-PE model.

HV-metric	Mean	SD	Max	Min
Case (I)				
MVSK-SE model	0.2228	0.0084	0.2330	0.2060
MVSK-YE model	0.2237	0.0175	0.2425	0.2137
MVSK-PE model	0.2758	0.0055	0.3092	0.2386
Case (II)				
MVSK-SE model	0.2276	0.0115	0.2399	0.1912
MVSK-YE model	0.2326	0.0221	0.2483	0.2222
MVSK-PE model	0.2863	0.0046	0.3293	0.2403
Case (III)				
MVSK-SE model	0.2144	0.0060	0.2200	0.2058
MVSK-YE model	0.2248	0.0129	0.2347	0.2097
MVSK-PE model	0.2637	0.0022	0.2922	0.2432

Table 5
The statistical index of GD metric between MVSK-SE, MVSK-YE and MVSK-PE model.

GD-metric	Mean	SD	Max	Min
Case (I)				
MVSK-SE model	0.3360	0.0047	0.3407	0.3306
MVSK-YE model	0.2501	0.0055	0.2591	0.2414
MVSK-PE model	0.2354	0.0030	0.2485	0.2243
Case (II)				
MVSK-SE model	0.3382	0.0066	0.3425	0.3325
MVSK-YE model	0.2537	0.0101	0.2892	0.2374
MVSK-PE model	0.2287	0.0027	0.2412	0.2132
Case (III)				
MVSK-SE model	0.3353	0.0043	0.3375	0.3220
MVSK-YE model	0.2479	0.0044	0.2624	0.2412
MVSK-PE model	0.2419	0.0014	0.2519	0.2333

Table 4 presents the index values of the HV metric of the MVSK-SE, MVSK-YE and MVSK-PE portfolio models by our algorithm. MVSK-PE model gets larger value of HV than the MVSK-SE and MVSK-YE models in three cases, which illustrates that the MVSK-PE model can provide more diversity solutions than the MVSK-SE and MVSK-YE models. Obviously, in both the aspects of convergence and diversity of solutions, the MVSK-PE model gives a better performance than the MVSK-SE and MVSK-YE models. This means that the MVSK-PE model can provide more diversified investment strategies for the investors comparing with the MVSK-SE and MVSK-YE models.

Table 5 presents the statistical index of GD metric value between MVSK-PE and one of MVSK-SE, MVSK-YE and MVSK-PE. It clearly shows that the index values of GD obtained by MVSK-PE model are smaller than those obtained by MVSK-SE and MVSK-YE models in three cases, which implies that the solutions obtained by MVSK-PE model can provide the better convergence and higher diversity of solutions than the MVSK-SE and MVSK-YE models in the PF.

5.5. Portfolio performance evaluation by using a specific performance metric

In real investment environment, the investors are most concerned about the risks and benefits. For the investors, the purpose of the establishment of the portfolio model is to reduce the risk and to improve the return. So, in order to accurately assess the benefits and risks, we use a specific metric based on the Euclidean norms [39] to measure the reliability of the solutions and to evaluate the performance of these portfolio models. The norm is calculated separately for both return and risk objectives. The aim is to measure the distance between the forecast risk/return for every portfolio in the efficient frontier $\left(V\left(x_p\right), M(x_p)\right)$ and the actual risk/return for the same portfolio be computed with the real parameters $\left(V'\left(x_p\right), M'(x_p)\right)$, that is, the norm between the estimates for t_n based on data from t_1 to t_{n-1}, and the actual values at t_n. The norm metric in terms of return and risk is formally defined for each efficient frontier in the following equations:

$$EN_{M(x)} = \left\| M\left(x_p\right) - M'(x_p)\right\|^2 /N \quad \text{and} \quad EN_{V(x)} = \left\| V\left(x_p\right) - V'\left(x_p\right)\right\|^2 /N \tag{28}$$

where N is the number of portfolio in the PF. According to the above mathematical expression, it can be seen that the smaller the value of $EN_{M(x)}$ and $EN_{V(x)}$ is, the closer the forecast risk/return will be to the actual risk/return for the portfolio, and the better the performance of the portfolio model will be.

The Mean, Maximum and Minimum values and Standard deviations of the norm metric values for MVSK-SE, MVSK-YE and MVSK-PE are recorded in three cases. In terms of the index values of the $EN_{M(x)}$ and $EN_{V(x)}$, MVSK-PE gets the smaller values than MVSK-SE and MVSK-YE in three cases, which illustrates that, the distances obtained by MVSK-PE from the forecasted risk and return to the actual ones are shorter than those obtained by MVSK-SE and MVSK-YE. That is, the portfolio obtained

Table 6
The metrics values for Euclidean distance between MVSK-SE, MVSK-YE and MVSK-PE model.

Evaluation-metric	$EN_{M(x)}$				$EN_{V(x)}$			
	Mean	SD	Max	Min	Mean	SD	Max	Min
Case (I)								
MVSK-SE model	0.0095	7.8751e−004	0.0096	0.0093	9.3389e−004	3.5319e−006	9.3779e−004	9.2728e−004
MVSK-YE model	0.0074	4.7159e−005	0.0075	0.0073	5.4625e−004	4.5061e−006	5.5327e−004	5.4014e−004
MVSK-PE model	0.0066	7.9868e−004	0.0067	0.0064	5.2197e−004	4.6813e−006	5.3083e−004	5.1390e−004
Case (II)								
MVSK-SE model	0.0096	8.8681e−005	0.0097	0.0094	9.3898e−004	4.8118e−006	9.4758e−004	9.3246e−004
MVSK-YE model	0.0075	5.6557e−005	0.0076	0.0074	5.4812e−004	5.7471e−006	5.5861e−004	5.3969e−004
MVSK-PE model	0.0068	602 403e−005	0.0069	0.0067	5.2951e−004	6.3610e−006	5.4061e−004	5.2064e−004
Case (III)								
MVSK-SE model	0.0083	8.0740e−005	0.0085	0.0082	7.8901e−004	2.8067e−006	7.9237e−004	7.8603e−004
MVSK-YE model	0.0072	5.5862e−005	0.0073	0.0071	5.2888e−004	3.1863e−006	5.3415e−004	5.2222e−004
MVSK-PE model	0.0065	5.4135e−005	0.0066	0.0064	5.0445e−004	2.0830e−006	5.0922e−004	5.0194e−004

Table 7
Summary statistics of ASR for MVSK-SE, MVSK-YE and MVSK-PE model.

ASR	Mean	Max	Min	SD
Case (I)				
MVSK-SE model	0.1140	0.1142	0.1138	1.3329e−004
MVSK-YE model	0.1305	0.1309	0.1297	1.9296e−004
MVSK-PE model	0.1403	0.1404	0.1401	4.9994e−004
Case (II)				
MVSK-SE model	0.1056	0.1057	0.1054	1.5465e−004
MVSK-YE model	0.1230	0.1237	0.1222	8.4012e−005
MVSK-PE model	0.1355	0.1358	0.1352	3.9787e−004
Case (III)				
MVSK-SE model	0.1211	0.1213	0.1208	1.6114e−004
MVSK-YE model	0.1362	0.1367	0.1357	5.4279e−005
MVSK-PE model	0.1445	0.1446	0.1444	3.0790e−004

by MVSK-PE is closer to the actual risk/return for the portfolio. Thus, the performance of the portfolio model MVSK-PE is better.

5.6. Portfolio performance evaluation by using the adjusted sharpe ratio

As a traditional performance measure, the Sharpe ratio (SR) is a commonly used measure of portfolio performance and its formula is given by the following general form:

$$\mathrm{SR} = \frac{E\left[r(x)\right]}{\sqrt{V(r(x))}} \tag{29}$$

where $r(x)$ is the return of portfolio. However, since the SR is based on the mean–variance theory, it is valid only for either normally distributed returns or quadratic preferences. Particularly, the SR can lead to misleading conclusions when the return distributions are skewed or kurtosis [40]. In order to overcome the drawbacks of the SR, Pezier et al. [41] proposed an Adjusted Sharpe Ratio (ASR), which can measure portfolio performance in the mean–variance–skewness–kurtosis framework of portfolio model. The ASR takes into accounts not only the mean and variance but also the skewness and kurtosis of portfolio, which is defined as follows:

$$\mathrm{ASR} = \mathrm{SR}\left[1 + (S/6)\,\mathrm{SR} - (K/24)\mathrm{SR}^2\right] \tag{30}$$

where S and K are the skewness and kurtosis of portfolio respectively. The value of ASR can reflect the level of maximum expected utility. The larger value of ASR is, the better the investment portfolio and performance of the model will be.

Table 6 presents the statistical results of ASR for each model in the three cases, including min, max, mean and standard deviation of the ASR. We can see that the various statistical indicators of ASR of MVSK-PE are larger than the corresponding MVSK-SE's and MVSK-YE's. That is, the MVSK-PE model outperforms the MVSK-SE and MVSK-YE models in terms of ASR performance measure.

5.7. The comparison of the total return among the models MVSK-PE, MVSK-SE and MVSK-YE

In the real investment circumstances, investors pay more attention to the profitability of executing these asset strategies. Furthermore, we discuss their economic benefits in portfolio management. Table 8 shows the comparison of realization total

Table 8
Summary statistics of total return among the models MVSK-PE, MVSK-SE and MVSK-YE in three cases.

Total return	Mean	Max	Min	SD
Case (I)				
MVSK-SE model	0.0048	0.0075	0.0034	0.0009
MVSK-YE model	0.0055	0.0077	0.0035	0.0010
MVSK-PE model	0.0060	0.0078	0.0035	0.0012
Case (II)				
MVSK-SE model	0.0040	0.0065	0.0027	0.0008
MVSK-YE model	0.0047	0.0066	0.0031	0.0009
MVSK-PE model	0.0053	0.0070	0.0033	0.0011
Case (III)				
MVSK-SE model	0.0056	0.0085	0.0040	0.0010
MVSK-YE model	0.0063	0.0086	0.0043	0.0012
MVSK-PE model	0.0067	0.0088	0.0045	0.0013

return among the models MVSK-PE, MVSK-SE and MVSK-YE in three cases respectively, including minimum, maximum, mean, and standard deviation.

From Table 7, the total return of the model MVSK-PE outperforms the models MVSK-SE and MVSK-YE in three cases. Furthermore, in the case of comparing the model MVSK-PE with models MVSK-SE and MVSK-YE, each statistical indicator of the total return is much larger than that of the models MVSK-SE and MVSK-YE. Taken together, the model MVSK-PE demonstrates more effectively than the models MVSK-SE and MVSK-YE. The results of numerical examples suggest that the model MVSK-PE can provide well-diversified Pareto optimal solutions than the models MVSK-SE and MVSK-YE in the decision space, and the model MVSK-PE can make asset allocation more feasible than the models MVSK-SE and MVSK-YE.

6. Conclusions

Portfolio optimization inherently involves conflicting criteria. In this sense, this work focuses on the study of the fuzzy higher moment portfolio selection that explicitly involves skewness and kurtosis in the multi-objective framework, furthermore, a new entropy function based on Minkowski metric is presented as an objective function to obtain better diversification portfolios. Next, two variations of mean–variance–skewness–kurtosis–entropy multi-objective portfolio models with Shannon's entropy and the Yager's entropy were also discussed. To solve the proposed higher moment multi-objective portfolio models, a new multi-objective evolutionary algorithm based on decomposition is designed. In addition, we use several portfolio performance evaluation techniques to evaluate the performance of these higher-moment multi-objective portfolio models. Finally, some numerical examples are presented to illustrate the practicality and effectiveness of the proposed model and the corresponding algorithm based on the data from Shanghai Stock Exchange.

For the future research, the multi-objective fuzzy portfolio selection problem and MOEAs will be applied to other asset allocation problems, mutual fund portfolio selection problems, combinational optimization models and multi-period problems. Asset returns can be considered as random fuzzy variables. Also the proposed model can be applied for other case studies. Moreover, the new proposed models of portfolio selection problems and their efficient solution methods will help us to solve more complicated problems in real situations under more imprecise and ambiguous conditions.

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