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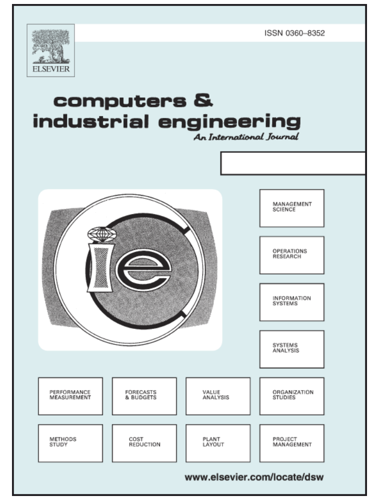

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Sequential Approximate Approach to the p-Median Problem

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Abstract

This paper deals with the problem of designing the optimal structure of most public service systems, which is often formulated as the p-median problem. The real instances of these problems are characterized by a considerably big number of possible service center locations, which can take the value of several thousands. Current exact approaches must face up to a big demand on computational time and they often fail when a large instance is being solved. This paper is focused on the approximate approach based on specific model reformulation. It uses the approximation of a common distance by some pre-determined distances given by so-called dividing points. The deployment of the dividing points influences the solution accuracy. To improve this approach, we have developed the sequential method of dividing points deployment. Hereby, we study the accuracy of the suggested method using the upper and lower distance approximations in comparison to the saved computational time.

Keywords: Location, Health service, Integer Programming, Optimization

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Abstract

This paper deals with the problem of designing the optimal structure of most public service systems, which is often formulated as the p-median problem. The real instances of these problems are characterized by a considerably big number of possible service center locations, which can take the value of several thousands. Current exact approaches must face up to a big demand on computational time and they often fail when a large instance is being solved. This paper is focused on the approximate approach based on specific model reformulation. It uses the approximation of a common distance by some pre-determined distances given by so-called dividing points. The deployment of the dividing points influences the solution accuracy. To improve this approach, we have developed the sequential method of dividing points deployment. Hereby, we study the accuracy of the suggested method using the upper and lower distance approximations in comparison to the saved computational time.

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1. Introduction

Public service systems play an important role in satisfaction of public demand for more secure life. The family of the public systems [1] includes medical emergency system [2, 3], fire-brigade deployment, system of police stations, public administration system [4] and many others. The quality of the service provided by the system is influenced by more or less suitable deployment of service centers, which serve individual communities situated at dwelling places of the served geographical area. The objective of the public service system design is to minimize total social costs, which are proportional to the distances from serviced communities to the nearest source of provided service. The mathematical model of the public service system design problem is often related to the p-median problem, which is formulated as the task of selection of at most p network nodes as service center locations, so that the sum of the distances between each served community location and the nearest located service center is minimal. When a real public service system is projected, the designer must face up to many difficulties and limitations. Besides the limited budget included into the problem formulation, they are subject to lack of time to develop a specific sufficient software tool for rational system design. That is why a general commercial IP-solver or some other ready-made software tool must be used to obtain a solution of the underlying large p-median problem. As concerns usage of a general IP-solver, the size of the solved integer programming problem must be taken into account. In the real problems, the number of serviced users takes the value of several thousands, and the number of possible service center locations can take this value as well [5]. The number of possible service center locations seriously impacts the computational time and the memory of computer due to used branch-and-bound method, which stores the unfathomed nodes of the inspected searching tree for the further processing. That is why the direct attempt at solving the problem described by a location-allocation model often fails, when larger instances are solved by a commercial IP-solver [6]. Another way of the p-median problem representation consists in the radial formulation either heterogeneous [7] or homogenous [8, 9]. This approach avoids assigning individual users to a located service center, and thus only information about the number of located centers in a given radius is dealt with. The radial approach leads to the model similar to the set covering problem, which is easy to solve in comparison to the location-allocation problems. Nevertheless, even if the radial model is used to obtain the exact optimal solution of a large instance by a commercial IP-solver, the memory and time limits can be exceeded.

In this paper, we suggest an approximate approach based on the radial problem formulation. The approximate approach can be easily implemented within a commercial IP-solver and it enables a trade-off between computational time and computer memory demands and accuracy of the resulting solution, what can considerably help to solve the designer?s time-table difficulties. Contrary to heuristics [10–13], the suggested approach provides a lower bound of the optimal solution together with the resulting feasible solution and its objective function value representing the upper bound of the unknown optimal solution. In addition, the suggested approximate approach enables to control the gap between both bounds, which is the substantial difference between the suggested approach and common approximate algorithms, where performance ratio is usually fixed [14]. The mentioned trade-off between time and accuracy consists in a reduction of the set of all distance values among possible center locations and user locations. The process of reduction is performed sequentially in several phases, where the current set of relevant distance values is selected from the preceding set according to a weight, which represents a current distance relevance. At the beginning of the reduction process, the relevance is initialized by an anticipated frequency of the distance values in the unknown optimal solution. Each next phase improves the relevance estimation gained through the previous phase. The reduction process is used not only for obtaining a good solution of the problem, but also for gaining a lower bound of the unknown optimal solution.

The remainder of this paper follows the below-mentioned scheme. Section 2 contains the location-allocation formulation of the p-median problem. Section 3 comprises the approximate approach based on a given sequence of dividing points, where a covering model is used to obtain both lower and upper bound of the optimal solution value together with a near-optimal solution. Section 4 presents the determination of dividing points, which enable to minimize the estimated deviation of the upper and lower bounds from the optimal solution value. Section 5 introduces a sequential approach, which adjusts the distances between the dividing points and improves the solution of the original problem step-by-step. Section 6 reports on performed numerical experiments and gives the information about the efficiency of the suggested approach.

2. Exact Approach to the p-Median Problem

To describe the p-median problem on a network, we denote a set of serviced nodes by J, similarly, we denote a set of possible service center locations by I. Here, we use only the formulation of the p-median problem, where at most p locations from the set I should be determined so that the sum of the network distances from each element of J to the nearest located service center is minimal. The network distance between the possible location i from I and the users location j from J is denoted as d_{ij}. The basic decisions in any solving process of the p-median problem concern the location of service centers at the network nodes from the set I. The p-median problem can be modeled by linear constraints and objective functions using zero-one variables $y_i \in \{0,1\}$ for $i \in I$, where the variable y_i takes the value of 1, if a service center should be located at the place i from I and it takes the value of 0 otherwise. In addition, the further defined allocation variables are used. The allocation variables $z_{ij} \in \{0,1\}$ for each $i \in I$ and $j \in J$ are introduced to assign the users location j to the possible location i by the value of one. Then the location-allocation model can be formulated as follows:

$$Minimize \qquad \sum_{i\in I}\sum_{j\in J} d_{ij}z_{ij} \tag{1}$$

$$Subject\ to: \qquad \sum_{i\in I} z_{ij} = 1 \qquad for\ \ j \in J \tag{2}$$

$$z_{ij} \leq y_i \qquad for\ \ i \in I\ \ and\ \ j \in J \tag{3}$$

$$\sum_{i\in I} y_i \leq p \tag{4}$$

$$z_{ij} \in \{0,1\} \qquad for\ \ i \in I\ \ and\ \ j \in J \tag{5}$$

$$y_i \in \{0,1\} \qquad for\ \ i \in I \tag{6}$$

In the above model, the allocation constraints (2) ensure that each users location is assigned to exactly one possible service center location. Link-up constraints (3) enable to assign a users location j to a possible location i only if the service center is located at this location, and constraint (4) bounds the number of located service centers. The problem described by the terms (1) - (6) can be rewritten into a form acceptable by the modeler of integrated optimization environment, and solved by the associated IP-solver. Due to the huge number of allocation variables z_{ij}, a commercial software usually fails when a very large instance of the problem (1) - (6) is being solved.

3. Radial Formulation of the *p*-Median Problem for Lower and Upper Bounds

The radial formulation of the p-median problem has appeared in two different versions. One of them [7] is based on the systems of radii, where each users location has their unique system, and an individual radius corresponds with some concrete distance between the users location and some possible location of a service center. Our approach is based on the second version [8], where the range of all considered distances is partitioned by so-called dividing points, and an individual radius corresponds with the position of a dividing point. In this version, the same system of radii is applied to each users location. In the radial formulation, we use the above-introduced notation. As above, the variable $y_i \in \{0,1\}$ models the decision of service center location at the place $i \in I$.

Presented approximate approach is based on the relaxation of the assignment of a service center to a system user [6]. Information about the number of service centers located in a given radius from the given users location is used instead of formalized knowledge of the nearest located service center. We use the fact that there is only a finite number of various distance values in the matrix $\{d_{ij}\}$ which can enter the optimal solution of the associated p-median problem. Note that none of the largest, second largest and so on to $p-1$ largest distances from given users location j to the set of all possible locations can be contained in any optimal solution. Let the mentioned set of $m+1$ different distance values form an increasing sequence $d_0 < d_1 < \cdots < d_m$. Without loss of generality, we can assume that d_0 is equal to zero; in the opposite case, we can reduce each item of the matrix subtracting the minimal value. To obtain the upper approximation of the original objective function value, the range $[d_0, d_m]$ is partitioned into $v+1$ zones. The zones are separated by a finite ascending sequence of the dividing points $D_1, D_2, \ldots, D_v$, which are chosen from the values $d_1 < d_2 < \cdots < d_{m-1}$. Let us denote $D_0 = d_0$ and $D_{v+1} = d_m$. Then the zone s corresponds with the interval $(D_s, D_{s+1}]$ for $s = 0, \ldots, v$. The length of the s-th interval is denoted by e_s.

In addition to the location variables y_i, the auxiliary zero-one variables x_{js} for $s = 0, \ldots, v$ are introduced. The variable x_{js} takes the value of 1, if the distance of the users location $j \in J$ from the nearest located service center is greater than D_s, and this variable takes the value of 0 otherwise. Then the expression $e_0 x_{j0} + e_1 x_{j1} + \cdots + e_v x_{jv}$ constitutes the upper approximation of the distance d_{j*} from the users location j to the nearest located service center. If the distance d_{j*} belongs to the interval $(D_s, D_{s+1}]$, it is estimated by the upper bound D_{s+1}.

Similarly to the covering model, we introduce a zero-one constant a_{ij}^s for each triple $[i,j,s] \in I \times J \times \{0, \ldots, v\}$. The constant a_{ij}^s is equal to 1, if the distance d_{ij} between the users location j and the possible service center location i is less than or equal to D_s, otherwise a_{ij}^s is equal to 0. Then the covering-type model can be formulated as follows:

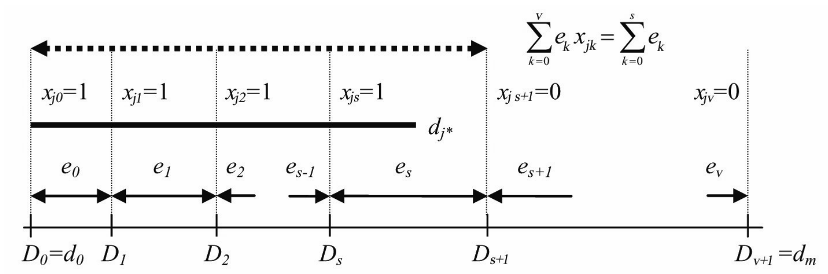

Figure 1: Upper approximation of d_{j*} using zone widths e_s and auxiliary variables x_{js}. The upper approximation of d_{j*} is denoted by thick dotted line at the top of figure.

$$Minimize \qquad \sum_{j\in J}\sum_{s=0}^{v} e_s x_{js} \tag{7}$$

$$Subject\ \ to: \qquad x_{js} + \sum_{i\in I} a_{ij}^{s} y_i \geq 1 \qquad for\ \ j\in J\ \ and\ \ s=0,\ldots,v \tag{8}$$

$$\sum_{i\in I} y_i \leq p \tag{9}$$

$$x_{js} \geq 0 \qquad for\ \ j\in J\ \ and\ \ s=0,\ldots,v \tag{10}$$

$$y_i \in \{0,1\} \qquad for\ \ i\in I \tag{11}$$

Objective function (7) gives the upper bound of the sum of the original distances from user locations to the located service centers. Constraints (8) ensure that the variables x_{js} are allowed to take the value of 0, if there is at least one center located in the radius D_s from the user location j. As the minimization process applied on (7) pushes all included variables x_{js} down to the zero value and each of these variables is limited from below either by the value of one or by the value of zero according to (8), the variable x_{js} can get only one of these values unless obligatory 0-1 constraints must be included into the model. Constraint (9) limits the number of located service centers by the integer p.

To obtain a lower bound of the original problem optimal solution, we can either use the above found dividing points and the associated zone widths, and express the lower bound of d_{j*} as $e_0x_{j1}+e_1x_{j2}+\cdots+e_{v-1}x_{jv}$, or suggest a tighter lower bound in the following way.

We realize that the interval $(D_s, D_{s+1}]$ given by a pair of successive dividing points contains a portion of successive elements of the sequence $d_0 < d_1 < \cdots < d_m$. Let us denote these elements from the interval $(D_s, D_{s+1}]$ as $D_s^1, D_s^2, \ldots, D_s^{v(s)}$, where $D_s^1 < D_s^2 < \cdots < D_s^{v(s)}$. These elements are strictly greater than D_s and less than or equal to D_{s+1}. If the distance d_{j*} between a user location and a possible service center location belongs to the interval $(D_s, D_{s+1}]$, then the lower and upper bounds of d_{j*} are D_s^1 and D_{s+1} respectively and the maximal deviation of d_{j*} from the lower estimation is $D_{s+1} - D_s^1$. As the variable x_{js} from the model (7) - (11) takes the value of 1, if the distance of the user location $j \in J$ from the nearest located service center is greater than D_s and this variable takes the value of 0 otherwise, we can replace the zone coefficients e_s with the coefficients $\underline{e}_s$, where $\underline{e}_0 = D_0^1 - D_0$ and $\underline{e}_s = D_{s+1}^1 - D_s^1$ for $s = 1, \ldots, v$.

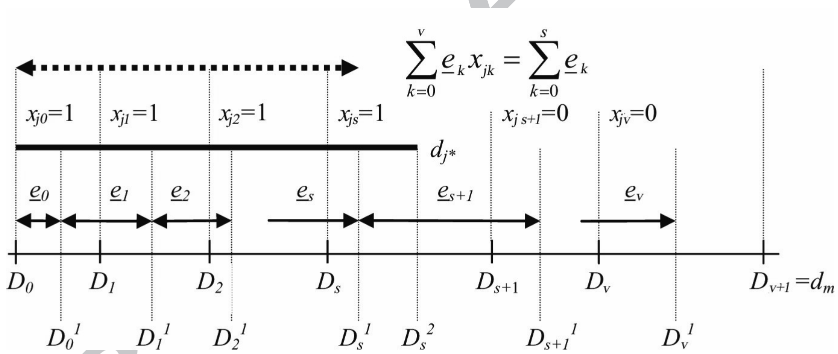

Figure 2: Lower approximation of d_{j*} using zone widths $\underline{e}_s$ and auxiliary variables x_{js}. The lower approximation of d_{j*} is denoted by thick dotted line at the top of figure.

Then the expression $\underline{e}_0x_{j0} + \underline{e}_1x_{j1} + \cdots + \underline{e}_vx_{jv}$ constitutes the lower approximation of d_{j*}, which corresponds to the distance of the node j from the nearest located service center. Then, the optimal objective function value of the following problem gives the lower bound of the objective function value of the original problem [15].

$$Minimize \qquad \sum_{j\in J}\sum_{s=0}^{v} \underline{e}_s x_{js} \tag{12}$$

$$Subject\ to: \qquad (8) - (11)$$

Having solved both problems, i.e. (7) - (11) and (12), (8) - (11), the better of two obtained solutions concerning the original objective function value (13) gives the resulting solution of this approach, and the optimal value of (12) gives the lower bound of the unknown optimal solution.

$$\sum_{j \in J} min \{d_{ij} : i \in I, \ \ y_i = 1\} \tag{13}$$

4. Optimal Deployment of Dividing Points

Obviously, the number v of the dividing points $D_1, D_2, \ldots, D_v$ influences the size of the covering model (7) - (11) as concerns either the number of the variables x_{js} or the number of the constraints (8). That is why the number v must be kept in a mediate extent to achieve the resulting solution quickly enough. On the other hand, the smaller the number of dividing points is, the bigger inaccuracy afflicts the approximate solution. Let us focus now on the problem of the efficient deployment of the given number of the dividing points in the set of the values $d_0 < d_1 < \cdots < d_m$. As before, let us denote $D_0 = d_0$ and $D_{v+1} = d_m$. Let the value d_h have the frequency N_h of its occurrence in the matrix $\{d_{ij}\}$.

We start from the hypothesis that the distance d_h from the sequence $d_0 < d_1 < \cdots < d_m$ occurs in the resulting solution n_h times, and that is why the deviation of this distance from its approximation encumbers the total deviation proportionally to n_h, where $n_h \leq N_h$.

When the distance d from a users location to the nearest located service center is estimated by some upper estimation, the nearest bigger dividing point D_{s+1} is used. The dividing point serves as the upper estimation for each distance of the sequence $d_0 < d_1 < \cdots < d_m$ which belongs to the interval $(D_s, D_{s+1}]$. It means that if the estimated distance is d_h, then the deviation from the upper bound is $D_{s+1} - d_h$. If n_h is the anticipated frequency of the distance d_h in the unknown optimal solution, then the difference $D_{s+1} - d_h$ encumbers the resulting deviation n_h times. Hence, the contribution of d_h estimation to the total deviation is $n_h(D_{s+1} - d_h)$. After these preliminaries, we determine the dividing point deployment so that the total deviation of the upper approximation from the unknown optimal solution is minimal. We introduce zero-one variables u_{ht} for each possible position t of the dividing point $d_t (t = 1 \ldots m)$ and for each possible position h of the preceding value $d_h (h = 0 \ldots t)$. If the distance d_h belongs to the interval ending by the

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dividing point d_t, then the variable u_{ht} takes the value of 1, otherwise 0. If u_{tt} is equal to one, then the distance d_t corresponds with the dividing point.

$$Minimize \qquad \sum_{t=1}^{m}\sum_{h=1}^{t}(d_t - d_h)n_h u_{ht} \tag{14}$$

$$Subject\ to: \quad u_{(h-1)t} \leq u_{ht} \qquad for\ \ t = 2,\dots,m\ \ and\ \ h = 2,\dots,t \tag{15}$$

$$\sum_{t=h}^{m} u_{ht} = 1 \qquad for\ \ h = 1,\dots,m \tag{16}$$

$$\sum_{t=1}^{m-1} u_{tt} = v \tag{17}$$

$$u_{ht} \in \{0,1\} \qquad for\ \ t = 1,\dots,m\ \ and\ \ h = 1,\dots,t \tag{18}$$

In the above model, link-up constraints (15) ensure that the distance d_{h-1} belongs to the interval ending with d_t only if each other distance between d_{h-1} and d_t belongs to this interval. Constraints (16) assure that each distance d_h belongs to some interval, and constraint (17) enables only the number v of dividing points. After the problem (14) - (18) is solved, the nonzero values of u_{tt} indicate the distances d_t which correspond to the dividing points.

Similar approach can be applied to obtain the efficient deployment of the dividing points to determine a good lower bound of the optimal solution of the original problem [15]. Nevertheless, several differences in the way of approximation must be taken into account. When the lower bound is computed, the expression $\underline{e}_0 x_{j0} + \underline{e}_1 x_{j1} + \cdots + \underline{e}_v x_{jv}$ is used as the lower approximation of d_{j*}. Here $\underline{e}_0 = D_0^1 - D_0$ and $\underline{e}_s = D_{s+1}^1 - D_s^1$ for $s = 1,\dots,v$. It follows that if the distance d_{j*} belongs to the interval $(D_s, D_{s+1}]$, the lower estimation of d_{j*} is not D_s, but D_s^1, which is the smallest value of the sequence which belongs to the interval. The value D_s^1 is the following element to the dividing point D_s concerning the sequence. It means that if the estimated distance is d_h, then the deviation from the lower bound is $d_h - D_s^1$. If n_h is the anticipated frequency of the distance d_h in the unknown optimal solution, then the difference $d_h - D_s^1$ encumbers the resulting deviation n_h times. Hence, the contribution of d_h estimation to the total deviation is $n_h(d_h - D_s^1)$. After these preliminaries, we determine the dividing point deployment so that the total deviation of the lower approximation from the unknown optimal solution is minimal. We introduce the zero-one variables w_{th} for each possible position t of the dividing point d_t and its successor $d_{t+1}(t = 0 \dots m-1)$ and for each possible position h of the succeeding value $d_h(h = t \dots m)$. If the distance d_h belongs to the interval starting with the dividing point d_t, then the decision variable w_{th} takes the value of 1, otherwise 0. If w_{tt} is equal to one, then the distance d_t corresponds to the dividing point. If $w_{th} = 1$ for $t < h$, then d_{h+1} is estimated by d_{t+1}, which corresponds to D_s^1 for some s.

$$Minimize \qquad \sum_{t=0}^{m-1} \sum_{h=t}^{m-1} (d_{h+1} - d_{t+1}) n_{h+1} w_{th} \tag{19}$$

$$Subject\ to: \quad w_{t(h+1)} \leq w_{th} \qquad for\ \ t = 0, \dots, m-1\ \ and\ \ h = t, \dots, m-1 \tag{20}$$

$$\sum_{t=0}^{h} w_{th} = 1 \qquad for\ \ h = 0, \dots, m-1 \tag{21}$$

$$\sum_{t=1}^{m-1} w_{tt} = v \tag{22}$$

$$w_{th} \in \{0,1\} \qquad for\ \ t = 0, \dots, m-1\ \ and\ \ h = t, \dots, m \tag{23}$$

Similarly to the previous model, link-up constraints (20) ensure that the distance d_{h+1} can belong to the interval starting with d_t only if each distance between d_{h+1} and d_t belongs to this interval. Constraints (21) assure that each distance d_h belongs to some interval, and the constraint (22) enables that only v dividing points will be chosen. After the problem (19) - (23) is solved, the nonzero values of w_{tt} indicate the distances d_t which correspond to the dividing points for the lower bounding process.

5. Sequential Improving of Distance Relevancies

The static approach to the dividing points determination comes from the hypothesis that the frequencies n_h of d_h in the unknown optimal solution may be proportional to N_h, and they decrease for longer distances. To formalize this hypothesis, the following expression was used [6]:

$$n_h = N_h e^{\frac{-d_h}{T}} \tag{24}$$

In the expression (24), T is a positive shaping parameter and N_h is the mentioned occurrence frequency, where only the $|I|-p+1$ smallest distances of each column of the matrix $\{d_{ij}\}$ are considered. After the anticipated frequencies n_h had been determined, the models (14) - (18) and (19) - (23) were used to obtain the series of the dividing points for the upper and lower bound respectively. Then the models (7) - (11) and (12), (8) - (11) were used to obtain the upper and lower bounds, and also the associated resulting solution of the original problem.

To the contrary with the static approach, the presented sequential improvement of the relevancies n_h is based on the idea of making the estimation of the individual distance d_h relevance more accurate [16]. The distance relevance here also means a measure of our expectation that this distance value is the distance between a users location and the nearest located service center, but this estimation is improved step-by-step by the following algorithm, which can be used either for the lower or upper bound determination. The input of the algorithm consists of the matrix $\{d_{ij}\}$, sequence $d_0 < d_1 < \cdots < d_m$ and the associated sequence of the frequencies N_h for $h = 0 \dots m$, and the number p of centers which are to be located. Further parameters T and v of the algorithm must be given, where T is the shaping parameter and v is the number of dividing points. Determination of suitable settings of the parameter T was studied in [17] and it was found that the value of 1 is the most suitable one.

The sequential algorithm

Step 1: Determine the initial values of the relevancies n_h according to (24).

Step 2: Compute the sequence of the dividing points $D_1 \dots D_v$ by the model (14) - (18).

Step 3: Using the sequence of the dividing points, determine the constants a_{ij}^s and e_s and solve the covering problem (7) - (11) to the optimality, to obtain the optimal values of the location variables y. Determine the value of the original objective function according to (13) and update the best found solution.

Step 4: If the stopping rule is met, terminate, otherwise go to Step 5.

Step 5: Determine the set I_1 of the active rows according to $I_1 = \{i \in I : y_i = 1\}$. Update the relevancies n_h so that each column of the matrix $\{d_{ij}\}$ is processed, and only minimal value over the active rows is included into the set of the relevant distances and their occurrence frequencies. Go to Step 2.

The above algorithm can be easily converted to the lower bound algorithm by replacing the model (14) - (18) in Step 2 by the model (19) - (23), and by replacing the model (7) - (11)) in Step 3 with the model (12), (8) - (11).

The effectiveness of suggested algorithm and the time necessary to find the resulting solution depend on the criteria of terminating the iteration process. The easiest way consists of the basic condition that the computing process is to be stopped whenever no better solution of the original problem is obtained. Since the sequential method may perform too many iterations with very little improvement of the objective function value, we suggest limiting the number of the performed iterations. In the following section, we study some additional rules that make the algorithm less time-consuming.

The above algorithm starts with an initial relevance estimation described in the previous section, and computes relevancies n_h in accordance to the hypothesis formalized by the expression (24). Having obtained the first optimal solution of the covering model following the dividing points deployed according to the initial relevancies, the algorithm updates the relevancies. For this purpose, a set of active matrix rows is defined so that the i-th row of the matrix $\{d_{ij}\}$ is denoted as active, if the location variable y_i of the problem (7) - (11) is equal to one. Then each column j of this matrix is processed, the minimal value over the active rows is included into the set of the relevant distances, and the associated frequency is increased. Thus a new sequence of the distance frequencies n_k is obtained. These new frequencies are used in the next iteration of the algorithm. This process can be repeated as long as better solution of the original problem keeps being obtained, or until the used stopping criterion is met.

6. Computational Study

6.1. Preliminary Numerical Experiments

We performed a sequence of numerical experiments to test the effectiveness of suggested sequential method. When the upper distance approximation was used, the following stopping criteria were applied: the iteration process was terminated if either ten iterations had been performed, or no improvement in the actual iteration had been achieved. Furthermore, one additional rule was defined. The computational process was stopped when the number of the distances d_h with $n_h > 0$ had decreased below the number of the dividing points v.

All experiments were performed using the optimization software XPRESS-IVE 1.17.12. The associated code was run on a personal computer equipped with the Intel Core 2 6700 processor with the parameters: 2.66 GHz and 3 GB RAM. The solved instances of the problem were obtained from the $OR - Lib$ set of the p-median benchmarks [18]. To make the results more comparable, we have changed some instances in the maximal number of the located service centers p, and grouped the problems by the number of possible service center locations $|I|$. This way, we obtained 9 sets of benchmarks, where the number of possible service center locations varied from 100 to 900 by hundreds. Each set of the test problems consists of 6 different instances, where the ratio of $|I|$ to p equals 2, 3, 4, 5, 10 and 20 respectively. Since the approximate approach is dedicated to large p-median instances, we decided to enlarge the set of test problems by using the data from the road network of Slovakia. According to the previous principle, we kept increasing the number $|I|$ from 1000 by hundreds up to 1200, when the exact approach based on the location-allocation model failed due to lack of disposable computer memory. The sequential approach was tested on even larger instances. Of course, it is not standard to compare exact and heuristic methods, but the exact solution was used here to evaluate the accuracy of the solution obtained by the sequential covering approach. To obtain the exact solution, we used common optimization environment XPRESS-IVE, which was used also for the approximate sequential method.

An individual experiment was organized so that the number v of the dividing points was set to 20. Since this is an important parameter of our approach, we have studied also the accuracy sensitivity on the number of dividing points. Our findings are summarized in the following subsection together with the reasons for using just 20 dividing points. The initial sequence of the dividing points was obtained according to the model (14) - (18) or (19) - (23), where the relevance n_h of each distance d_h was computed in accordance to the formula (24) for $T = 1$. The obtained results for the upper distance approximation are shown below. Time comparison of the exact and sequential approximate approach is shown in the Figure 3. The solved instances are grouped by the number of possible service center locations $|I|$. There are two columns with the average computational time for each size of the set I represented by 6 different instances. Each column represents one solving method. The achieved results are interesting from various points of view. As it is shown, the exact location-allocation approach failed in large instances due to enormous demands on the memory capacity, but our suggested sequential approach based on the radial formulation can overcome this weakness thanks to lower number of variables and simpler model structure. Furthermore, the average computational time of the sequential method does not grow with increasing size of solved instance as rapidly as in the location-allocation approach. This feature plays a very important role. It destines the sequential method for solving large problem instances.

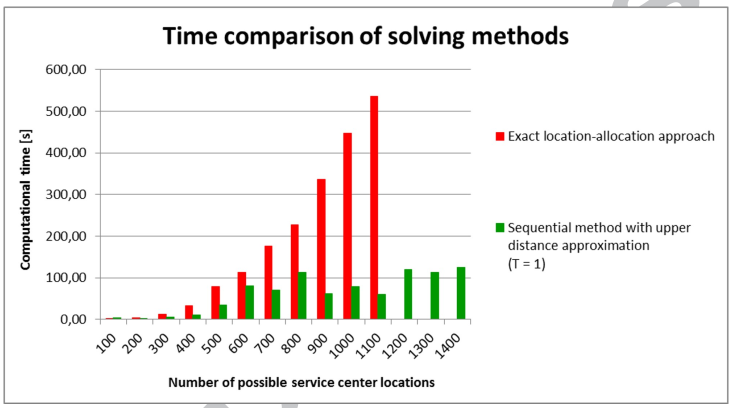

Figure 3: Time comparison of the exact location-allocation approach to the sequential method with the upper distance approximation

The sequential covering approach with the upper distance approximation was compared to the exact method based on the location-allocation formulation also from the viewpoint of the solution accuracy. Everywhere it was possible, the exact solution obtained by the location-allocation model (1) - (6) was used to evaluate the quality of the result provided by the sequential method. Generally, the quality of the approximate solution is evaluated by so-called gap_{BFSE}, which is defined as follows: Let ES denote the objective function value of the exact solution obtained as the result of the model (1) - (6). If BFS (Best Found Solution) denotes the real objective function value of the approximate solution computed according to the formula (13), then the gap_{BFSE} expresses the difference between the best found solution and the exact one in percentage of the exact solution. Its value is defined by (25).

$$gap_{BFSE} = \frac{|BFS - ES|}{ES} * 100 \tag{25}$$

The associated results of numerical experiments are given in Table 1, which contains the average and the maximal value of gap_{BFSE} for each size of the set I represented by 6 different instances.

$\|I\|$	100	200	300	400	500	600	700	800	900	1000	1100
Maximal gap_{BFSE}	0.74	0.57	0.30	0.40	4.46	0.58	4.05	0.24	0.56	0.26	0.00
Average gap_{BFSE}	0.24	0.23	0.13	0.11	0.78	0.13	0.73	0.06	0.10	0.04	0.00

Table 1: Accuracy of the solution obtained by the sequential approach with upper distance approximation

The obtained results reported in Table 1 show that the sequential approach with the upper distance approximation provides very good accuracy of the resulting solution mainly in cases of medium-sized p-median instances. The results in Table 1 seemingly contradict to the common opinion that the approximate method with limited number of dividing points must bring worse results on the bigger problems. Nevertheless, the presented results show that the size of the problem does not play so important role as other characteristics of the network, especially the distance matrix. It is necessary to realize, that the suggested approximate approach can reach the optimal solution in the case, when the dividing points cover all distance values from the sequence $d_0 < d_1 < \cdots < d_m$, which are contained in the optimal solution. The proximity of the obtained solution to the optimal one depends here more on the portion of distances covered by dividing points rather than on the size of the matrix $\{d_{ij}\}$. The mentioned portion is influenced by specific of the network graph of the solved benchmark. We have to point out that the problems from 100 to 900 possible service center locations originated from the Beasleys benchmarks [18] and the last cases were derived from real road network of Slovakia. This may explain the extremely small gap in the columns 1100 and 1300. The average gaps for Beasleys benchmarks arise randomly and obviously do not correlate with the size of the problem.

The main disadvantage of the upper distance approximation consists in the fact that it is not possible to evaluate the solution accuracy in a general case, because the exact solution is usually unknown and the upper bound does not represent proper information to evaluate the quality of the obtained result. Therefore we recommend the upper distance approximation as a complement to the lower distance approximation. As we will show, the lower distance approximation brings more interesting features mainly in cases where there is no other possibility to evaluate the result accuracy.

When the lower distance approximation is employed, then the objective function value of the model (12), (8) - (11) gives the lower bound of the unknown optimal solution. As before, we obtain here the corresponding real objective function value according to the formula (13). Thus, we can evaluate not only the result accuracy by gap_{BFSE}, but we can also measure the quality of the lower bound. Let the best lower bound be denoted as BLB. If we know the exact solution ES, then we can compute the gap_{BLBE} which expresses the difference between the best lower bound BLB and the exact solution ES in percentage of the exact solution. Its value is given by (26).

$$gap_{BLBE} = \frac{|BLB - ES|}{ES} * 100 \tag{26}$$

It is generally known that the location-allocation approach usually fails when a very large p-median instance is being solved. If we wanted to evaluate the quality of the sequential covering solution in a general case, it would not be possible due to the fact that the optimal solution is unknown. Therefore, we suggest measuring also the difference between the best lower bound BLB and the best found solution BFS given by (13). Since BFS constitutes the upper bound of the unknown optimal solution, the difference computed according to (27) represents the maximal deviation of the covering solution from the exact one. If the optimal solution is unknown, this is the only way of evaluating the accuracy of the resulting approximate solution.

$$gap_{BLBBFS} = \frac{|BFS - BLB|}{BFS} * 100 \tag{27}$$

As concerns the computational time, the situation becomes a little bit complicated in comparison to the presented preliminary experiments with the upper distance approximation. As we have mentioned in previous sections, the suggested sequential approach may perform too many iterations with very little improvement and thus the computational process may take too long time. This phenomenon deserved further research [19], which was focused on the possibility of an "exchange" of the loss of accuracy for lower computational time. The exchange follows the fact that the suggested sequential approach is accurate enough to admit a mild increase in the resulting gap for improving the computational time. Possible modifications of the sequential method are based on a deep analysis of the obtained results and the whole sequential process. The analysis led to the conclusion that the enormous time demands were required by the last iteration. In most cases the optimal solution was found, and one additional iteration was performed with no improvement, so the computational process was stopped. If the gap_{BLBBFS} equals to zero, then it is obvious that no better solution can be found, and no more iterations are needed. This observation plays an important role in modifying the stopping criteria. Generalization of this idea consisted in a special adjustment of the stopping criterion, where reaching of a tolerable deviation of the lower bound below given threshold in percentage of the best found solution was taken as a complementary condition for the loop termination. So, when the lower distance approximation was used, the following stopping criteria were applied: the iteration process was terminated if either ten iterations had been performed, or no improvement of the best found solution in the actual iteration had been achieved or the gap_{BLBBFS} had dropped below given 2-percent threshold. Applying these rules, the considerable reduction of the lower bound computational time was achieved at the negligible decrease of the accuracy.

The associated testing was performed with the same pool of the benchmarks as the preliminary experiments with the upper distance approximation. All numerical experiments were performed for 20 dividing points and $T = 1$. The achieved results are plotted in the Table 2 and Figure 4. The Table 2 contains the average gaps between the best found solution, the exact one and also the evaluation of the result by gap_{BLBBFS}. Each column corresponds to the size of 6 benchmarks solved by the sequential approach with lower distance approximation. The Figure 4 shows the time comparison of the sequential approach to the location-allocation one.

$\|I\|$	100	300	500	600	700	800	900	1000	1100	1200	1300	1400
Average gap_{BFSE}	0.82	0.33	0.09	0.12	0.02	0.06	0.03	0.11	0.00			
Average gap_{BLBE}	3.42	1.80	0.72	0.43	0.33	0.35	0.16	0.16	0.01			
Average gap_{BLBBFS}	4.20	2.12	0.80	0.55	0.36	0.41	0.20	0.26	0.17	0.16	0.24	0.13

Table 2: Accuracy of the solution obtained by the sequential approach with lower distance approximation

The results reported in the Table 2 and also the time comparison prove, that the suggested sequential approach provides very good accuracy of the resulting solution. As we can see, the average gap does not exceed one percent. Thus we can conclude, that presented approach is a suitable tool for such instances, where the location-allocation approach fails. It provides very good solution in a short time without the necessity of developing special software tool. Concerning the solution accuracy, it must be noted, that the quality of the result may be sensitive to the start-up settings, mainly to the

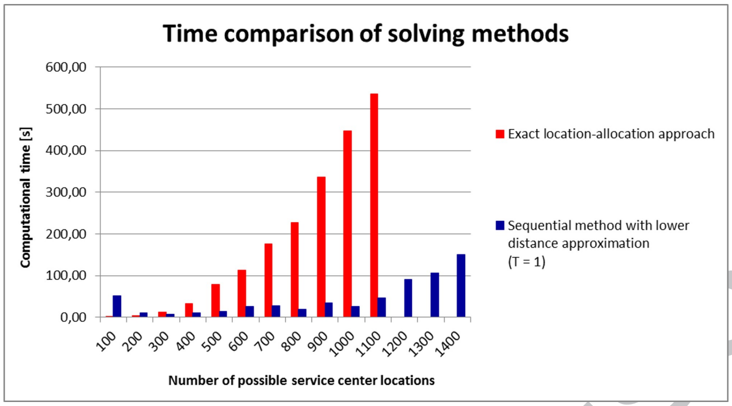

Figure 4: Time comparison of the exact location-allocation approach to the sequential method with the lower distance approximation

value of parameter T. Therefore the initial phase of the sequential method should become an interesting topic of future possible research in this area. Another parameter, which can significantly influence the quality of the result, is the number of dividing points. Therefore we focus on its suitable value in the following subsection.

6.2. Accuracy sensitivity on the number of dividing pointss

This subsection is focused on the exploration of the solution accuracy depending on the number of dividing points. It must be realized that the number of zones used in the distance approximation does not influence only the size of the radial model (7) - (11), but it directly affects the accuracy of the result and also the computational time. To explore the accuracy sensitivity on the number of dividing points v, we have suggested a set of numerical experiments on medium-sized benchmarks used in the previous computational study. For each size $|I|$ of the set of possible service center locations I, 6 different instances were solved. As described in the subsection 6.1, the instances differ in the value of parameter p, which defines the number of located service centers. The number of dividing points was set to the value 5, 10, 15, 20 and 25 respectively. For each cardinality of the set I and each number of dividing points v, the average gap_{BLBBFS} was computed. The value of mentioned gap is defined by (27). The achieved results are reported in the following Table 3 and Table 4.

According to the reported results, it is obvious that the number of dividing points v significantly influences the accuracy of the solution obtained by the sequential method. It can be noticed in the Table 3 that for $v = 15, 20$ the values of gap drop below 1 percent, which represents satisfactory accuracy. Contrary to the progress of gap, behavior of the computational time value non-linearly grows with increasing number of dividing points v. On average, we can state that the stronger slope of the dependency starts roughly above the value $v = 20$.

$\|I\|$	Number of dividing points				
	$v = 5$	$v = 10$	$v = 15$	$v = 20$	$v = 25$
600	15.37	5.19	1.64	0.47	0.09
700	13.09	3.46	0.91	0.28	0.08
800	11.63	2.98	0.80	0.22	0.03
900	10.39	2.20	0.42	0.06	0.00
1000	7.90	1.36	0.33	0.06	0.01
1100	7.43	1.41	0.33	0.07	0.01
1200	6.71	1.31	0.29	0.08	0.01
1300	7.23	1.42	0.34	0.06	0.01
1400	7.28	1.48	0.45	0.09	0.02
1500	6.93	1.24	0.35	0.05	0.01
AVERAGE	9.40	2.20	0.59	0.14	0.03

Table 3: Table of average gaps in percent of the best found solution for different number of dividing points

$\|I\|$	Number of dividing points				
	$v = 5$	$v = 10$	$v = 15$	$v = 20$	$v = 25$
600	3.85	13.90	7.61	5.16	12.31
700	4.83	19.20	8.62	28.76	45.30
800	3.46	10.03	5.36	17.02	32.71
900	3.32	22.99	11.41	60.23	58.64
1000	21.11	25.56	26.11	33.63	25.18
1100	24.70	22.58	27.62	33.82	91.34
1200	44.17	34.57	48.75	31.58	188.88
1300	64.90	58.61	103.19	82.28	233.59
1400	94.41	63.47	78.09	87.17	341.35
1500	127.35	132.18	107.91	113.00	161.23
AVERAGE	39.21	40.31	42.47	49.27	119.05

Table 4: Table of average computational times in seconds for different number of dividing points

To analyze behavior of two dependencies, we computed ratio of computational time to gap to express the value of the one percent gap accuracy in time units. We computed the average values for each number v of dividing points, performed scaling of these results, named them as "Scaled time/gap" and plotted them in the Figure 5. Used scaling consists in linear mapping of the obtained values to the interval [0, 10], where the lowest value of the result corresponds with the value of 0 and the biggest result corresponds to the value of 10. The same scaling was applied on average gaps computed separately for each number of dividing points and these values are plotted also in the Figure 5.

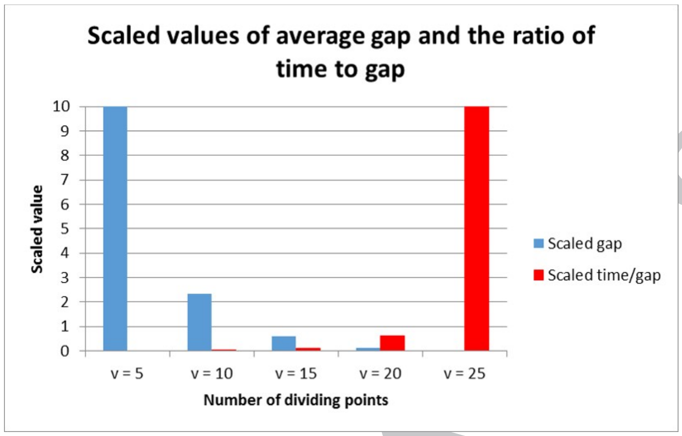

Figure 5: Scaled values of average gap and the ratio of time to gap for different number of dividing points.

Based on reported results we can confirm that the number of 20 dividing points is a suitable choice of this parameter. It was shown that this setting keeps the model of solved problems in tolerable size, which issues in acceptable computational time of the method. Furthermore, the number of 20 dividing points proved to be suitable to achieve satisfactory accuracy.

6.3. Numerical Experiments with Medium and Large Instances

To test the effectiveness of the suggested approximate sequential approach to the p-median problem, especially to verify the stopping rules for obtaining the lower bound, we have suggested and realized another sequence of numerical experiments. The solved instances of the problem were divided into two sets. The first set of test problems originates from the Slovak road network for the cardinality of I from 1500 to 2800 possible service center locations. The cardinality of J is the same as the cardinality of I, and the maximal number p of possible service center locations was set so that the ratio of $|I|$ to p equals 2, 3, 4, 5, 10 and 20 respectively. All instances were solved by both sequential approaches (lower and upper distance approximation) and the better solution concerning the objective function value (13) was taken as the result. The associated average results are plotted in the Table 5 and

Figure 6. All numerical experiments were performed for 20 dividing points and $T = 1$.

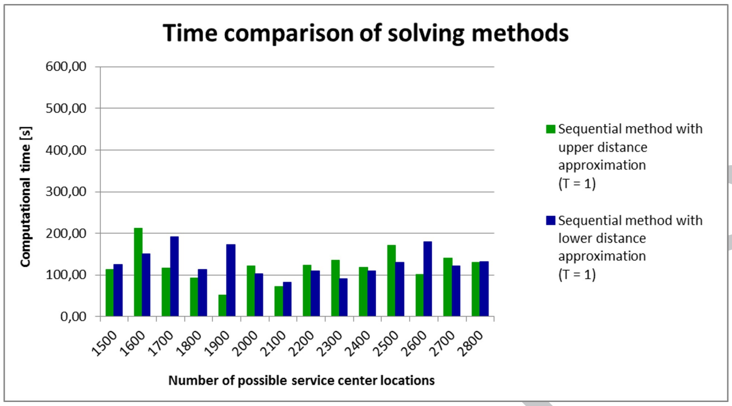

Figure 6: Time comparison of the sequential methods with the upper and lower distance approximation for large p-median instances

$\|I\|$	1500	1600	1700	1800	1900	2000	2100	2200	2300	2400	2500	2600	2700	2800
gap_{BLBBFS}	0.21	0.31	0.32	0.32	0.25	0.24	0.23	0.27	0.21	0.19	0.19	0.21	0.19	0.19

Table 5: Accuracy of the solution obtained by the sequential approach for large p-median instances

The last set of numerical experiments was organized to verify the suggested stopping criteria, and to show the effectiveness of the sequential method on large instances. The input data were taken from the benchmarks commonly used in available literature. Furthermore, the exact solution of all benchmarks was given. This fact presents an important advantage for evaluating the solution accuracy.

We have compared the efficiency of the sequential method (*Approx.*) using lower distance approximation, also with the state of the art for the p-median problem that is given by Avella, Sassano and Vasilev [10]. Their approach is denoted as AVS algorithm. Furthermore, we have compared our results to the *Zebra* approach (Z-Enlarge-and-BRanch-Algorithm) by García, Labbé and Marín [7]. Even if it is not standard to compare an approximate method to the exact one, the exact solution is used in our case to evaluate the accuracy of the approximate one. Exact methods usually spend the majority of computing time verifying the optimality of the solution, while approximate methods try to improve the solution as much as possible. Particular exact results were taken from the literature. It is important to note that the $TSBLIB$ instances (available online: http://www.iwr.uni-heidelberg.de/groups/comopt/software/TSPLIB95/) differ from the Slovak road benchmarks in the size of d_h sequence. These benchmarks contain thousands of different values, while the Slovak p-median instances only hundreds of different distances. If we had taken all possible distances d_h into consideration, the sequential approach would have been much time consuming. Therefore, we reduced the set of the values so that we took the first 200 distances d_h from the original sequence with their occurrence frequencies N_h and the rest of the values were ranked into 100 values uniformly. The occurrence frequencies N_h for $h \geq 200$ were set to the sum of N_r where $d_r \geq d_h$ and $d_r < d_{h+1}$ in the original sequence. Thus a new set of 300 distances d_h with their occurrence frequencies N_h was obtained and used as the input of the approximate sequential method. The zone coefficients e_s were computed according to particular dividing points, where the distance D_s^1 for each zone s was taken from the original sequence of the distances. The comparison of the solving methods for the selected $TSPLIB$ instances is shown in Table 3. Each row of the table represents one solved instance of the p-median problem. The best upper bound is denoted as BUB, the best lower bound is given in BLB columns, and the computational time in seconds is denoted as $Time$. The symbol $*$ means that the computer ran out of memory while solving the problem. When available, the best information up to that moment is shown. The instances with a small value of p (5, 10, 20) were solved with $T = 1000$, and the instances with higher value of p were solved with $T = 1$. The difference between the best lower bound and the best found solution is expressed in the percentage of the best found solution, and this value is denoted as GAP. The column $GGAP$ (General GAP) is dedicated to the difference between the best found *Approx.* solution and the optimal one obtained by AVS algorithm expressed in the percentage of the optimal solution. All experiments were performed using the optimization software FICO XPRESS 7.3 (64-bit, release 2012). The associated code was run on a PC equipped with the Intel Core i7 2630QM processor with the parameters: 2.0 GHz and 8 GB RAM.

The results of the numerical experiments prove the usefulness of the suggested sequential approach. Even if the difference between the lower bound and the exact solution is quite high in some instances, the covering solution is near to the optimal one. This fact makes the sequential approximate approach very successful.

$File$	$\|I\|$	p	AVS		$Zebra$		$Approx.$				
			$Time$	BUB	$Time$	BUB	$Time$	BLB	BUB	GAP	$GGAP$
$rl1304$	1304	5	32	3099073	3579	3099073	43.49	2908780	3099253	6.15	0.01
$rl1304$	1304	10	1614	2134295	4015	2134295	117.73	1998600	2135714	6.42	0.07
$rl1304$	1304	20	22	1412108	170	1412108	45.07	1289580	1415218	8.88	0.22
$rl1304$	1304	50	14	795012	24	795012	126.04	674993	798756	15.49	0.47
$rl1304$	1304	200	16	268573	10	268573	4.2	256547	270343	5.1	0.66
$rl1304$	1304	300	18	177326	11	177326	6.52	169561	178090	4.79	0.43
$rl1304$	1304	400	14	128332	12	128332	4.47	125586	128552	2.31	0.17
$rl1304$	1304	500	20	97024	13	97024	4.16	96494	97066	0.59	0.04
$fl1400$	1400	5	45	174877	598	174877	21.71	141453	175741	19.51	0.49
$fl1400$	1400	10	33	100601	140	100601	27.82	84095	101478	17.13	0.87
$fl1400$	1400	20	24	57191	24	57191	30.64	50801	58584	13.29	2.44
$u1432$	1432	5	41	1210126	412	1210126	205.9	1179730	1210482	2.54	0.03
$u1432$	1432	10	26	849759	172	849759	147.4	834492	850300	1.86	0.06
$u1432$	1432	200	58	159887	22	159887	27.49	159887	159887	0	0.00
$u1432$	1432	300	43	123689	21	123689	24.55	123689	123689	0	0.00
$u1432$	1432	500	36	93200	10	93200	7.21	93200	93200	0	0.00
$v1748$	1748	5	59	4479421	3870*	4479421	45.9	4199630	4482483	6.31	0.07
$v1748$	1748	10	478	2983645	4245	2983645	74.37	2790800	2985774	6.53	0.07
$v1748$	1748	200	22	390350	20	390350	8.9	340493	428248	20.49	9.71
$v1748$	1748	300	24	286039	24	286039	3.41	262880	299649	12.27	4.76
$v1748$	1748	400	155	221526	22	221526	6.96	211006	229846	8.2	3.76
$v1748$	1748	500	74	176986	22	176986	9.05	171925	177229	2.99	0.14
$d2103$	2103	5	96	1005136	2872*	1005136	146.92	902993	1005201	10.17	0.01
$d2103$	2103	10	260	687321	3143	687321	233.98	595052	687891	13.5	0.08
$d2103$	2103	20	733	482926	1759	482926	988.04	448603	484660	7.44	0.36
$d2103$	2103	200	1828	117753	55	117753	46.96	116410	118620	1.86	0.74
$d2103$	2103	300	1133	90471	305	90471	30.45	89705	90537	0.92	0.07
$d2103$	2103	400	235	75356	8917	75356	40.84	75004	75368	0.48	0.02
$pcb3038$	3038	5	1114	1777835	109*	1777835	566.61	1625920	1779865	8.65	0.11
$pcb3038$	3038	10	134	1211704	64*	1211704	1102.6	1046490	1213089	13.73	0.11
$pcb3038$	3038	200	2562	237399	564	237399	279.31	227578	240569	5.4	1.34
$pcb3038$	3038	300	2977	186833	274	186833	169.78	180167	187162	3.74	0.18
$pcb3038$	3038	400	454	156276	106	156276	104.06	151232	156753	3.52	0.31
$pcb3038$	3038	500	704	134798	115	134798	106.84	130757	135435	3.45	0.47

Table 6: Comparison of the sequential approximate approach to other solving methods for selected $TSPLIB$ benchmarks

7. Conclusions

We have presented a dynamic method for the approximate solving of large p-median problem instances. The method is based on step-by-step improving the anticipated distance relevance, and on the concept of the dividing points which distinguish the suggested method from other approaches based on the radial formulation of the p-median problem. Our suggested method proved to be very efficient as far as the accuracy is concerned, when solving the instance up to one thousand users locations or middle and larger instances originated from real transportation networks.

The comparison of the suggested method to other approaches performed with middle and large benchmarks shows that it can be taken as a compromise approach enabling a trade-off between the accuracy and computational time. It was found that in most cases the suggested method finishes its search in smaller computational time than each of the compared methods, and, in addition, the computational time was better in order in several cases. Therefore, we can conclude that the proposed method is a suitable complement to the state-of-the-art methods. The main contribution of the suggested approach lies in its simple implementation without the necessity of programming several algorithms. Just two different models are enough to obtain a good solution in a short time. Furthermore, common optimization tools can be used instead of special ones.

Further research connected with the suggested sequential approach will be focused on the initial stage of the approach, where we found a possibility of making the computational process more efficient by a suitable adjustment of the mentioned parameter T or by the development of other forms of the distance relevance estimation.

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Highlights

- We study the public service system design formulated as the p-median problem.
- We focus on the approximate radial approach using dividing points.
- To improve the accuracy, a sequential method is introduced.
- Presented approach enables simple implementation in common optimization software.
- The proposed method is a suitable complement to the state-of-the-art methods.