



# A greedy algorithm for solving ordinary transportation problem with capacity constraints

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## ARTICLE INFO

### Article history:

Received 10 January 2017

Received in revised form 30 May 2017

Accepted 30 May 2017

Available online 15 June 2017

### Keywords:

Transportation problem

Greedy algorithm

Duality

Supermodularity

Matroid

## ABSTRACT

Consider the ordinary transportation problem with the objective to minimize the cost of transporting a single commodity from  $M$  warehouses to  $N$  demand locations. Each warehouse  $i$  has a finite capacity  $k_i$ . We convert the above problem into a dual problem and construct a greedy algorithm to solve it.

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## 1. Introduction

Consider ordinary transportation problem for a single commodity with  $M$  warehouses and  $N$  demand locations. Denote  $\mathcal{M} = \{1, \dots, M\}$  as the set of all warehouses and  $\mathcal{N} = \{1, \dots, N\}$  as the set of all demand locations. At each warehouse  $i \in \mathcal{M}$  the maximum amount of product available is  $k_i \geq 0$  and at each demand location  $j \in \mathcal{N}$  the demand is  $d_j \geq 0$ . The cost of transporting one unit of the product from warehouse  $i$  to demand location  $j$  is  $c_{ij} \geq 0$ . The transportation problem can be formulated into a standard linear programming problem as follows:

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{i=1}^M \sum_{j=1}^N c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^N x_{ij} \leq k_i \quad \forall i \in \mathcal{M}, \\ & \sum_{i=1}^M x_{ij} \geq d_j \quad \forall j \in \mathcal{N}, \\ & x_{ij} \geq 0, \quad \forall i \in \mathcal{M}, j \in \mathcal{N}. \end{aligned} \quad (1)$$

The primal transportation problem (1) has the following dual format

$$\begin{aligned} \max_{u_i, v_j} \quad & \sum_{j=1}^N d_j v_j - \sum_{i=1}^M k_i u_i \\ \text{s.t.} \quad & v_j - u_i \leq c_{ij}, \quad \forall i \in \mathcal{M}, j \in \mathcal{N} \\ & v_j, u_i \geq 0, \quad \forall i \in \mathcal{M}, j \in \mathcal{N}. \end{aligned} \quad (2)$$

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Many works have studied conditions under which transportation problems with no capacity constraints can be solved by greedy algorithms (see for example Queyranne et al. [2]). One approach to show greedy algorithms lead to optimality is to verify that the transportation problems satisfy Monge conditions (Hoffman [9], Bein et al. [3]) and the other approach is to consider its dual (Lovász [11], Fujishige and Tomizawa [8], and Edmonds [5]). In Faigle and Fujishige [6], they observe that “a class of greedy algorithm solvable problems have optimal solutions that have the structure of chains”. When warehouses have finite capacities, in general greedy algorithm does not lead to optimal solutions and simplex method is not efficient. As a result, many works focus on finding other efficient algorithms to reduce the computational time (Charnes and Cooper [4], Arsham and Kahn [1], Ji and Chu [10]). Within this stream of literature, the most commonly applied algorithm is the stepping stone algorithm that finds the optimal solution by iteratively updating the transportation schedule with feasible cycles of positive cost reductions (Charnes and Cooper [4]). Also, Ji and Chu [10] developed a linear programming approach to compute the optimal solution using the dual formulation. In our paper, we show that the dual problem can be solved by a greedy algorithm. However, different from the greedy algorithm defined in Federgruen and Groenevelt [7] where the increase direction in each iteration is a coordinate direction with the largest increment, in this paper the increase direction is a set of coordinates that results in the largest increment if each coordinate in the set is increased by the same amount.

## 2. The greedy algorithm

We first convert the dual problem to a concave maximization problem. Note that for any feasible  $u_i$  and  $v_j$ , we have  $v_j \leq c_{ij} + u_i$ .

Because  $d_j \geq 0$  the optimal  $v_j^*$  must satisfy  $v_j^* = \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$  for all  $j \in \mathcal{N}$ . Substitute  $v_j^* = \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$  into Eq. (2), we convert a linear programming problem into a concave optimization problem:

$$\max_{u_i \geq 0, i \in \mathcal{M}} - \sum_{i=1}^M k_i u_i + \sum_{j=1}^N d_j \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}. \quad (3)$$

Denote  $\mathbf{u} = (u_1, \dots, u_M)$  and the objective function of (3) as  $f(\mathbf{u}) = -\sum_{i=1}^M k_i u_i + \sum_{j=1}^N d_j \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$ . We have the following lemma that characterizes properties of the objective function  $f(\mathbf{u})$ .

**Lemma 1.**  $f(\mathbf{u})$  is jointly concave and supermodular in  $\mathbf{u}$ .

Lemma 1 shows basic properties of  $f(\mathbf{u})$ . However, these properties are not enough to ensure that the greedy algorithm defined in Federgruen and Groenevelt [7], i.e., increase  $u_i$  that leads to the largest increment, is optimal. To see this, consider the following example.

**Example 1.** Let

$$\begin{aligned} f(u_1, u_2, u_3, u_4) &= -2u_1 - 3u_2 - 2u_3 - 7u_4 \\ &\quad + \min\{u_1, u_2 + 1, u_3 + 1, u_4 + 3\} \\ &\quad + 10 \min\{u_1, u_2, u_3, u_4 + 1\} \\ &\quad + \min\{u_1 + 1, u_2 + 1, u_3, u_4 + 3\}. \end{aligned}$$

At  $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$ , the marginal increment of  $f(\mathbf{u})$  in the directions  $u_1, u_2, u_3$  and  $u_4$  are all negative, implying that none of the direction  $u_i$  will lead to a positive increment in the objective function. However, in the direction  $(1, 1, 1, 0)$  the increment of  $f(\mathbf{u})$  is 5. Thus, simply increasing the  $u_i$  with the largest increment is not optimal.

Example 1 implies that a greedy algorithm only increasing in one coordinate direction at a time may not be optimal. Therefore, we generalize the single-coordinate-greedy-algorithm to include all subsets of coordinate directions. Denote  $2^{\mathcal{M}}$  as the class of all subsets of  $\mathcal{M}$  and  $2^{\mathcal{N}}$  as the class of all subsets of  $\mathcal{N}$ . Given a feasible  $\mathbf{u}$ , we define a mapping  $\sigma_{\mathbf{u}} : 2^{\mathcal{M}} \mapsto 2^{\mathcal{N}}$  as  $\sigma_{\mathbf{u}}(\mathcal{A}) = \{j \mid \arg \min_i \{c_{ij} + u_i\} \subseteq \mathcal{A}, j \in \mathcal{N}\}$ . In other words,  $\sigma_{\mathbf{u}}(\mathcal{A})$  is the set of  $j$ 's that  $\min_i \{c_{ij} + u_i\}$  will increase if we increase all  $u_i$ 's,  $i \in \mathcal{A}$  by the same relatively small amount. Let  $\mathbf{e}_{\mathcal{A}}$  be an  $M$  dimension vector with the  $i$ th element equals 1 if  $i \in \mathcal{A}$ , and zero otherwise. For any  $\mathcal{A} \in 2^{\mathcal{M}}$ , define  $\Delta_{\mathbf{u}}(\mathcal{A}) = -\sum_{i \in \mathcal{A}} k_i + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A})} d_j$  as the increment of  $f(\mathbf{u})$  if the increasing direction is  $\mathbf{e}_{\mathcal{A}}$ . Let  $\mathbf{u}_{\mathcal{A}}$  denote an  $M$  dimension vector in which the  $i$ th element equals  $u_i$  if  $i \in \mathcal{A}$  and zero otherwise. For two vectors  $\mathbf{u}$  and  $\mathbf{v}$ ,  $\mathbf{u}, \mathbf{v} \in \mathcal{M}$ ,  $\mathbf{u} \leq \mathbf{v}$  iff  $u_i \leq v_i, i \in \mathcal{M}$ . Similarly, we can define “ $\geq$ ” and “ $=$ ” for vectors. We have the following lemma that characterizes properties of  $\sigma_{\mathbf{u}}(\mathcal{A})$  and  $\Delta_{\mathbf{u}}(\mathcal{A})$ .

**Lemma 2.** (1)  $\sigma_{\mathbf{u}}(\emptyset) = \emptyset$ .  $\sigma_{\mathbf{u}}(\mathcal{A})$  is monotone, that is, for any two sets  $\mathcal{A} \subseteq \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}}(\mathcal{B})$ .

(2)  $\sigma_{\mathbf{u}}(\mathcal{A})$  is superadditive in union, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \cup \sigma_{\mathbf{u}}(\mathcal{B}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$ . Therefore,  $\sigma_{\mathbf{u}}(\mathcal{A})$  is also supermodular.

(3)  $\sigma_{\mathbf{u}}(\mathcal{A})$  is additive in interception, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B}) = \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ .

(4)  $\Delta_{\mathbf{u}}(\emptyset) = 0$ .  $\Delta_{\mathbf{u}}(\mathcal{A})$  is supermodular, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$  we have  $\Delta_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{B}) \leq \Delta_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B}) + \Delta_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ .

(5) For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$  and  $\mathcal{A} \cap \mathcal{B} = \emptyset$ , we have  $\Delta_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{B}) \leq \Delta_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$ .

(6) For any  $\mathbf{u}^1$  and  $\mathbf{u}^2$ , if  $\mathbf{u}_{\mathcal{A}}^1 = \mathbf{u}_{\mathcal{A}}^2$  and  $\mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^1 \leq \mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^2$ , then  $\sigma_{\mathbf{u}^1}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}^2}(\mathcal{A})$  and  $\Delta_{\mathbf{u}^1}(\mathcal{A}) \leq \Delta_{\mathbf{u}^2}(\mathcal{A})$ .

We define the greedy algorithm as follows.

**Algorithm 1.** (1) Initialize:  $\mathbf{u} = \mathbf{0}, \mathcal{A}_{max} \in \arg \max_{\mathcal{A} \in 2^{\mathcal{M}}} \Delta_{\mathbf{u}}(\mathcal{A})$ ;  
 (2) While  $\Delta_{\mathbf{u}}(\mathcal{A}_{max}) > 0$ , do  
 $\mathbf{u} = \mathbf{u} + \sup \{u \mid \sigma_{\mathbf{u}} = \sigma_{\mathbf{u} + u \mathbf{e}_{\mathcal{A}_{max}}}\} \mathbf{e}_{\mathcal{A}_{max}}, \mathcal{A}_{max} \in \arg \max_{\mathcal{A} \in 2^{\mathcal{M}}} \Delta_{\mathbf{u}}(\mathcal{A})$ ;  
 return  $\mathbf{u}^* = \mathbf{u}$ .

Next, we show that  $\mathbf{u}^*$ , the solution returned by Algorithm 1, is always optimal. Since  $f(\mathbf{u})$  is concave in  $\mathbf{u}$ , it is sufficient to show that  $\mathbf{u}^*$  is a local maximum. Similar to  $\sigma_{\mathbf{u}}$ , we define a mapping  $\pi_{\mathbf{u}} : 2^{\mathcal{M}} \mapsto 2^{\mathcal{N}}$  as  $\pi_{\mathbf{u}}(\mathcal{A}) = \{j \mid \arg \min_i \{c_{ij} + u_i\} \cap \mathcal{A} \neq \emptyset, j \in \mathcal{N}\}$ . In other words,  $\pi_{\mathbf{u}}(\mathcal{A})$  is the set of  $j$ s that  $\min_i \{c_{ij} + u_i\}$  will decrease if we decrease all  $u_i$ 's,  $i \in \mathcal{A}$  for a same significantly small amount. For any  $\mathcal{A} \in 2^{\mathcal{M}}$ , define  $\Lambda_{\mathbf{u}}(\mathcal{A}) = \sum_{i \in \mathcal{A}} k_i - \sum_{j \in \pi_{\mathbf{u}}(\mathcal{A})} d_j$  as the decrement of  $f(\mathbf{u})$  if the decreasing direction is  $\mathcal{A}$ . We have the following lemma that characterizes properties of  $\pi_{\mathbf{u}}(\mathcal{A})$  and  $\Lambda_{\mathbf{u}}(\mathcal{A})$ .

**Lemma 3.** (1)  $\pi_{\mathbf{u}}(\emptyset) = \emptyset$ .  $\pi_{\mathbf{u}}(\mathcal{A})$  is monotone, that is, for any two sets  $\mathcal{A} \subseteq \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\pi_{\mathbf{u}}(\mathcal{A}) \subseteq \pi_{\mathbf{u}}(\mathcal{B})$ .

(2)  $\pi_{\mathbf{u}}(\mathcal{A})$  is additive in union, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\pi_{\mathbf{u}}(\mathcal{A}) \cup \pi_{\mathbf{u}}(\mathcal{B}) = \pi_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$ .

(3)  $\pi_{\mathbf{u}}(\mathcal{A})$  is subadditive in interception, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , we have  $\pi_{\mathbf{u}}(\mathcal{A}) \cap \pi_{\mathbf{u}}(\mathcal{B}) \subseteq \pi_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ .

(4)  $\Lambda_{\mathbf{u}}(\emptyset) = 0$ .  $\Lambda_{\mathbf{u}}(\mathcal{A})$  is supermodular, that is, for any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$  we have  $\Lambda_{\mathbf{u}}(\mathcal{A}) + \Lambda_{\mathbf{u}}(\mathcal{B}) \leq \Lambda_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B}) + \Lambda_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ .

(5) For any  $\mathbf{u}^1$  and  $\mathbf{u}^2$ , if  $\mathbf{u}_{\mathcal{A}}^1 \leq \mathbf{u}_{\mathcal{A}}^2$  and  $\mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^1 = \mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^2$ , then  $\pi_{\mathbf{u}^1}(\mathcal{A}) \supseteq \pi_{\mathbf{u}^2}(\mathcal{A})$  and  $\Lambda_{\mathbf{u}^1}(\mathcal{A}) \leq \Lambda_{\mathbf{u}^2}(\mathcal{A})$ .

(6) For any  $\mathbf{u}^1$  and  $\mathbf{u}^2$ , if  $\mathbf{u}_{\mathcal{A}}^1 = \mathbf{u}_{\mathcal{A}}^2$  and  $\mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^1 \leq \mathbf{u}_{\mathcal{M} \setminus \mathcal{A}}^2$ , then  $\pi_{\mathbf{u}^1}(\mathcal{A}) \subseteq \pi_{\mathbf{u}^2}(\mathcal{A})$  and  $\Lambda_{\mathbf{u}^1}(\mathcal{A}) \geq \Lambda_{\mathbf{u}^2}(\mathcal{A})$ .

(7) For all  $\mathcal{A} \in 2^{\mathcal{M}}$ ,  $\pi_{\mathbf{u}}(\mathcal{A}) \supseteq \sigma_{\mathbf{u}}(\mathcal{A})$  and  $\Lambda_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{A}) \leq 0$ .

The following theorem shows that Algorithm 1 returns an optimal solution of the dual problem.

**Theorem 1.** If Algorithm 1 stops at some  $\mathbf{u}$ , then it returns the optimal policy of the dual problem,  $\mathbf{u} = \mathbf{u}^*$ . Otherwise, the dual problem is unbounded and the primal problem is infeasible. In each iteration,  $\mathcal{A}_{max}$  can be found by maximizing a supermodular set function. Thus, (2) can be solved by a greedy algorithm.

**Proof.** First, it is clear that when the algorithm stops at some  $\mathbf{u}$ , the objective function cannot increase in any direction by adding a positive amount to  $\mathbf{u}$ , that is,  $\Delta_{\mathbf{u}}(\mathcal{A}) \leq 0$  for all  $\mathcal{A} \subseteq 2^{\mathcal{M}}$ . To show Algorithm 1 is optimal, we only need to show that  $\mathbf{u}$  is a local maximum, which is equivalent to verifying that the algorithm stops at some  $\mathbf{u} = \mathbf{u}^*$  such that  $\Lambda_{\mathbf{u}^*}(\mathcal{A}) \leq 0$  for all  $\mathcal{A} \subseteq 2^{\mathcal{M}}$ .

Now suppose the above is not true, that is, there exists a set  $\mathcal{R}$  such that  $\Lambda_{\mathbf{u}^*}(\mathcal{R}) > 0$ . Denote the set  $\mathcal{A}_{max}^* \subseteq 2^{\mathcal{M}}$  as the set that is picked for increment before reaching  $\mathbf{u}^*$ . If  $\mathcal{R} \subseteq \mathcal{A}_{max}^*$ , then for an arbitrary small  $\epsilon > 0$ , from the definitions of  $\Delta_{\mathbf{u}}(\mathcal{A})$  and  $\Gamma_{\mathbf{u}}(\mathcal{A})$ , we have

$$\begin{aligned} \Delta_{\mathbf{u}^* - \epsilon \mathbf{u}_{\mathcal{A}_{max}^*}}(\mathcal{A}_{max}^* \setminus \mathcal{R}) &= \Delta_{\mathbf{u}^* - \epsilon \mathbf{u}_{\mathcal{A}_{max}^*}}(\mathcal{A}_{max}^*) + \Lambda_{\mathbf{u}^* - \epsilon \mathbf{u}_{\mathcal{A}_{max}^*}}(\mathcal{R}) \\ &> \Delta_{\mathbf{u}^* - \epsilon \mathbf{u}_{\mathcal{A}_{max}^*}}(\mathcal{A}_{max}^*), \end{aligned}$$

which contradicts with  $\mathcal{A}_{max}^* \in \arg \max_{\mathcal{A} \in 2^{\mathcal{M}}} \Delta_{\mathbf{u}^* - \epsilon \mathbf{u}_{\mathcal{A}_{max}^*}}(\mathcal{A})$ .

Therefore, we can safely assume  $\mathcal{R} \not\subseteq \mathcal{A}_{max}^*$ . Then consider the last iteration that some elements in  $\mathcal{R}$  are selected in Algorithm 1 for increment. That is, all iterations afterwards only increases  $u_i$  that  $i \notin \mathcal{R}$ . Suppose  $\mathbf{u}^0$  is the vector at the beginning of that iteration,  $\mathbf{u}^1$  is the vector at the end of the iteration and  $\mathcal{A}_{max}^1$  is the corresponding increment set. Then according to Lemma 2(6) part two,

$$\Delta_{\mathbf{u}^1}(\mathcal{A}_{max}^1) \geq \Delta_{\mathbf{u}^*}(\mathcal{A}_{max}^1) > 0.$$

This implies that it is profitable to reduce the elements in set  $\mathcal{R}$  in this iteration. Thus, by similar argument we must have  $\mathcal{R} \not\subseteq \mathcal{A}_{max}^1$ . For an arbitrary small  $\epsilon > 0$ , from the definitions of  $\Delta_{\mathbf{u}}(\mathcal{A})$  and  $\Gamma_{\mathbf{u}}(\mathcal{A})$ , we have

$$\Delta_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{A}_{max}^1 \setminus \mathcal{R}) + \Lambda_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{R} \setminus \mathcal{A}_{max}^1) \geq \Delta_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{A}_{max}^1) + \Lambda_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{R}),$$

hence

$$\Lambda_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{R} \setminus \mathcal{A}_{max}^1) \geq \Lambda_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{R}) \geq 0.$$

From Lemma 3 part (6),

$$\Lambda_{\mathbf{u}^0}(\mathcal{R} \setminus \mathcal{A}_{max}^1) \geq \Lambda_{\mathbf{u}^{1-\epsilon\mathbf{u}}}(\mathcal{R} \setminus \mathcal{A}_{max}^1) \geq 0,$$

implying that it is profitable to reduce the elements in set  $\mathcal{R}^1 = \mathcal{R} \setminus \mathcal{A}_{max}^1$  at  $\mathbf{u}^0$ .

Repeating the above process, we can construct a sequence of sets with the latter ones contained in the former ones, such that it is profitable to reduce all the elements in these sets by the same significant small amount. Because the number of elements in  $\mathcal{A}$  is finite, there exists an iteration  $k$  in which the increment set  $\mathcal{A}_{max}$  contains all the elements in the reduction set  $\mathcal{R}^k$ , i.e.,  $\mathcal{A}_{max} \supseteq \mathcal{R}^k$ . Hence, we have a contradiction.  $\square$

**Example 2.** Using the same function given in Example 1,

$$\begin{aligned} f(u_1, u_2, u_3, u_4) &= -2u_1 - 3u_2 - 2u_3 - 7u_4 \\ &\quad + \min\{u_1, u_2 + 1, u_3 + 1, u_4 + 3\} \\ &\quad + 10 \min\{u_1, u_2, u_3, u_4 + 1\} \\ &\quad + \min\{u_1 + 1, u_2 + 1, u_3, u_4 + 3\}, \end{aligned}$$

we solve

$$\max_{u_i \geq 0, i \in \mathcal{M}} f(u_1, u_2, u_3, u_4).$$

At  $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$ , the marginal increment of  $f(\mathbf{u})$  in the direction of  $\mathbf{e}_{u_1, u_2, u_3} = (1, 1, 1, 0)$  is  $\Delta_{(0,0,0,0)}(\{u_1, u_2, u_3\}) = 5$ . It is the largest increment among all possible directions  $\mathbf{e}_{\mathcal{A}}$ . We increase in the direction by the largest amount  $\sup\{u | \sigma_{(0,0,0,0)} = \sigma_{(u,u,u,0)}\} = 1$  so that  $\mathbf{u} = (1, 1, 1, 0)$  and  $\mathcal{A}_{max}$  changes from  $\{u_1, u_2, u_3\}$  to  $\emptyset$ .  $\Delta_{\mathbf{u}}(\mathcal{A}_{max}) = 0$ . Thus, Algorithm 1 stops at  $\mathbf{u}^* = (1, 1, 1, 0)$  with  $f(\mathbf{u}^*) = 5$ .

The following corollary can be obtained directly from Algorithm 1.

**Corollary 1.** Algorithm 1 returns a vector with all of its elements in integer values if all parameters in the constraints of the dual problem are integers.

We also have the following result directly from Lovász [11] and Lemma 2 part (4).

**Corollary 2.**  $\mathcal{A}_{max}$  can be found by maximizing  $\Delta_{\mathbf{u}}(\mathcal{A})$ 's concave Lovász extension.

Corollary 2 ensures that in each iteration of Algorithm 1, finding  $\mathcal{A}_{max}$  can be done in polynomial time, which implies that each iteration of Algorithm 1 only takes polynomial time to complete. Note that in the stepping stone algorithm, the rule to identify the feasible cycle is problem specific, which may significantly affect the computation time of the optimal solution. Here, Algorithm 1 specifies a unique increment direction  $\mathbf{e}_{\mathcal{A}_{max}}$  in each iteration, which significantly reduces the variance of computational time.

This paper shows that the ordinary transportation problem can be solved by a greedy algorithm. The constraints in the transportation problem studied in this paper can be represented by a matroid

with exactly two elements in each base. To see this, denote  $\kappa_i$  as the  $i$ th capacity constraint  $\sum_{j=1}^N x_{ij} \leq \kappa_i$  and  $\delta_j$  as the  $j$ th demand constraint  $\sum_{i=1}^M x_{ij} \leq \delta_j$ . Denote  $\mathcal{C}$  as the set of all demand and capacity constraints, i.e.,  $\mathcal{C} = \{\kappa_i, \delta_j | i \in \mathcal{M}, j \in \mathcal{N}\}$ . Then,  $\mathcal{X} = \{x^{ij} | x^{ij} = \{\kappa_i, \delta_j\}, i \in \mathcal{M}, j \in \mathcal{N}\}$  defines a matroid on  $\mathcal{C}$  with base  $\mathcal{X}$  and the number of elements in each base is two. Thus, after taking the dual, the set of constraints defines a matroid on the set of variables with exactly two elements in each base. This may shed some lights on using simple greedy-type algorithms to solve linear programming models with constraints that can be represented by matroids.

### Acknowledgments

I would like to thank the Editor-In-Chief for his support during the review process, and also the reviewers for their helpful comments. I would also like to thank Long He for his constructive feedback during our discussion involving the paper. Finally, I am thankful to Arvind Sainathan for his assistance with the writing and constructive ideas on the examples.

### Appendix

**Proof of Lemma 1.** Note that  $\min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$  is concave and supermodular in  $\mathbf{u}$ , therefore, its linear combination with positive weights  $\sum_{j=1}^N d_j \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$  is also concave and supermodular.

Adding a linear term  $-\sum_{i=1}^M \kappa_i u_i$  to  $\sum_{j=1}^N d_j \min_{i \in \mathcal{M}} \{c_{ij} + u_i\}$  is also concave and supermodular. Thus,  $f(\mathbf{u})$  is concave and supermodular.  $\square$

**Proof of Lemma 2.** (1) From definition, we have  $\sigma_{\mathbf{u}}(\emptyset) = \emptyset$ . For any two sets  $\mathcal{A} \subseteq \mathcal{B} \in 2^{\mathcal{M}}$ , if  $j \in \sigma_{\mathbf{u}}(\mathcal{A})$ , then  $\arg \min_i \{c_{ij} + u_i\} \subseteq \mathcal{A} \subseteq \mathcal{B}$ . Therefore,  $j \in \sigma_{\mathbf{u}}(\mathcal{B})$ . Thus,  $\sigma_{\mathbf{u}}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}}(\mathcal{B})$ .

(2) For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , because  $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B}$ , from (1) we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$  and  $\sigma_{\mathbf{u}}(\mathcal{B}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$ . Therefore,  $\sigma_{\mathbf{u}}(\mathcal{A}) \cup \sigma_{\mathbf{u}}(\mathcal{B}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})$ .

(3) For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , because  $\mathcal{A} \supseteq \mathcal{A} \cap \mathcal{B}$  and  $\mathcal{B} \supseteq \mathcal{A} \cap \mathcal{B}$ , from (1) we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \supseteq \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$  and  $\sigma_{\mathbf{u}}(\mathcal{B}) \supseteq \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ . Therefore,  $\sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B}) \supseteq \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ . For any  $j \in \sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B})$ , we have  $\arg \min_i \{c_{ij} + u_i\} \subseteq \mathcal{A}$  and  $\arg \min_i \{c_{ij} + u_i\} \subseteq \mathcal{B}$ , therefore,  $\arg \min_i \{c_{ij} + u_i\} \subseteq \mathcal{A} \cap \mathcal{B}$ . That is,  $j \in \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ . Thus,  $\sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ . Combine the two inequalities, we have  $\sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B}) = \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})$ .

(4) From definition, we have  $\Delta_{\mathbf{u}}(\emptyset) = 0$ . For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ ,

$$\begin{aligned} \Delta_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{B}) &= - \sum_{i \in \mathcal{A}} \kappa_i + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A})} d_j - \sum_{i \in \mathcal{B}} \kappa_i + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{B})} d_j \\ &= - \sum_{i \in \mathcal{A} \cup \mathcal{B}} \kappa_i + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A}) \cup \sigma_{\mathbf{u}}(\mathcal{B})} d_j - \sum_{i \in \mathcal{A} \cap \mathcal{B}} \kappa_i \\ &\quad + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B})} d_j \\ &\leq - \sum_{i \in \mathcal{A} \cup \mathcal{B}} \kappa_i + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B})} d_j - \sum_{i \in \mathcal{A} \cap \mathcal{B}} \kappa_i \\ &\quad + \sum_{j \in \sigma_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B})} d_j \\ &= \Delta_{\mathbf{u}}(\mathcal{A} \cup \mathcal{B}) + \Delta_{\mathbf{u}}(\mathcal{A} \cap \mathcal{B}). \end{aligned}$$

The inequality holds because of (2).

(5) The statement is true directly follows (4).

(6) If  $j \in \sigma_{u^1}(\mathcal{A})$ , then  $\arg \min_i \{c_{ij} + u_i^1\} \subseteq \mathcal{A}$ . For any  $i^0 \in \arg \min_i \{c_{ij} + u_i^1\} \subseteq \mathcal{A}$ , we have  $c_{i^0j} + u_{i^0}^2 = c_{i^0j} + u_{i^0}^1 \leq c_{ij} + u_i^1 \leq c_{ij} + u_i^2$  holds for all  $i \in \mathcal{M}$  and hence  $i^0 \in \arg \min_i \{c_{ij} + u_i^2\}$  and  $\min_i \{c_{ij} + u_i^1\} = \min_i \{c_{ij} + u_i^2\}$ . For any  $i^0 \notin \arg \min_i \{c_{ij} + u_i^1\}$ , there exists  $i_{\min} \in \mathcal{A}$  such that  $c_{i_{\min}j} + u_{i_{\min}}^2 = c_{i_{\min}j} + u_{i_{\min}}^1 < c_{i^0j} + u_{i^0}^1 \leq c_{i^0j} + u_{i^0}^2$ , thus  $i^0 \notin \arg \min_i \{c_{ij} + u_i^2\}$ . This hence leads to  $\arg \min_i \{c_{ij} + u_i^1\} = \arg \min_i \{c_{ij} + u_i^2\}$ . Thus,  $\arg \min_i \{c_{ij} + u_i^2\} \subseteq \mathcal{A}$ . This proves that  $\sigma_{u^1}(\mathcal{A}) \subseteq \sigma_{u^2}(\mathcal{A})$ .

Following the above analysis,  $\Delta_{u^1}(\mathcal{A}) = -\sum_{i \in \mathcal{A}} k_i + \sum_{j \in \sigma_{u^1}(\mathcal{A})} d_j \leq -\sum_{i \in \mathcal{A}} k_i + \sum_{j \in \sigma_{u^2}(\mathcal{A})} d_j = \Delta_{u^2}(\mathcal{A})$ .  $\square$

**Proof of Lemma 3.** (1) From definition, we have  $\pi_u(\emptyset) = \emptyset$ . For any two sets  $\mathcal{A} \subseteq \mathcal{B} \in 2^{\mathcal{M}}$ , if  $j \in \pi_u(\mathcal{A})$ , then  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{A} \neq \emptyset$  and  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{A} \subseteq \mathcal{B}$ . Therefore,  $j \in \mathcal{B}$ . Thus,  $\pi_u(\mathcal{A}) \subseteq \pi_u(\mathcal{B})$ .

(2) For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ ,  $j \in \pi_u(\mathcal{A}) \cup \pi_u(\mathcal{B})$ , if and only if  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{A} \neq \emptyset$  or  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{B} \neq \emptyset$ , if and only if  $j \in \pi_u(\mathcal{A} \cup \mathcal{B})$  and hence  $\pi_u(\mathcal{A}) \cup \pi_u(\mathcal{B}) = \pi_u(\mathcal{A} \cup \mathcal{B})$ .

(3) For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ , if  $j \in \pi_u(\mathcal{A} \cap \mathcal{B})$ , then  $\arg \min_i \{c_{ij} + u_i\} \cap (\mathcal{A} \cap \mathcal{B}) \neq \emptyset$ . Thus,  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{A} \neq \emptyset$  and  $\arg \min_i \{c_{ij} + u_i\} \cap \mathcal{B} \neq \emptyset$ . That is,  $j \in \pi_u(\mathcal{A}) \cap \pi_u(\mathcal{B})$ .

(4) From definition, we have  $\Lambda_u(\emptyset) = 0$ . For any two sets  $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{M}}$ ,

$$\begin{aligned} \Lambda_u(\mathcal{A}) + \Lambda_u(\mathcal{B}) &= \sum_{i \in \mathcal{A}} k_i - \sum_{j \in \pi_u(\mathcal{A})} d_j + \sum_{i \in \mathcal{B}} k_i - \sum_{j \in \pi_u(\mathcal{B})} d_j \\ &= \sum_{i \in \mathcal{A} \cup \mathcal{B}} k_i - \sum_{j \in \pi_u(\mathcal{A}) \cup \pi_u(\mathcal{B})} d_j + \sum_{i \in \mathcal{A} \cap \mathcal{B}} k_i \\ &\quad - \sum_{j \in \pi_u(\mathcal{A}) \cap \pi_u(\mathcal{B})} d_j \\ &\leq \sum_{i \in \mathcal{A} \cup \mathcal{B}} k_i - \sum_{j \in \sigma_u(\mathcal{A} \cup \mathcal{B})} d_j + \sum_{i \in \mathcal{A} \cap \mathcal{B}} k_i \\ &\quad - \sum_{j \in \sigma_u(\mathcal{A} \cap \mathcal{B})} d_j \\ &= \Lambda_u(\mathcal{A} \cup \mathcal{B}) + \Lambda_u(\mathcal{A} \cap \mathcal{B}). \end{aligned}$$

The inequality holds because of (3).

(5) If  $j \in \pi_{u^2}(\mathcal{A})$ , then  $\arg \min_i \{c_{ij} + u_i^2\} \cap \mathcal{A} \neq \emptyset$ . For any  $i^0 \in \arg \min_i \{c_{ij} + u_i^2\} \cap \mathcal{A}$ , we have  $c_{i^0j} + u_{i^0}^1 \leq c_{i^0j} + u_{i^0}^2 \leq c_{ij} + u_i^2 = c_{ij} + u_i^1$  holds for all  $i \in \mathcal{M} \setminus \mathcal{A}$ . Thus, for all  $i \in \mathcal{M} \setminus \mathcal{A}$ ,  $i \notin \arg \min_i \{c_{ij} + u_i^1\}$ . This hence implies that  $\arg \min_i \{c_{ij} + u_i^1\} \cap \mathcal{A} \neq \emptyset$ . This proves that  $\pi_{u^1}(\mathcal{A}) \supseteq \pi_{u^2}(\mathcal{A})$ . Thus,  $\Lambda_{u^1}(\mathcal{A}) \leq \Lambda_{u^2}(\mathcal{A})$ .

(6) If  $j \in \pi_{u^1}(\mathcal{A})$ , then  $\arg \min_i \{c_{ij} + u_i^1\} \cap \mathcal{A} \neq \emptyset$ . For any  $i^0 \in \arg \min_i \{c_{ij} + u_i^1\} \cap \mathcal{A}$ , we have  $c_{i^0j} + u_{i^0}^2 = c_{i^0j} + u_{i^0}^1 \leq c_{ij} + u_i^1 \leq c_{ij} + u_i^2$ . This hence implies that  $\arg \min_i \{c_{ij} + u_i^2\} \cap \mathcal{A} \neq \emptyset$ . This proves that  $\pi_{u^1}(\mathcal{A}) \subseteq \pi_{u^2}(\mathcal{A})$  and  $\Lambda_{u^1}(\mathcal{A}) \geq \Lambda_{u^2}(\mathcal{A})$ .

(7) From the definition of  $\pi_u(\mathcal{A})$  and  $\sigma_u(\mathcal{A})$ , we have  $\pi_u(\mathcal{A}) \supseteq \sigma_u(\mathcal{A})$ . This hence leads to  $\Lambda_u(\mathcal{A}) + \Delta_u(\mathcal{A}) \leq 0$ .  $\square$

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