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Complex Analysis and Operator Theory

Unbounded Weighted Conditional Expectation Operators

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Abstract In this note some basic properties of unbounded weighted conditional expectation operators are investigated. A description of polar decomposition, spectrum, normality, quasi-normality and self-adjointness in this context are provided. Also, we give some necessary and sufficient conditions for EM_u to leave invariant its domain. Finally, some examples are provided to illustrate concrete application of the main results of the paper.

Keywords Conditional expectation · Unbounded operator · Spectrum · Polar decomposition

Mathematics Subject Classification 47B25 · 47B38

1 Introduction and Preliminaries

Theory of weighted conditional expectation type operators is one of important arguments in the connection of operator theory and measure theory. Weighted conditional expectation type operators have been studied in an operator theoretic setting, by, for example, de Pagter and Grobler [7] and Rao [12,13], as positive operators acting on L^p-spaces or Banach function spaces. Moy [11] characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded. Also, some results about weighted conditional expectation type operators can be found in [2,8,10]. Dodds et al. [3]

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showed that lots of operators are of the form of weighted conditional type operators. The class of weighted conditional type operators includes the classes of composition operators, multiplication operators, weighted composition operators, some integral type operators and etc. These are some reasons that stimulate us to consider weighted conditional type operators in our work. So, in [4–6] we studied bounded multiplication conditional expectation operators M_wEM_u on L^p spaces.

As far as I know, unbounded weighted conditional type operators weren't investigated till now. This paper is devoted to the study of unbounded weighted conditional type operators on $L^2(X, \Sigma, \mu)$, such that (X, Σ, μ) is a σ-finite measure space. In general closed operators are important classes of unbounded linear operators which are large enough to cover all interesting operators occurring in applications. In Sect. 2, we develop basic concepts and results about weighted conditional type operators on Hilbert space $L^2(\Sigma)$. First we investigate some basic properties of unbounded weighted conditional type operators. We show that a densely-defined weighted conditional type operators are always closed. In general closed operators are important classes of unbounded linear operators which are large enough to cover all interesting operators occurring in applications.

Let $\mathcal{H}$ stand for a Hilbert space and $\mathcal{B}(\mathcal{H})$ for the Banach algebra of all bounded operators on $\mathcal{H}$. By an operator in $\mathcal{H}$ we understand a linear mapping $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ defined on a linear subspace $\mathcal{D}(T)$ of $\mathcal{H}$ which is called the domain of T. Set $\mathcal{D}^\infty(T) = \cap_{n=1}^{\infty}\mathcal{D}(T^n)$. Denote by $\mathcal{N}(T)$, $\mathcal{R}(T)$ and T^* the kernel, the range and the adjoint of T respectively. Given an operator T in $\mathcal{H}$, we write the graph norm $\|.\|_T$ on $\mathcal{D}(T)$ by

$$\|f\|_T^2 = \|f\|^2 + \|Tf\|^2, \quad f \in \mathcal{D}(T).$$

A densely defined operator T on $\mathcal{H}$ is said to be normal if T is closed and $T^*T = TT^*$.

2 Unbounded Weighted Conditional Expectation

Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-subalgebra of Σ such that $(X, \mathcal{A}, \mu)$ is also σ-finite. We denote the collection of (equivalence classes modulo sets of zero measure of) Σ-measurable complex-valued functions on X by $L^0(\Sigma)$ and the support of a function $f \in L^0(\Sigma)$ is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. Moreover, we set $L^2(\Sigma) = L^2(X, \Sigma, \mu)$. We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

For each σ-finite subalgebra $\mathcal{A}$ of Σ, the conditional expectation, $E^{\mathcal{A}}(f)$, of f with respect to $\mathcal{A}$ is defined whenever $f \geq 0$ almost everywhere or $f \in L^2$. In any case, $E^{\mathcal{A}}(f)$ is the unique $\mathcal{A}$-measurable function for which

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is an idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. If there is no possibility of confusion we write $E(f)$ in place of $E^{\mathcal{A}}(f)$ [14,16].

Let u be a complex Σ-measurable function on X. Define the measure $\mu_u : \Sigma \to [0, \infty]$ by

$$\mu_u(E) = \int_E |u|^2 d\mu, \quad E \in \Sigma.$$

It is clear that the measure μ_u is also σ-finite.

In this paper we consider u is conditionable (i.e., $E(u)$ is defined). Operators of the form $EM_u(f) = E(u.f)$ acting in $L^2(\mu) = L^2(X, \Sigma, \mu)$ with $\mathcal{D}(EM_u) = \{f \in L^2(\mu) : E(u.f) \in L^2(\mu)\}$ are called weighted conditional expectation type operators. First we give some conditions under which the operator EM_u is densely defined.

Lemma 2.1 *Let $w = 1 + E(|u|^2)$, μ and $d\nu = wd\mu$. We get that*

(i) *$S(w) = X$ and $L^2(\nu) = L^2(X, \Sigma, \nu) \subseteq \mathcal{D}(EM_u)$.*
(ii) *If $E(|u|^2) < \infty$ a.e., then*

$$\overline{L^2(\nu)}^{\|.\|_\mu} = \overline{\mathcal{D}(EM_u)}^{\|.\|_\mu} = L^2(\mu).$$

Proof Let $f \in L^2(\nu)$. Then

$$\begin{aligned}\|f\|_\mu^2 &= \int_X |f|^2 d\mu \\ &\le \int_X |f|^2 d\mu + \int_X E(|u|^2)|f|^2 d\mu \\ &= \|f\|_\nu^2 < \infty,\end{aligned}$$

so $f \in L^2(\mu)$. Also, by conditional-type Hölder-inequality we have

$$\begin{aligned}\|EM_u(f)\|_\mu^2 &= \int_X |E(uf)|^2 d\mu \\ &\le \int_X E(|u|^2)E(|f|^2) d\mu \\ &= \int_X E(|u|^2)|f|^2 d\mu \\ &\le \|f\|_\nu^2 < \infty,\end{aligned}$$

this implies that $f \in \mathcal{D}(EM_u)$.

Now we prove that $L^2(\nu)$ is dense in $L^2(\mu)$. Suppose that $f \in L^2(\mu)$ such that $\langle f, g\rangle = \int_X f.\bar{g} d\mu = 0$ for all $g \in L^2(\nu)$. For $A \in \Sigma$ we set $A_n = \{x \in A : w(x) \le n\}$. It is clear that $A_n \subseteq A_{n+1}$ and $X = \cup_{n=1}^\infty A_n$. Also, X is σ-finite, hence $X = \cup_{n=1}^\infty X_n$ with $\mu(X_n) < \infty$. If we set $B_n = A_n \cap X_n$, then $B_n \nearrow A$ and so $f.\chi_{B_n} \nearrow f.\chi_A$ a.e., μ. Since $\nu(B_n) \le (n+1)\mu(B_n) < \infty$, we have $\chi_{B_n} \in L^2(\nu)$ and by our assumption $\int_{B_n} f d\mu = 0$. Therefore by Fatou's lemma we get that $\int_A f d\mu = 0$.

Thus for all $A \in \Sigma$ we have $\int_A f d\mu = 0$. This means that $f = 0$ a.e., μ and so $L^2(\nu)$ is dense in $L^2(\mu)$.

Proposition 2.2 *If $u : X \to \mathbb{C}$ is Σ-measurable, then the following conditions are equivalent*:

(i) *EM_u is densely defined,*
(ii) *$w - 1 = E(|u|^2) < \infty$ a.e., μ,*
(iii) *$\mu_{E(|u|^2)} |_{\mathcal{A}}$ is σ-finite.*

Proof (i)→(ii) Set $E = \{E(|u|^2) = \infty\}$. Clearly by Lemma 2.1, $f |_E = 0$ a.e., μ for every $f \in L^2(\nu)$. This implies that $f.w |_E = 0$ a.e., μ for every $f \in L^2(\mu)$. So we have $w.\chi_{A\cap E} = 0$ a.e., μ for all $A \in \Sigma$ with $\mu(A) < \infty$. By the σ-finiteness of μ we have $w.\chi_E = 0$ a.e., μ. Since $S(w) = X$, we get that $\mu(E) = 0$.

(ii) → (i) It is clear by Lemma 2.1.

(ii) → (iii) Since $(X, \mathcal{A}, \mu)$ is a σ-finite measure space and $E(|u|^2) < \infty$ a.e., μ, then there exists a sequence $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $\mu(X_n) < \infty$ and $E(|u|^2) < n$ a.e., μ on X_n for every $n \in \mathbb{N}$ and $X_n \nearrow X$ as $n \to \infty$.

So we have

$$\mu_{(w-1)} |_{\mathcal{A}} (X_n) = \int_{X_n} E(|u|^2) d\mu \leq n\mu(X_n) < \infty, \quad n \in \mathbb{N}.$$

This yields (iii).

(iii) → (i) Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence such that $A_n \nearrow X$ as $n \to \infty$ and $\mu_{w-1} |_{\mathcal{A}} (A_n) < \infty$ for every $k \in \mathbb{N}$. It follows from the definition of μ_{w-1} that $w - 1 = E(|u|^2) < \infty$ a.e., μ on X. Applying Lemma 2.1 we obtain (i).

It is easily seen that $\mathcal{D}(M_{\bar{u}}E) \subseteq \mathcal{D}(EM_u)$. Also, by the previous proposition easily we get that: the operator EM_u is densely defined if and only if the operator $M_{\bar{u}}E$ is densely defined. In this case we get that $\overline{\mathcal{D}(EM_u)}^{\|\cdot\|_2} = \overline{\mathcal{D}(M_{\bar{u}}E)}^{\|\cdot\|_2}$.

Now in the next lemma we obtain the adjoint of EM_u.

Lemma 2.3 *Let the linear transformation $T = EM_u$ be densely defined on $L^2(\Sigma)$, then $T^* = M_{\bar{u}}E$.*

Proof Let $f \in \mathcal{D}(T)$ and $g \in \mathcal{D}(T^*)$. So we have

$$\begin{aligned}\langle Tf, g\rangle &= \int_X E(uf)\bar{g} d\mu \\ &= \int_X E(\bar{g})E(uf) d\mu \\ &= \int_X fuE(\bar{g}) d\mu \\ &= \int_X f\overline{\bar{u}E(g)} d\mu \\ &= \langle f, M_{\bar{u}}Eg\rangle.\end{aligned}$$

From this relation we conclude that $M_{\bar{u}}E \subseteq (EM_u)^*$ and so $\mathcal{D}((EM_u)^*) \subseteq \mathcal{D}(M_{\bar{u}}E)$. Since the inner product function is continuous and EM_u is densely defined, then $\mathcal{D}((EM_u)^*) = \mathcal{D}(M_{\bar{u}}E)$. This completes the proof of the equality $(EM_u)^* = M_{\bar{u}}E$.

By using Lemma 2.3 we show that all densely defined weighted conditional expectation operators are closed.

Proposition 2.4 *If $E(|u|^2) < \infty$ a.e., μ. Then the linear transformation $EM_u : \mathcal{D}(EM_u) \to L^2(\Sigma)$ is closed.*

Proof Assume that $f_n \in \mathcal{D}(EM_u)$, $f_n \to f$, $E(uf_n) \to g$, and let $h \in \mathcal{D}(M_{\bar{u}}E)$. Then

$$\begin{aligned}\langle f, M_{\bar{u}}Eh\rangle &= \lim_{n\to\infty}\langle f_n, M_{\bar{u}}Eh\rangle\\ &= \lim_{n\to\infty}\langle E(uf_n), h\rangle\\ &= \langle g, h\rangle.\end{aligned}$$

This calculation (which uses the continuity of the inner product and the fact that $f_n \in \mathcal{D}(EM_u)$) shows that $f \in \mathcal{D}(EM_u)$ and $E(uf) = g$, as required.

In the next two propositions we characterize normal and self-adjoint weighted conditional expectation operators.

Proposition 2.5 *If $\mathcal{D}(EM_u)$ is dense in $L^2(\Sigma)$ and u is almost every where finite valued, then the operator EM_u is normal if and only if $u \in L^0(\mathcal{A})$.*

Proof Let the operator EM_u be normal. Then for every $f \in \mathcal{D}(M_{\bar{u}}EM_u) = \mathcal{D}(EM_{E(|u|^2)})$ we have

$$T^*Tf = TT^*f \ \ \mu,\ a.e., \Rightarrow \bar{u}E(uf) = E(|u|^2)E(f) \ \ \mu,\ a.e.,$$

by taking E over both side of the equality, for a positive element a of $L^2(X, \mathcal{A}, \nu)$ we get that $|E(u.\chi_{A_n})|^2 a = E(|u|^2.\chi_{A_n})a \quad \mu,\ a.e.$, in which $A_n = \{x \in X : |u(x)| \leq n\}$ and $\mu(A_n) < \infty$. This implies that $|E(u.\chi_{A_n})|^2 = E(|u|^2.\chi_{A_n})$. Since $E(|u.\chi_{A_n} - E(u.\chi_{A_n})|^2) = E(|u.\chi_{A_n}|^2) - |E(u.\chi_{A_n})|^2$, then $u.\chi_{A_n} = E(u.\chi_{A_n})$ and so $u = E(u)$, i.e, $u \in L^0(\mathcal{A})$.

Conversely. If EM_u is densely defined, then by the same method of Lemma 2.1 we get that

$$L^2(\nu) \subseteq \mathcal{D}(M_{\bar{u}}EM_u), \quad L^2(\nu) \subseteq \mathcal{D}(EM_{E(|u|^2)}), \quad \mathcal{D}(EM_{E(|u|^2)}) \subseteq \mathcal{D}(M_{\bar{u}}EM_u)$$

and

$$\overline{\mathcal{D}(M_{\bar{u}}EM_u)} = \overline{L^2(\nu)} = \overline{\mathcal{D}(EM_{E(|u|^2)})} = L^2(\mu),$$

where $d\nu = (1 + (E(|u|^2))^2)d\mu$. Suppose that $u \in L^0(\mathcal{A})$. Then $\mathcal{D}(M_{\bar{u}}EM_u) = \mathcal{D}(EM_{E(|u|^2)}) = \mathcal{D}(M_{|u|^2}E)$ and for every $f \in \mathcal{D}(M_{|u|^2}E)$ we have

$$T^*Tf = |u|^2 E(f) = TT^*f \ \ \mu, \ a.e.,$$

On the other hand, since EM_u is densely defined. Then by Proposition 2.4 it is closed. This implies that the operator EM_u is normal.

Proposition 2.6 *If $\mathcal{D}(EM_u)$ is dense in $L^2(\Sigma)$, then the operator EM_u is self-adjoint if and only if $u \in L^0(\mathcal{A})$ is real valued.*

Proof Suppose that the operator EM_u is self-adjoint, then it is normal. So by Proposition 2.5 we get that $u \in L^0(\mathcal{A})$. Let a be a positive $\mathcal{A}$-measurable function in $\mathcal{D}(EM_u)$. Hence

$$\bar{u}a = \bar{u}E(a) = T^*(a) = T(a) = E(ua) = ua,$$

this implies that $\bar{u} = u$ i.e, u is real valued.

Conversely, if $u \in L^0(\mathcal{A})$ is real valued, then for $f, g \in \mathcal{D}(M_{\bar{u}}E) \subseteq \mathcal{D}(EM_u)$,

$$\begin{aligned}\langle E(uf), g\rangle &= \int_X E(uf)\bar{g}d\mu \\ &= \int_X uE(f)\bar{g}d\mu \\ &= \int_X \bar{u}E(f)\bar{g}d\mu \\ &= \langle M_{\bar{u}}E(f), g\rangle.\end{aligned}$$

This implies that $\mathcal{D}(EM_u) = \mathcal{D}(M_{\bar{u}}E)$ and $T^*(f) = T(f)$ for all $f \in \mathcal{D}(EM_u)$.

It is well-known that for a densely defined closed operator T of $\mathcal{H}_1$ into $\mathcal{H}_2$, there exists a partial isometry U_T with initial space $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(|T|)}$ and final space $\mathcal{N}(T^*)^{\perp} = \overline{\mathcal{R}(T)}$ such that

$$T = U_T|T|.$$

A closed densely defined operator T in $\mathcal{H}$ is said to be quasinormal if $U|T| \subseteq |T|U$, where $T = U|T|$ is the polar decomposition of T [1,15]. As is shown in [9, Theorem 3.1], A closed densely defined operator T in $\mathcal{H}$ is quasinormal if and only if $T|T|^2 = |T|^2T$.

Proposition 2.7 *Suppose that $\mathcal{D}(EM_u)$ is dense in $L^2(\Sigma)$. Let $EM_u = U|EM_u|$ be the polar decomposition of EM_u. Then*

(i) $|EM_u| = M_{u'}EM_u$, *where* $u' = (E(|u|^2))^{\frac{-1}{2}}.\chi_S.\bar{u}$ *and* $S = S(E(|u|^2))$,

(ii) $U = EM_{\widetilde{u}}$, *where* $\widetilde{u} : X \to \mathbb{C}$ *is an a.e.,* μ *well-defined* Σ*-measurable function such that*

$$\widetilde{u} = u.\frac{1}{(E(|u|^2))^{\frac{1}{2}}}.\chi_S.$$

Proof (i). For every $f \in \mathcal{D}(M_{u'}EM_u)$ we have

$$\begin{aligned}\|M_{u'}EM_u(f)(f)\|^2 &= \int_X ((|u|^2))^{-1}.\chi_S|u|^2|E(uf)|^2d\mu \\ &= \int_X ((|u|^2))^{-1}.\chi_S E(|u|^2)|E(uf)|^2d\mu \\ &= \||EM_u|(f)\|^2.\end{aligned}$$

Also, by Lemma 2.1 we conclude that $\mathcal{D}(M_{u'}EM_u) = \mathcal{D}(|EM_u|)$ and it is easily seen that $M_{u'}EM_u$ is a positive operator. This observations imply that $|EM_u| = M_{u'}EM_u$.

(ii). For $f \in L^2(\Sigma)$ we have

$$\int_X |E(\widetilde{u}f)|^2d\mu = \int_X \frac{1}{(E(|u|^2))}.\chi_S|E(uf)|^2d\mu,$$

which implies that the operator $EM_{\widetilde{u}}$ is well-defined and $\mathcal{N}(EM_u) = \mathcal{N}(EM_{\widetilde{u}})$. Also, for $f \in \mathcal{D}(EM_u) \ominus \mathcal{N}(EM_u)$ we have

$$\begin{aligned}U(|EM_u|(f)) &= \frac{1}{(E(|u|^2))^{\frac{1}{2}}}.\chi_S E(u(E(|u|^2))^{\frac{-1}{2}}\chi_S\bar{u}E(uf)) \\ &= \frac{1}{E(|u|^2)}.\chi_S E(|u|^2)E(uf) \\ &= E(uf).\end{aligned}$$

Thus $\|U(f)\| = \|f\|$ for all $f \in \mathcal{R}(|EM_u|)$ and since U is a contraction, then it holds for all $f \in \mathcal{N}(EM_u)^{\perp} = \overline{\mathcal{R}(|EM_u|)}$.

Here we remind that: A complex number λ belongs to the resolvent set $\rho(T)$ of the closed linear operator T on a Hilbert space $\mathcal{H}$, if the operator $\lambda I - T$ has a bounded everywhere on $\mathcal{H}$ defined inverse $(\lambda I - T)^{-1}$, called the resolvent of T at λ and denoted by $R_\lambda(T)$.

The set $\sigma(T) := \mathbb{C}\backslash\rho(T)$ is called the spectrum of the operator T. Here we investigate the spectrum of the densely defined weighted conditional expectation operators.

Theorem 2.8 *Let* EM_u *be densely defined and* $\mathcal{A} \subsetneq \Sigma$, *then*

(i) essrange(E(u)) $\cup\{0\} \subseteq \sigma(EM_u)$,
(ii) *If* $L^2(\mathcal{A}) \subseteq \mathcal{D}(EM_u)$, *then* $\sigma(EM_u) \subseteq$ essrange(E(u)) $\cup\{0\}$.

Proof (i) Since $\mathcal{A} \neq \Sigma$, we get that EM_u is not surjective. So $0 \in \sigma(EM_u)$. Let $0 \neq \lambda \in$ essrange(E(u)). Then the measure of the set

$$N_n := \{x \in X : |E(u)(x) - \lambda| < n^{-1}\}$$

is positive for each $n \in \mathbb{N}$. Because μ is σ-finite, upon replacing N_n by a subset, we can assume that $0 < \mu(N_n) < \infty$. Since $E(u)$ is bounded on N_n, $\chi_{N_n} \in \mathcal{D}(EM_u)$ and

$$\begin{aligned}\|(\lambda I - EM_u)\chi_{N_n}\|^2 &= \int_X |\lambda\chi_{N_n} - E(u\chi_{N_n})|^2 d\mu \\ &= \int_X |\lambda\chi_{N_n} - E(u)\chi_{N_n}|^2 d\mu \\ &= \int_{N_n} |(\lambda - E(u))|^2 d\mu \\ &\leq n^{-2}\|\chi_{N_n}\|^2.\end{aligned}$$

Hence λ is not a regular point. Therefore, $\lambda \notin \rho(EM_u)$, and so $\lambda \in \sigma(EM_u)$.

(ii) Suppose that $\lambda \notin$ essrange(E(u)). Then there exists $n \in \mathbb{N}$ such that $\mu(N_n) = 0$. We show that $T - \lambda I$ is invertible. If $Tf - \lambda f = 0$ for $f \in \mathcal{D}(EM_u)$, then $E(uf) = \lambda f$. So f is $\mathcal{A}-$measurable. Thus $(E(u) - \lambda)f = E(uf) - \lambda f = 0$. Since $\lambda \notin$ essrange(E(u)), then $E(u) - \lambda \geq \varepsilon$ a.e., for some $\varepsilon > 0$. So $f = 0$ a.e., This implies that $T - \lambda I$ is injective and so is invertible.

Now we show that $T - \lambda I$ is surjective. Let $g \in L^2(\Sigma)$. We can write

$$g = g - E(g) + E(g), \quad g_1 = g - E(g), \quad g_2 = E(g).$$

Clearly we have $g_2 \in L^2(\mathcal{A}) \subseteq \mathcal{D}(EM_u)$ and $g_1 \in L^2(\Sigma)$, $(E(g_1) = 0$. Let

$$f_1 = \frac{\lambda g_1 + T(g_2)}{\lambda(E(u) - \lambda)}, \quad f_2 = \frac{-g_2}{\lambda}.$$

Since $\lambda \notin$ essrange(E(u)), we get $E(u) - \lambda \geq \varepsilon$ a.e., for some $\varepsilon > 0$. So $\|\frac{1}{E(u)-\lambda}\|_\infty \leq \frac{1}{\varepsilon}$. Thus $f_2 \in L^2(\mathcal{A})$, $f_1 \in L^2(\Sigma)$ and $f = f_1 + f_2 \in L^2(\Sigma)$. Direct computation shows that $T(f) - \lambda f = g$. This implies that $T - \lambda I$ is invertible and so $\lambda \notin \sigma(T)$.

As a remark we obtain the spectrum the operator EM_u when it's everywhere defined.

Remark If EM_u is everywhere defined, then essrange(E(u)) $\cup\{0\} = \sigma(EM_u)$.

Here we discuss about quasi-normal weighted conditional expectation operators.

Proposition 2.9 *Let EM_u be densely defined on $L^2(\Sigma)$. Then we have*

(i) *If $\bar{u}E(u) = E(|u|^2)$, μ, a.e., on $S(E(|u|^2))$, then the operator EM_u is quasi-normal.*
(ii) *If the operator EM_u is quasi-normal, then $\bar{u}E(u) = E(|u|^2)$, μ, a.e., on $S(E(u))$.*

Proof (i) Let $\nu_1 = 1 + (E(|u|^2))^3$ and $\nu_2 = 1 + |E(u)|^2(E(|u|^2))^2$. It is obvious that $\nu_2 \leq \nu_1$, μ, a.e., So $L^2(\nu_1) \subseteq L^2(\nu_2) \subseteq L^2(\mu)$. If we set $T_1 = EM_u|EM_u|^2 = EM_{uE(|u|^2)}$ and $T_2 = |EM_u|^2EM_u = M_{\bar{u}E(u)}EM_u$, then by Proposition 2.2 we conclude that EM_u is densely defined if and only if T_1 is densely defined. Also, it is easily seen that $\mathcal{D}(T_1) \subseteq \mathcal{D}(T_2)$. So,

$$\overline{L^2(\nu_1)} = \overline{L^2(\nu_2)} = \overline{\mathcal{D}(T_1)} = \overline{\mathcal{D}(T_2)} = L^2(\mu).$$

Since $\bar{u}E(u) = E(|u|^2)$, μ, a.e., on $S(E(|u|^2))$, then $|E(u)|^2 = E(|u|^2)$. So for all $f \in \mathcal{D}(T_2)$ we get that

$$\begin{aligned}\|T_1(f)\|^2 &= \int_X (E(|u|^2))^2|E(uf)|^2 d\mu \\ &= \int_X |E(u)|^2E(|u|^2)|E(uf)|^2 d\mu \\ &= \|T_2(f)\|^2.\end{aligned}$$

This implies that $\mathcal{D}(T_1) = \mathcal{D}(T_2)$. Also, for all $f \in \mathcal{D}(T_2)$ we have

$$\begin{aligned}T_2 &= M_{\bar{u}E(u)}EM_u(f) \\ &= \bar{u}E(u)E(uf) \\ &= E(|u|^2)E(uf) \\ &= T_1(f).\end{aligned}$$

(ii) Suppose that the operator EM_u is quasi-normal, then for all $f \in \mathcal{D}(|EM_u|^2 EM_u)$ we have

$$\begin{aligned}EM_u|EM_u|^2(f) &= E(uE(|u|^2)f) \\ &= E(|u|^2)E(uf) \\ &= \bar{u}E(u)E(uf) \\ &= |EM_u|^2EM_u,\end{aligned}$$

So, for a positive $\mathcal{A}$-measurable function $a \in L^2(\nu_1)$ we get that $\bar{u}E(u)E(u)a = E(|u|^2)E(u)a$ and then $\bar{u}E(u) = E(|u|^2)$, μ, a.e., on $S(E(u))$.

Corollary 2.10 *Let EM_u be densely defined on $L^2(\Sigma)$ and $S(E(u)) = S(E(|u|^2))$. Then the operator EM_u is quasi-normal if and only if $\bar{u}E(u) = E(|u|^2)$, μ, a.e.,*

Elementary properties of conditional expectation E help us to prove that if $u \geq 0$, then $S(E(u)) = S(E(|u|^2))$. This implies that if $u \geq 0$, then EM_u is quasi-normal if and only if $\bar{u}E(u) = E(|u|^2)$, μ, a.e., The next proposition can be easily deduced from the closed graph theorem.

Proposition 2.11 *If T is a closed operator on $\mathcal{H}$ such that $T(\mathcal{D}(T) \subseteq (\mathcal{D}(T)$, then T is a bounded operator on the Hilbert space $(\mathcal{D}(T), \|.\|_T)$.*

Proposition 2.12 *Let $\mathcal{D}(EM_u)$ is dense in $L^2(\Sigma)$. Then the following conditions are valid*:

(i) *If $EM_u(\mathcal{D}(EM_u)) \subseteq \mathcal{D}(EM_u)$, then $|E(u)|^4 \leq c(1 + |E(u)|^2)$ a.e., μ.*
(ii) *Assume that, there exists $c > 0$ such that $|E(u)|^2 E(|u|^2) \leq c(1 + |E(u)|^2)$ a.e., μ and $\int_X |E(u)f|^2 d\mu \leq \int_X |E(uf)|^2 d\mu$, for all $f \in \mathcal{D}(EM_u)$. Then $EM_u(\mathcal{D}(EM_u)) \subseteq \mathcal{D}(EM_u)$.*

Proof (i). Since EM_u is closed, densely defined and $EM_u(\mathcal{D}(EM_u)) \subseteq \mathcal{D}(EM_u)$, then by closed graph theorem EM_u is a bounded operator on $(\mathcal{D}(EM_u), \|.\|_{EM_u})$. Hence there exists $c > 0$ such that $\|EM_u(f)\|^2_{EM_u} \leq c\|f\|^2_{EM_u}$ for $f \in \mathcal{D}(EM_u)$. By replacing f with $EM_u(f)$ we have

$$\begin{aligned}\|(EM_u)^2(f)\|^2 &\leq \|EM_u(f)\|^2 + \|(EM_u)^2(f)\|^2 \\ &\leq c(\|f\|^2 + \|EM_u(f)\|^2),\end{aligned}$$

i.e,

$$\int_X |E(u)|^2 |E(uf)|^2 d\mu \leq c\left(\int_X |f|^2 d\mu + \int_X |E(uf)|^2 d\mu\right).$$

Thus for every $A \in \mathcal{A}$ with $\chi_A \in \mathcal{D}(EM_u)$ we have

$$\int_A |E(u)|^2 |E(u)|^2 d\mu \leq c\left(\int_A d\mu + \int_A |E(u)|^2 d\mu\right).$$

Since EM_u is densely-defined, we get that $|E(u)|^4 \leq c(1 + |E(u)|^2)$.

(ii). Let $f \in \mathcal{D}(EM_u)$. Then by assumptions $|E(u)|^2 E(|u|^2) \leq c(1 + |E(u)|^2)$ a.e., μ and $\int_X |E(u)f|^2 d\mu \leq \int_X |E(uf)|^2 d\mu$, a.e., μ, we have

$$\begin{aligned}\int_X |(EM_u)^2(f)|^2 d\mu &= \int_X |E(u)|^2 |E(uf)|^2 d\mu \\ &\leq \int_X |E(u)|^2 E(|u|^2)|f|^2 d\mu \\ &\leq c\left(\int_X |f|^2 d\mu + \int_X |E(u)f|^2 d\mu\right) \\ &\leq c\left(\int_X |f|^2 d\mu + \int_X |E(uf)|^2 d\mu\right) \\ &\leq c(\|f\|^2 + \|E(uf)\|^2) \\ &< \infty.\end{aligned}$$

Therefore $EM_u(f) \in \mathcal{D}(EM_u)$.

Corollary 2.13 *If $\mathcal{D}(M_u)$ is dense in $L^2(\Sigma)$, then the following conditions are equivalent*:

(i) $M_u(\mathcal{D}(M_u)) \subseteq \mathcal{D}(M_u)$.
(ii) *There exists $c > 0$ such that $u^4 \leq c(1 + u^2)$ a.e., μ.*

Here we present some examples of conditional expectations and corresponding multiplication operators to illustrate concrete application of the main results of the paper in this section.

Example 1 In this example we consider some cases of sub-algebras of the σ-algebra Σ when $\mu(X) < \infty$. In this cases, we restrict our attention to weighted conditional type operators on sub-spaces of $L^2(\Sigma)$.
Case 1. As our first cases, take the extreme case when $\mathcal{A} = \Sigma$. Then $E(f) = f$ for all $f \in L^2(\Sigma)$; that is, $E = I$, where I is the identity operator on $L^2(\Sigma)$. In this situation the weighted conditional type operators are simply multiplication operators: $EM_u = M_u$. Therefore our results contain the similar standard results regarding multiplication operators as follows:

(a) M_u is densely defined if and only if $u < \infty$ a.e., with respect to μ,
(b) If $u < \infty$ a.e., μ, then
  (b)$_1$ M_u is self adjoint if and only if u is real valued,
  (b)$_2$ M_u is closed,
  (b)$_3$ $\mathrm{essrange(u)} = \sigma(M_u)$,

etc.

Case 2. If $\mathcal{A} = \{X, \emptyset\}$. Here $\mathcal{A}$-measurable functions are constant on X and

$$E(f) = (\mu(X))^{-1} \int_X f d\mu, \quad f \in L^2(\Sigma).$$

In this case the weighted conditional type operator EM_u acts as

$$EM_u(f) = (\mu(X))^{-1} \int_X uf d\mu, \quad f \in \mathcal{D}(EM_u).$$

Again, we have:

(a) EM_u is densely defined if and only if $E(|u|^2) = (\mu(X))^{-1} \int_X |u|^2 d\mu < \infty$ a.e., μ (equivalently; EM_u is densely defined if and only if u is an L^2 function),
(b) If u is an L^2 function, then
  (b)$_1$ EM_u is normal if and only if $u = (\mu(X))^{-1} \int_X u d\mu$ a.e., μ,
  (b)$_2$ EM_u is self adjoint if and only if $u = (\mu(X))^{-1} \int_X u d\mu$ a.e., μ and $(\mu(X))^{-1} \int_X u d\mu$ is real,
  (b)$_3$ EM_u is closed,
  (b)$_4$ $\sigma(EM_u) = \{(\mu(X))^{-1} \int_X u d\mu\}$,

etc.

Case 3. Suppose that the σ-sub-algebra $\mathcal{A}$ is generated by a countable partition $\{A_n : n \in \mathbb{N}\}$ of X into disjoint sets of finite measure. It is known that the conditional expectation of any $f \in L^2(\Sigma)$ relative to $\mathcal{A}$ is:

$$E(f) = \sum_{n=1}^{\infty} (\mu(A_n))^{-1} \left(\int_{A_n} f d\mu \right) . \chi_{A_n}.$$

In this case the weighted conditional type operator EM_u acts as

$$EM_u(f) = \sum_{n=1}^{\infty} (\mu(A_n))^{-1} \left(\int_{A_n} uf d\mu \right) . \chi_{A_n}, \quad f \in \mathcal{D}(EM_u).$$

Then we have:

(a) EM_u is densely defined if and only if $E(|u|^2) = \sum_{n=1}^{\infty} \beta_n \chi_{A_n} < \infty$ a.e., μ, where $\beta_n = (\mu(A_n))^{-1}(\int_{A_n} |u|^2 d\mu)$,

(b) If $\sum_{n=1}^{\infty} \beta_n \chi_{A_n} < \infty$ a.e., μ, then

(b)$_1$ EM_u is normal if and only if $u = \sum_{n=1}^{\infty} (\mu(A_n))^{-1}(\int_{A_n} ud\mu).\chi_{A_n}$ a.e., μ,

(b)$_2$ EM_u is self adjoint if and only if $u = \sum_{n=1}^{\infty} (\mu(A_n))^{-1}(\int_{A_n} ud\mu).\chi_{A_n}$ a.e., μ and u is real,

(b)$_3$ EM_u is closed,

(b)$_4$ $\sigma(EM_u) = \{\beta_n : n \in \mathbb{N}\}$,

etc.

Example 2 Let $X = [0, 1] \times [0, 1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0, 1] : A$ is a Lebesgue set in $[0, 1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(x, y) = \int_0^1 f(x, t)dt$, which is independent of the second coordinate. Then we have:

(a) EM_u is densely defined if and only if $E(|u|^2)(x, y) = \int_0^1 |u(x, t)|^2 dt < \infty$ a.e., μ,

(b) If $\int_0^1 u(x, t)dt < \infty$ a.e., μ, then

(b)$_1$ EM_u is normal if and only if $u(x, y) = \int_0^1 u(x, t)dt$ a.e., μ,

(b)$_2$ EM_u is self adjoint if and only if $u(x, y) = \int_0^1 u(x, t)dt$ a.e., μ and u is real valued,

(b)$_3$ EM_u is closed,

(b)$_4$ $\sigma(EM_u) = \{\int_0^1 u(x, t)dt : x \in E\}$, where $E = \{0 \leq x \leq 1 : \int_0^1 u(x, t)dt < \infty\}$,

etc.

Example 3 Let $\Omega = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} =< \{(-a, a) : 0 \leq a \leq 1\} >$ (Sigma algebra generated by symmetric intervals). Then

$$E^{\mathcal{A}}(f)(x) = \frac{f(x) + f(-x)}{2}, \quad x \in \Omega,$$

where $E^{\mathcal{A}}(f)$ is defined. If we set $u(x) = e^x$, then $E(|u|^2)(x) = \cosh(2x)$ and so

(a) EM_u is densely defined,
(b) EM_u can not be normal at all, since $e^x \neq \cosh(x)$ for every $x \neq 0$,
(c) EM_u can not be self adjoint at all, since $e^x \neq \cosh(x)$ for every $x \neq 0$,
(d) EM_u is closed,
(e) $\sigma(EM_u) = \{\cosh(x) : -1 \leq x \leq 1\}$,

etc.

Example 4 Let $X = \mathbb{N} \cup \{0\}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{x\}) = \frac{e^{-\theta}\theta^x}{x!}$, for each $x \in X$ and $\theta \geq 0$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B = \{\emptyset, X, \{0\}, X_1 = \{1, 3, 5, 7, 9, \ldots.\}, X_2 = \{2, 4, 6, 8, \ldots.\}, \}$ of $\mathbb{N}$. Note that $\mathcal{A}$ is a sub-σ-finite algebra of Σ and each of element of $\mathcal{A}$ is an $\mathcal{A}$-atom. Thus the conditional expectation of any $f \in \mathcal{D}(E)$ relative to $\mathcal{A}$ is constant on $\mathcal{A}$-atoms. Hence there exists scalars a_1, a_2, a_3 such that

$$E(f) = a_1\chi_{\{0\}} + a_2\chi_{X_1} + a_3\chi_{X_2}.$$

So

$$E(f)(0) = a_1, \quad E(f)(2n-1) = a_2, \quad E(f)(2n) = a_3,$$

for all $n \in \mathbb{N}$. By definition of conditional expectation with respect to $\mathcal{A}$, we have

$$a_1 = f(0), \quad a_2 = \frac{\sum_{n\in\mathbb{N}} f(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}, \quad a_3 = \frac{\sum_{n\in\mathbb{N}} f(2n)\frac{e^{-\theta}\theta^{2n}}{(2n)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n}}{(2n)!}}.$$

For example, if we set $u(x) = x$, then $E(u)$ is a special function as follows;

$$E(u) = \theta\coth(\theta)\chi_{X_1} + \frac{\cosh(\theta) - 1}{\cosh(\theta)}\chi_{X_2}.$$

And so we have:

(a) EM_u is densely defined,
(b) EM_u is normal if and only if EM_u is self adjoint if and only if $\theta = \theta\coth(\theta)$ when $\theta \in X_1$ and $\theta = \frac{\cosh(\theta)-1}{\cosh(\theta)}$ when $\theta \in X_2$.
(c) EM_u is closed,
(d) $\sigma(EM_u) = \{\theta\coth(\theta), \frac{\cosh(\theta)-1}{\cosh(\theta)}\}$,

etc.

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