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Semianalytical structural analysis based on combined application of finite element method and discrete-continual finite element method Part 2: Three-dimensional theory of elasticity

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Abstract

This paper is devoted to semianalytical structural analysis, based on combined application of finite element method (FEM) [1] and discrete-continual finite element method (DCFEM) [2-9]. Boundary problems of three-dimensional theory of elasticity (static analysis of three-dimensional structure [1]) are under consideration, the given domain is embordered by extended one. The field of application of DCFEM comprises structures with regular (constant or piecewise constant) physical and geometrical parameters in some dimension ("basic" dimension). DCFEM presupposes finite element mesh approximation for non-basic dimensions of extended domain while in the basic dimension problem remains continual. Corresponding discrete and discrete-continual approximation models for subdomains and coupled multilevel approximation model for extended domain are under consideration. Brief information about software and verification sample are presented as well.

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Keywords: discrete-continual finite element method; finite element method; semianalytical strctural analysis; three-dimensional theory of elasticity

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1. Formulation of the problem and notation system

Let's consider problem of static analysis of three-dimensional structure loaded by concentrated force with hinged ends (cross-sections) along basic dimension (Fig. 1, 2). Some elements of notation system is presented at Fig. 1, 2.

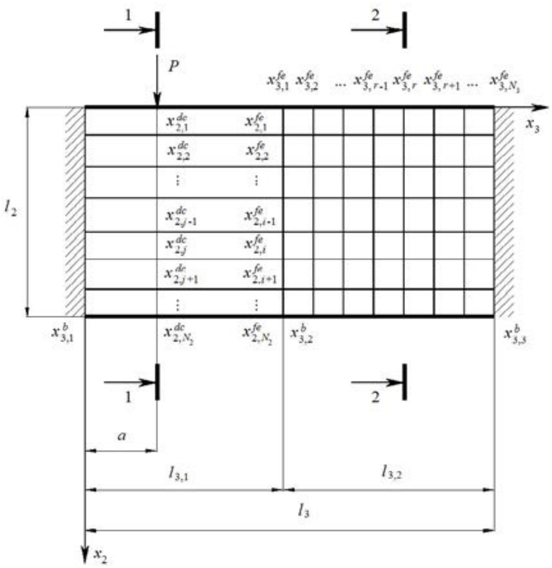

Fig. 1. Considering three-dimensional structure

Let's Ω be domain occupied by structure, $\Omega = \{(x_1, x_2): 0 < x_1 < l_1,\ 0 < x_2 < l_2,\ 0 < x_3 < l_3\}$, where $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_k = \{(x_1, x_2, x_3): 0 < x_1 < l_1,\ 0 < x_2 < l_2,\ x_{3,k}^b < x_3 < x_{3,k+1}^b\},\ k = 1, 2$; x_1, x_2, x_3 are coordinates (x_3 corresponds to basic dimension); $x_{3,1}^b = 0$, $x_{3,2}^b = l_{3,1}$, $x_{3,3}^b = l_{3,1} + l_{3,2} = l_3$ are coordinates of corresponding boundary points (cross-sections) along basic dimension; Ω_1 and Ω_2 are subdomains of Ω ; ω_1 and ω_2 are extended subdomains, embordering subdomains $\Omega_1 \subset \omega_1$ and $\Omega_2 \subset \omega_2$; $\omega = \omega_1 \cup \omega_2$; $x_{1,i,j}^{dc},\ x_{2,i,j}^{dc},\ i = 1, 2, ..., N_1^{dc}$, $j = 1, 2, ..., N_2^{dc}$ are coordinates (along x_1 and x_2) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain ω_1 ; $(N_1^{dc} - 1)$ and $(N_2^{dc} - 1)$ are the numbers of discrete-continual finite elements along coordinates x_1 and x_2 ; $x_{1,i,j,r}^{fe},\ x_{2,i,j,r}^{fe}\ x_{3,i,j,r}^{fe},\ i = 1, 2, ..., N_1^{fe},\ j = 1, 2, ..., N_2^{fe},\ r = 1, 2, ..., N_3^{fe}$ are coordinates (along x_1 , x_2 and x_3) of nodes of finite elements, which are used for approximation of domain ω_2 ; $(N_1^{fe} - 1)$, $(N_2^{fe} - 1)$ and $(N_3^{fe} - 1)$ is the number of finite elements along coordinates x_1 , x_2 and x_3 .

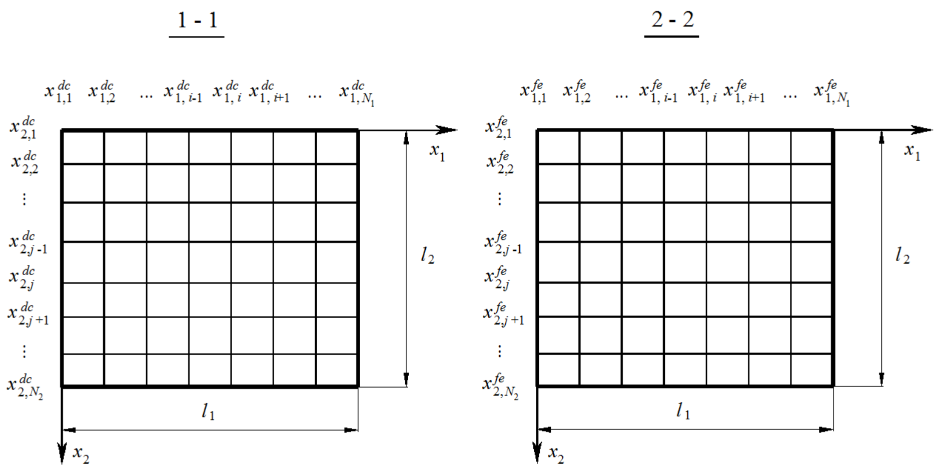

Fig. 2. Cross-sections of considering three-dimensional structure

Let’s consider the case of rectangular mesh approximation of domain ω_2 (Fig. 2):

$$x^{fe}_{1,i,j,r} = x^{fe}_{1,i}, \quad x^{fe}_{21,i,j,r} = x^{fe}_{2,j}, \quad x^{fe}_{31,i,j,r} = x^{fe}_{3,r}, \quad i = 1, 2, ..., N^{fe}_1, \quad j = 1, 2, ..., N^{fe}_2, \quad r = 1, 2, ..., N^{fe}_3 . \tag{1}$$

Three-index notation system is used for numbering of discrete-continual finite elements. Typical number of has the form (k, i, j), where k is the number of subdomain, i and j are numbers of elements (along x_1 and x_2). Four-index system is used for numbering of finite elements. Typical number of has the form (k, i, j, r), where k is the number of subdomain, i, j and k are numbers of elements (along x_1, x_2 and x_3). Let’s $N^{fe}_1 = N^{dc}_1 = N_1$ and $N^{fe}_2 = N^{dc}_2 = N_2$, $x^{dc}_{q,i,j} = x^{fe}_{q,i,j,r}$, $i = 1, 2, ..., N_1$, $j = 1, 2, ..., N_2$, $r = 1, 2, ..., N^{fe}_3$, $q = 1, 2, 3$.

Let’s consider the simples case:

$$x_{q,i,j} = x^{dc}_{q,i,j} = x^{fe}_{q,i,j,r}, \quad i = 1, 2, ..., N_1, \quad j = 1, 2, ..., N_2, \quad r = 1, 2, ..., N^{fe}_3, \quad q = 1, 2, 3; \tag{2}$$

$$x_{1,i} = x_{1,i,j}, \quad i = 1, 2, ..., N_1, \quad j = 1, 2, ..., N_2; \quad x_{2,j} = x_{2,i,j}, \quad i = 1, 2, ..., N_1, \quad j = 1, 2, ..., N_2 . \tag{3}$$

2. Discrete-continual approximation model for subdomain

Discrete-continual approximation model is used for three-dimensional problems. It presupposes mesh approximation for non-basic dimensions of extended domain (along x_1 and x_2) while in the basic dimension (along x_3) problem remains continual. Thus extended subdomain ω_1 is divided into discrete-continual finite elements

$$\omega_1 = \bigcup_{i=1}^{N_1-1} \bigcup_{j=1}^{N_2-1} \omega_{1,i,j} ; \quad \omega_{1,i,j} = \{ (x_1, x_2, x_3) : \; x_{1,i} < x_1 < x_{1,i+1}, \quad x_{2,j} < x_2 < x_{2,j+1}, \quad x^b_{3,1} < x_3 < x^b_{3,2} \} . \tag{4}$$

Lame constants for discrete-continual finite element are defined by formulas:

$$\bar{\lambda}_{1,i,j} = \theta_{1,i,j} \lambda; \quad \bar{\mu}_{1,i,j} = \theta_{1,i,j} \mu , \tag{5}$$

$$\theta_{1,i,j} = \begin{cases} 1, & \omega_{1,i,j} \subset \Omega_1; \\ 0, & \omega_{1,i,j} \not\subset \Omega_1. \end{cases} \quad (6)$$

where $\theta_{1,i,j}$ is the characteristic function of element $\omega_{1,i,j}$.

Basic nodal unknown functions are displacement components $u_1^{(1)}, u_2^{(1)}, u_3^{(1)}$ and their derivatives $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}$ with respect to x_3 (superscript hereinafter corresponds to the number of considered subdomain i.e. ω_1). Thus for node $(1, i, j)$ we have the following unknown functions: $u_1^{(1,i,j)}, u_2^{(1,i,j)}, u_3^{(1,i,j)}$ and $v_1^{(1,i,j)}, v_2^{(1,i,j)}, v_3^{(1,i,j)}$.

Bilinear approximation is used for unknown functions within discrete-continual finite element.

DCFEM is reduced at some stage to the solution of systems of $6N_1N_2$ first-order ordinary differential equations:

$$\overline{U}_1'(x_3) = A_1\overline{U}_1(x_3) + \overline{\tilde{R}}_1(x_3), \quad (7)$$

where $\overline{U}_1(x_2)$ is global vector of nodal unknown functions (subscript corresponds to the number of subdomain ω_1),

$$\overline{U}_1 = \overline{U}_1(x_2) = [\,(\overline{u}_1)^T \quad (\overline{v}_1)^T\,]^T; \quad (8)$$

$$\overline{u}_1 = \overline{u}_1(x_3) = [\,(\overline{u}_n^{(1,1,1)})^T \quad (\overline{u}_n^{(1,2,1)})^T \quad ... \quad (\overline{u}_n^{(1,N_1,1)})^T \quad \dots \quad (\overline{u}_n^{(1,1,2)})^T \quad (\overline{u}_n^{(1,2,2)})^T \quad ... \quad (\overline{u}_n^{(1,N_1,2)})^T \quad \dots$$
$$\dots \quad (\overline{u}_n^{(1,1,N_2)})^T \quad (\overline{u}_n^{(1,2,N_2)})^T \quad ... \quad (\overline{u}_n^{(1,N_1,N_2)})^T\,]^T;$$

(9)

$$\overline{v}_1 = \overline{v}_1(x_3) = [\,(\overline{v}_n^{(1,1,1)})^T \quad (\overline{v}_n^{(1,2,1)})^T \quad ... \quad (\overline{v}_n^{(1,N_1,1)})^T \quad \dots \quad (\overline{v}_n^{(1,1,2)})^T \quad (\overline{v}_n^{(1,2,2)})^T \quad ... \quad (\overline{v}_n^{(1,N_1,2)})^T \quad \dots$$
$$\dots \quad (\overline{v}_n^{(1,1,N_2)})^T \quad (\overline{v}_n^{(1,2,N_2)})^T \quad ... \quad (\overline{v}_n^{(1,N_1,N_2)})^T\,]^T;$$

(10)

$$\overline{u}_n^{(1,i,j)} = \overline{u}_n^{(1,i,j)}(x_3) = [\,u_1^{(1,i,j)} \quad u_2^{(1,i,j)} \quad u_3^{(1,i,j)}\,]^T; \quad \overline{v}_n^{(1,i,j)} = \overline{v}_n^{(1,i,j)}(x_3) = [\,v_1^{(1,i,j)} \quad v_2^{(1,i,j)} \quad v_3^{(1,i,j)}\,]^T; \quad (11)$$

A_1 is global matrix of coefficients of order $6N_1N_2$; $\overline{\tilde{R}}_1(x_2)$ is the right-side vector of order $6N_1N_2$.

Correct analytical solution of (7) is defined by formula

$$\overline{U}_1(x_3) = E_1(x_3)\overline{C}_1 + \overline{S}_1(x_3), \quad (12)$$

where

$$E_1(x_3) = \varepsilon_1(x_3 - x_{3,1}^b) - \varepsilon_1(x_3 - x_{3,2}^b); \quad \overline{S}_1(x_3) = \varepsilon_1(x_3) * \overline{\tilde{R}}_1(x_3); \quad (13)$$

$\varepsilon_1(x_3)$ is the fundamental matrix-function of system (4), which is constructed in the special form convenient for problems of structural mechanics [2]; $*$ is convolution notation; $\overline{C}_1$ is the vector of constants of order $6N_1N_2$.

3. Discrete (finite element) approximation model for subdomain

Discrete (finite element) approximation model for the considering three-dimensional problems presupposes finite element approximation along x_1, x_2 and x_3. Thus extended subdomain ω_2 is divided into finite elements

$$\omega_2 = \bigcup_{i=1}^{N_1-1}\bigcup_{j=1}^{N_2-1}\bigcup_{r=1}^{N_3-1}\omega_{2,i,j,r}; \quad \omega_{2,i,j,r} = \{\,(x_1, x_2):\; x_{1,i} < x_1 < x_{1,i+1}, \;\; x_{2,j} < x_2 < x_{2,j+1}, \;\; x_{3,r}^{fe} < x_3 < x_{3,r+1}^{fe}\,\}. \quad (14)$$

Lame constants for finite element are defined by formulas:

$$\bar{\lambda}_{2,i,j,r} = \theta_{2,i,j,r}\lambda; \quad \bar{\mu}_{2,i,j,r} = \theta_{2,i,j,r}\mu\,; \tag{15}$$

$$\theta_{2,i,j,r} = \begin{cases} 1, & \omega_{2,i,j,r} \subset \Omega_2\,; \\ 0, & \omega_{2,i,j,r} \not\subset \Omega_2\,, \end{cases} \tag{16}$$

where $\theta_{2,i,j,r}$ is the characteristic function of element $\omega_{2,i,j,r}$.

Basic nodal unknowns are displacement components $u_1^{(2)}, u_2^{(2)}, u_3^{(2)}$ (superscript hereinafter corresponds to the number of considered subdomain i.e. ω_2). Thus for node $(2,i,j,r)$ we have the following unknowns: $u_1^{(2,i,j,r)}, u_2^{(2,i,j,r)}, u_3^{(2,i,j,r)}$.

Bilinear approximation of unknowns is used within finite element (conventional three-dimensional parallelepipedic 8-node finite element of three-dimensional problem of elasticity theory).

As known, FEM is reduced to the solution of systems of $3N_1N_2N_3$ linear algebraic equations:

$$K_2\bar{U}_2 = \bar{R}_2\,, \tag{17}$$

where $\bar{U}_2$ is global vector of nodal unknowns (subscript corresponds to the number of subdomain ω_2),

$$\{\bar{U}_2\}_{i_g} = u_q^{(k,i,j,r)}\,; \tag{18}$$

i_g is the global index of element of vector $\bar{U}_2$; k, i, j, r, q are corresponding local indexes,

$$k = 2\,; \quad r = \left[\frac{i_g}{3N_1N_2}\right] + 1\,; \quad j = \left[\frac{i_g - 3(r-1)N_1N_2}{3N_1}\right] + 1\,; \quad i = \left[\frac{i_g - 3(r-1)N_1N_2 - 3(j-1)N_1}{3}\right] + 1\,; \tag{19}$$

$$q = i_g - 3(r-1)N_1N_2 - 3(j-1)N_1 - 3i\,; \tag{20}$$

K_2 is global stiffness matrix of order $3N_1N_2N_3$; $\bar{R}_2$ is global right-side vector of order $3N_1N_2N_3$ (global load vector); $[a]$ is the integral part of a.

4. Multilevel approximation model for domain

System (17) can be rewritten for all nodes with indexes $1 < r < N_3$ (i.e. $x_{3,2}^b < x_3 < x_{3,3}^b$) in the following form (resolving system of $2N_1(N_2-2)$ linear algebraic equations):

$$\tilde{K}_2\bar{U}_2 = \tilde{\bar{R}}_2\,, \tag{21}$$

where $\tilde{K}_2$ is reduced global stiffness matrix of size $[3N_1N_2(N_3-2)]\times[3N_1N_2N_3]$; $\tilde{\bar{R}}_2$ is reduced right-side vector of order $3N_1N_2(N_3-2)$.

Boundary conditions at section $x_3 = x_{3,1}^b$ (hinged edge) has the form ($3N_1N_2$ equations):

$$u_1^{(1,i,j)}(x_{3,1}^b+0) = 0, \quad u_2^{(1,i,j)}(x_{3,1}^b+0) = 0, \quad u_3^{(1,i,j)}(x_{3,1}^b+0) = 0, \quad i = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2\,. \tag{22}$$

Equations (22) can be rewritten in matrix form:

$$B_1^+\overline{U}_1(x_{2,1}^b+0)=\overline{g}_1^+\,, \tag{23}$$

where B_1^+ is matrix of boundary conditions of size $3N_1N_2\times 6N_1N_2$ with elements defined by formula

$$\{B_1^+\}_{p,q}=\delta_{p,q}\,,\quad p=1,2,...,3N_1N_2\,,\quad q=1,2,...,6N_1N_2\,; \tag{24}$$

$\overline{g}_1^+$ is the zero vector of order $3N_1N_2$ (i.e. $\overline{g}_1^+=0$); $\delta_{p,q}$ is Kronecker delta

$$\delta_{p,q}=\begin{cases}1, & p=q\\ 0, & p\neq q.\end{cases} \tag{25}$$

After substitution of (12) into (23) it can be obtained that

$$B_1^+E_1(x_{3,1}^b+0)\overline{C}_1=\overline{g}_1^+-B_1^+\overline{S}_1(x_{3,1}^b+0)\quad\text{or}\quad Q_1\overline{C}_1=\overline{G}_1\,, \tag{26}$$

where Q_1 is the matrix of size $3N_1N_2\times 6N_1N_2$; $\overline{G}_1$ is the vector of order $3N_1N_2$;

$$Q_1=B_1^+E_1(x_{3,1}^b+0)\quad;\qquad \overline{G}_1=\overline{g}_1^+-B_1^+\overline{S}_1(x_{3,1}^b+0)\quad. \tag{27}$$

Boundary conditions at section $x_3=x_{3,2}^b$ (perfect contact) has the form ($6N_1N_2$ equations):

$$u_q^{(1,i,j)}(x_{3,2}^b-0)=u_q^{(2,i,j,r)}\,,\quad i=1,2,...,N_1\,,\quad j=1,2,...,N_2\,,\quad r=1,\quad q=1,2,3\,; \tag{28}$$

$$\sigma_{1,3}^{(1,i,j)}(x_{3,2}^b-0)=\sigma_{1,3}^{(2,i,j,r)}\,,\quad \sigma_{2,3}^{(1,i,j)}(x_{3,2}^b-0)=\sigma_{2,3}^{(2,i,j,r)}\,,\quad \sigma_{3,3}^{(1,i,j)}(x_{3,2}^b-0)=\sigma_{3,3}^{(2,i,j,r)}\,,\\ i=1,2,...,N_1\,,\quad j=1,2,...,N_2\,,\quad r=1; \tag{29}$$

where $\sigma_{1,3}^{(1,i,j)}(x_3)$, $\sigma_{1,3}^{(2,i,j,r)}$ and $\sigma_{3,3}^{(1,i,j)}(x_3)$ are nodal functions (after corresponding averaging) of stress components $\sigma_{1,3}(x_3)$, $\sigma_{2,3}(x_3)$ and $\sigma_{3,3}(x_3)$ for discrete-continual finite element $(1,i,\,j)$; $\sigma_{1,3}^{(2,i,j,r)}$, $\sigma_{2,3}^{(2,i,j,r)}$ and $\sigma_{3,3}^{(2,i,j,r)}$ are nodal stress components $\sigma_{1,3}$, $\sigma_{2,3}$ and $\sigma_{3,3}$ (after corresponding averaging) for finite element $(2,i,\,j,r)$; $r=1$.

Equations (21) and (22) can be rewritten in matrix form:

$$B_2^-\overline{U}_1(x_{3,2}^b-0)=B_2^+\overline{U}_2\,, \tag{30}$$

where B_2^- is matrix of boundary conditions of size $6N_1N_2\times 6N_1N_2$, which can be constructed in accordance with so-called method of basis variations [2-9]; B_2^+ is matrix of boundary conditions of size $6N_1N_2\times 3N_1N_2N_3$, which can be constructed in accordance with method of basis variations [2-9].

After substitution of (12) into (30) it can be obtained that

$$B_2^-E_1(x_{3,2}^b-0)\overline{C}_1-B_2^+\overline{U}_2=-B_2^-\overline{S}_1(x_{3,2}^b-0)\quad\text{or}\quad Q_{2,1}\overline{C}_1+Q_{2,2}\overline{U}_2=\overline{G}_2\,, \tag{31}$$

where $Q_{2,1}$ is the matrix of size $6N_1N_2\times 6N_1N_2$; $Q_{2,2}$ is the matrix of size $6N_1N_2\times 3N_1N_2N_3$; $\overline{G}_2$ is the vector of order $6N_1N_2$,

$$Q_{2,1} = B_2^- E_1(x_{3,2}^b - 0)\ ;\quad Q_{2,2} = -B_2^+\ ;\quad \overline{G}_2 = -B_2^- \overline{S}_1(x_{3,2}^b - 0)\ . \tag{32}$$

Boundary conditions at section $x_3 = x_{3,3}^b$ (hinged edge) has the form ($3N_1N_2$ equations):

$$u_1^{(2,i,j,r)} = 0,\ \ u_2^{(2,i,j,r)} = 0,\ \ u_3^{(2,i,j,r)} = 0,\ \ i = 1, 2, ..., N_1,\ \ j = 1, 2, ..., N_2,\ \ r = N_3\ . \tag{33}$$

Equations (33) can be rewritten in matrix form:

$$B_3^- \overline{U}_2 = \overline{g}_3^-\ , \tag{34}$$

where B_3^- is matrix of boundary conditions of size $3N_1N_2 \times 3N_1N_2N_3$ with elements defined by formula

$$\{B_3^-\}_{p,q} = \delta_{p,q},\ \ p = 1, 2, ..., 3N_1N_2,\ \ q = 1, 2, ..., 3N_1N_2N_3\ ; \tag{35}$$

$\overline{g}_3^-$ is the zero vector of order $3N_1N_2$ (i.e. $\overline{g}_3^- = 0$).

Thus, the total number of equation is equal to $3N_1N_2N_3 + 6N_1N_2$. Corresponding coupled system of $3N_1N_2N_3 + 6N_1N_2$ linear algebraic equations with $3N_1N_2N_3 + 6N_1N_2$ unknowns has the form:

$$\begin{bmatrix} Q_1 & 0 \\ Q_{2,1} & Q_{2,2} \\ 0 & \tilde{\tilde{K}}_2 \\ 0 & B_3^- \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \overline{\tilde{\tilde{R}}}_2 \\ \overline{g}_3^- \end{bmatrix}. \tag{36}$$

It should be noted that boundary conditions (34) can be taken into account automatically within construction of global stiffness matrix and global right-side vector corresponding to subdomain ω_2 . Then we get (instead of (36)):

$$\begin{bmatrix} Q_1 & 0 \\ Q_{2,1} & Q_{2,2} \\ 0 & \tilde{\tilde{K}}_2 \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \overline{\tilde{\tilde{R}}}_2 \end{bmatrix}, \tag{37}$$

where $\tilde{\tilde{K}}_2$ is corresponding reduced global stiffness matrix of size $[3N_1N_2(N_3 - 1)] \times [3N_1N_2N_3]$; $\overline{\tilde{\tilde{R}}}_2$ is corresponding reduced global right-side vector of order $3N_1N_2(N_3 - 1)$.

Strain and stress components are computed according to well-known formulas after solving of system (37).

5. Software and verification samples

We should stress that all methods and algorithms considered in this paper have been realized in software. The main purpose of Analysis system CSASA3D (DCFEM + FEM) is semianalytical structural analysis (static analysis of three-dimensional structures within three-dimensional theory of elasticity), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio 2013 Community and Intel Parallel Studio 2015XE [10] with Intel MKL Library. Software is designed for Microsoft Windows 8.1/10.

Corresponding verification samples (ANSYS Mechanical 15.0 [6,7] was used for verification purposes) proved that DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of the so-called edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can't be adequately taken into account by the standard FEM [1].

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