

## Uniform Bounds for Black–Scholes Implied Volatility\*

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**Abstract.** In this note, Black–Scholes implied volatility is expressed in terms of various optimization problems. From these representations, upper and lower bounds are derived which hold uniformly across moneyness and call price. Various symmetries of the Black–Scholes formula are exploited to derive new bounds from old. These bounds are used to reprove asymptotic formulas for implied volatility at extreme strikes and/or maturities.

**Key words.** implied volatility, put-call symmetry, asymptotic formulas

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**1. Introduction.** We define the Black–Scholes call price function  $C_{\text{BS}} : \mathbb{R} \times [0, \infty) \rightarrow [0, 1)$  by the formula

$$\begin{aligned} C_{\text{BS}}(k, y) &= \int_{-\infty}^{\infty} (e^{yz-y^2/2} - e^k)^+ \phi(z) dz \\ &= \begin{cases} \Phi\left(-\frac{k}{y} + \frac{y}{2}\right) - e^k \Phi\left(-\frac{k}{y} - \frac{y}{2}\right) & \text{if } y > 0, \\ (1 - e^k)^+ & \text{if } y = 0, \end{cases} \end{aligned}$$

where  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$  is the standard normal density and  $\Phi(x) = \int_{-\infty}^x \phi(z) dz$  is its distribution function. As is well known, the financial significance of the function  $C_{\text{BS}}$  is that, within the context of the Black–Scholes model [4], the minimal replication cost of a European call option with strike  $K$  and maturity  $T$  written on a stock with initial price  $S_0$  is given by

$$\text{replication cost} = S_0 e^{-\delta T} C_{\text{BS}} \left[ \log \left( \frac{K e^{-rT}}{S_0 e^{-\delta T}} \right), \sigma \sqrt{T} \right],$$

where  $\delta$  is the dividend rate,  $r$  is the interest rate, and  $\sigma$  is the volatility of the stock. Therefore, in the definition of  $C_{\text{BS}}(k, y)$ , the first argument  $k$  plays the role of log-moneyness of the option and the second argument  $y$  is the total standard deviation of the terminal log stock price.

Of the six parameters appearing in the Black–Scholes formula for the replication cost, five are readily observed in the market. Indeed, the strike  $K$  and maturity date  $T$  are specified by the option contract, and the initial stock price  $S_0$  is quoted. The interest rate is the yield of a zero-coupon bond  $B_{0,T}$  with maturity  $T$  and unit face value, and can be computed from

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the initial bond price  $B_{0,T} = e^{-rT}$ . Similarly, the dividend rate can be computed from the stock's initial time- $T$  forward price  $F_{0,T} = S_0 e^{(r-\delta)T}$ .

As suggested by Latané and Rendleman [17] in 1976, the remaining parameter, the volatility  $\sigma$ , can also be inferred from the market, assuming that the call has a quoted price  $C_{0,T,K}$ . Indeed, note that for fixed  $k$ , the map  $C_{BS}(k, \cdot)$  is strictly increasing and continuous, so we can define the inverse function

$$Y_{BS}(k, \cdot) : [(1 - e^k)^+, 1) \rightarrow [0, \infty)$$

by

$$y = Y_{BS}(k, c) \Leftrightarrow C_{BS}(k, y) = c.$$

The implied volatility of the call option is then defined to be

$$\sigma^{\text{implied}} = \frac{1}{\sqrt{T}} Y_{BS} \left[ \log \left( \frac{K e^{-rT}}{S_0 e^{-\delta T}} \right), \frac{C_{0,T,K}}{S_0 e^{-\delta T}} \right].$$

Because of its financial significance, the function  $Y_{BS}$  has been the subject of much interest. For instance, approximations for  $Y_{BS}$  can be found in several papers [5, 7, 19, 22]. Unfortunately, there seems to be only one case where the function  $Y_{BS}$  can be expressed explicitly in terms of elementary functions: when  $k = 0$  we have

$$\begin{aligned} C_{BS}(0, y) &= 2 \Phi \left( \frac{y}{2} \right) - 1 \\ &= 1 - 2 \Phi \left( -\frac{y}{2} \right) \end{aligned}$$

and, hence,

$$\begin{aligned} Y_{BS}(0, c) &= 2 \Phi^{-1} \left( \frac{1+c}{2} \right) \\ &= -2 \Phi^{-1} \left( \frac{1-c}{2} \right). \end{aligned}$$

The main contribution of this article is to provide bounds on the quantity  $Y_{BS}(k, c)$  in terms of elementary functions of  $(k, c)$ . As an example, in Proposition 4.3 below we will see that

$$(1) \quad Y_{BS}(k, c) \leq -2\Phi^{-1} \left( \frac{1-c}{1+e^k} \right)$$

for every  $(k, c)$  such that  $(1 - e^k)^+ \leq c < 1$ .

We list here two possible applications of such bounds. When  $k \neq 0$ , the function  $Y_{BS}$  can be evaluated numerically. A simple way to do so is to implement the bisection method for finding the root of the map  $y \mapsto C_{BS}(k, y) - c$ . That is to say, for fixed  $(k, c)$  pick two points  $\ell < u$  such that  $C_{BS}(k, \ell) < c$  and  $C_{BS}(k, u) > c$ . By the intermediate value theorem, we know that the root is in the interval  $(\ell, u)$ . We then let  $m = \frac{1}{2}(\ell + u)$  be the midpoint. If  $C_{BS}(k, m) > c$  we know that the root  $Y_{BS}(k, c)$  is in the interval  $(\ell, m)$ , in which case

we relabel  $m$  as  $u$ . Similarly, if  $C_{\text{BS}}(k, m) < c$  we relabel  $m$  as  $\ell$ . This process is repeated until  $|C_{\text{BS}}(k, m) - c| < \varepsilon$ , where  $\varepsilon > 0$  is a given tolerance level whereupon we declare  $Y_{\text{BS}}(k, c) \approx m$ . (We note here that a more sophisticated idea is to apply the Newton–Raphson method as suggested by Manaster and Koehler [21] in 1982. We will return to this idea in section 3.)

In order to implement the bisection method, we need a lower bound  $\ell$  and upper bound  $u$  to initialize the algorithm. However, aside from the obvious lower bound  $\ell = 0$ , there do not seem to be many well-known explicit upper and lower bounds on the quantity  $Y_{\text{BS}}(k, c)$  which hold *uniformly* in  $(k, c)$ . This note provides such bounds and, indeed, (1) is an example.

We now consider another application of our bounds. Consider a market model with a zero-coupon bond with maturity date  $T$  whose time- $t$  price is  $B_{t,T}$  and a stock with time  $t$  price  $S_t$ . Suppose the initial price of a call option with strike  $K$  and maturity  $T$  is given by

$$C_{0,T,K} = B_{0,T} \mathbb{E}^T[(S_T - K)^+],$$

where the expectation is under a fixed  $T$ -forward measure. Further, suppose the stock's initial time- $T$  forward price is given by

$$F_{0,T} = \mathbb{E}^T[S_T].$$

(If the stock pays no dividend, static arbitrage considerations would imply  $F_{0,T} = S_0/B_{0,T}$ . We do not need this formula here so the stock is allowed to pay dividends in the present discussion; however, we will return to this point in Remark 4.13 below.) Now, (1) implies that the implied volatility is bounded by

$$(2) \quad \begin{aligned} \sigma^{\text{implied}} &= \frac{1}{\sqrt{T}} Y_{\text{BS}} \left[ \log \left( \frac{K}{F_{0,T}} \right), \frac{C_{0,T,K}}{F_{0,T} B_{0,T}} \right] \\ &\leq -\frac{2}{\sqrt{T}} \Phi^{-1} \left( \frac{\mathbb{E}^T[S_T \wedge K]}{\mathbb{E}^T[S_T] + K} \right). \end{aligned}$$

Note that the above bound is the composition of two ingredients: the model-*dependent* formulas for the quantities  $C_{0,T,K}$  and  $F_{0,T}$ , and a uniform and model-*independent* bound on the function  $Y_{\text{BS}}$ .

There has been much recent interest in implied volatility asymptotics. See, for instance, the papers [2, 3, 6, 9, 12, 13, 14, 15, 18, 23, 26] for asymptotic formulas which depend on minimal model data, such as the distribution function or the moment generating function of the returns of the underlying stock. Paralleling the discussion above, such asymptotic formulas can be seen as compositions of two limits: first, the asymptotic shape of the call surface as predicted by the model at, for instance, extreme strikes and/or maturities; and second, asymptotics of the model-independent function  $Y_{\text{BS}}$ . The uniform bounds on  $Y_{\text{BS}}$  that are presented in this note are used to provide short, new proofs of these second model-independent asymptotic formulas.

In their long survey article, Andersen and Lipton [1] warn that many of the asymptotic implied volatility formulas that have appeared in recent years may not be applicable in practice, since typical market parameters are usually not in the range of validity of any of the proposed asymptotic regimes. Our new bounds on the function  $Y_{\text{BS}}$  are uniform, and hence sidestep the critique of Andersen and Lipton.

The rest of the note is organized as follows. In section 2 we discuss various symmetries of the Black–Scholes pricing function  $C_{BS}$ . These symmetries will be used repeatedly throughout the remainder of the note. In section 3 the Black–Scholes implied total standard deviation function  $Y_{BS}$  is represented as the value function of several optimization problems. These results constitute the main contribution of this note since they allow  $Y_{BS}$  to be bounded arbitrarily well from above and below by choosing suitable controls to insert into the respective objective functions. In section 4 these bounds are used to reprove some known asymptotic formulas. As a by-product, we derive formulas which have the same asymptotic behavior as the known formulas, but are guaranteed to bound  $Y_{BS}$  either from above or below.

**2. Put-call and close-far symmetries.** The Black–Scholes call price function  $C_{BS}$  contains a certain amount of symmetry. In order to streamline the presentation of our bounds, we begin with an exploration of two of these symmetries.

To treat the two cases  $k \geq 0$  and  $k < 0$  as efficiently as possible, we begin with an observation. Suppose  $c$  is the normalized price of a call option with log-moneyness  $k$ . Then by the usual put-call parity formula, the corresponding normalized price of a put option with the same log-moneyness is

$$p = c + e^k - 1.$$

Now if  $c = C_{BS}(k, y)$  is for some  $y > 0$ , then we have

$$\begin{aligned} p &= C_{BS}(k, y) + e^k - 1 \\ &= e^k \Phi\left(\frac{k}{y} + \frac{y}{2}\right) - \Phi\left(\frac{k}{y} - \frac{y}{2}\right) \\ &= e^k C_{BS}(-k, y). \end{aligned}$$

The above calculation is the well-known Black–Scholes put-call symmetry formula. We have just proven the following result.

**Proposition 2.1.** *For any  $k \in \mathbb{R}$  and  $c \in [(1 - e^k)^+, 1)$  we have*

$$Y_{BS}(k, c) = Y_{BS}(-k, e^{-k}c + 1 - e^{-k}).$$

One conclusion of Proposition 2.1 is that it is sufficient to study the function  $Y_{BS}(k, \cdot)$  only in the case  $k \geq 0$ . Indeed, to study the case  $k < 0$  one simply applies the above put-call symmetry formula.

We now come to another, less well-known, symmetry of the Black–Scholes formula. While put-call symmetry involves replacing the log-moneyness  $k$  with  $-k$ , the symmetry discussed here involves replacing the total standard deviation  $y$  with  $2|k|/y$ . By put-call symmetry, we can confine our discussion to the case  $k > 0$ .

**Proposition 2.2.** *For all  $k > 0$  and  $0 < c < 1$ , let*

$$\hat{C}(k, c) = 1 - \int_0^c \frac{2k}{[Y_{BS}(k, u)]^2} du.$$

*Then  $\hat{C}(k, c) > 0$  and we have*

$$Y_{BS}(k, c) = \frac{2k}{Y_{BS}(k, \hat{C}(k, c))}.$$

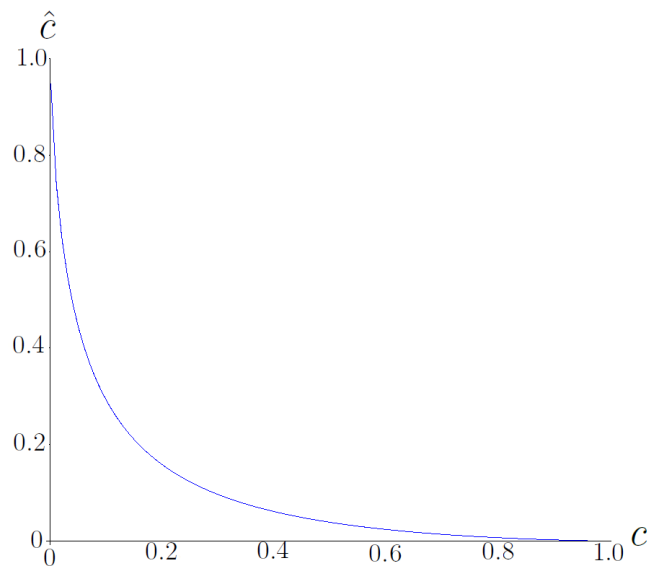


Figure 1. The function  $\hat{C}(k, \cdot)$ .

Figure 1 shows the graph of  $c \mapsto \hat{C}(k, c)$  when  $k = 0.2$ .

*Proof.* We must prove that

$$\hat{C}(k, c) = C_{\text{BS}}\left(k, \frac{2k}{Y_{\text{BS}}(k, c)}\right)$$

or, equivalently,

$$\hat{C}\left(k, C_{\text{BS}}(k, y)\right) = C_{\text{BS}}\left(k, \frac{2k}{y}\right).$$

The above identity can be verified by differentiating both sides with respect to  $y$ , and using the Black–Scholes vega formula: for  $k > 0$ , we have

$$(3) \quad C_{\text{BS}}(k, y) = \int_0^y \phi(-k/x + x/2) dx. \quad \blacksquare$$

*Remark 2.3.* Fix  $k > 0$  and  $y > 0$ , and let  $c = C_{\text{BS}}(k, y)$ . Note that  $c \approx 0$  when  $y$  is very small, and indeed it is a straightforward exercise to verify (see section 4) that

$$\log c = -\frac{k^2}{2y^2} + O(\log y) \text{ as } y \downarrow 0.$$

On the other hand, we have  $c \approx 1$  when  $y$  is very large and, furthermore,

$$\log(1 - c) = -\frac{y^2}{8} + O(\log y) \text{ as } y \uparrow \infty.$$

Now, let  $\hat{c} = C_{\text{BS}}(k, 2k/y)$  so that by Proposition 2.2 we have  $\hat{c} = \hat{C}(k, c)$ . From the above calculations we have

$$\log(1 - \hat{c}) = -\frac{k^2}{2y^2} + O(\log y) \text{ as } y \downarrow 0$$

and

$$\log \hat{c} = -\frac{y^2}{8} + O(\log y) \text{ as } y \uparrow \infty.$$

In the context of the Black–Scholes model, the quantity  $y$  has the interpretation as the total standard deviation  $y = \sigma\sqrt{T}$ , where  $\sigma$  is the volatility and  $T$  is the maturity date of the option. Proposition 2.2 then is a symmetry relation between the prices of short-dated and long-dated options.

We conclude this section with some easy observations which we will use later.

**Proposition 2.4.** *For all  $k > 0$ , the function  $\hat{C}(k, \cdot)$  is convex and satisfies the functional equation*

$$\hat{C}(k, \hat{C}(k, c)) = c$$

holds for all  $0 < c < 1$ .

*Proof.* It is easy to see that  $Y_{\text{BS}}(k, \cdot)$  is strictly increasing. That  $\hat{C}(k, \cdot)$  is convex follows from the fact that its gradient  $-2k/Y_{\text{BS}}(k, \cdot)^2$  is increasing.

The functional equation is proven by noting

$$Y_{\text{BS}}(k, c) = \frac{2k}{Y_{\text{BS}}(k, \hat{C}(k, c))} = Y_{\text{BS}}(k, \hat{C}(k, \hat{C}(k, c)))$$

and using the fact that  $Y_{\text{BS}}(k, \cdot)$  is strictly increasing. ■

**Proposition 2.5.** *For  $k > 0$ , let*

$$J(k, c) = \int_0^c \frac{1}{Y_{\text{BS}}(k, u)} du.$$

Then

$$J(k, c) + J(k, \hat{c}) = J(k, 1),$$

where  $\hat{c} = \hat{C}(k, c)$ .

*Proof.* By setting  $c = C_{\text{BS}}(k, y)$  and hence  $\hat{c} = C_{\text{BS}}(k, 2k/y)$ , the identity can be proven by computing the derivative with respect to  $y$  of the left-hand side, and note that it vanishes identically. ■

**Remark 2.6.** By changing variables, we have the identities

$$\begin{aligned} J(k, 1) &= \int_0^\infty \frac{\phi(-k/y + y/2)}{y} dy \\ &= \int_{-\infty}^\infty \frac{\phi(x)}{\sqrt{x^2 + 2k}} dx \\ &= \frac{e^{k/2}}{\sqrt{2\pi}} K_0(k/2), \end{aligned}$$

where  $K_0$  is a modified Bessel function; see [11].

**3. Various optimization problems.** This section contains one of the main results of this note, formulas for the function  $Y_{BS}$  in terms of various optimization problems. The first result is that  $Y_{BS}(k, c)$  can be calculated by solving a minimization problem. In particular, we can use this formula to find an upper bound simply by evaluating the objective function at a feasible control.

**Theorem 3.1.** *For all  $k \in \mathbb{R}$  and  $(1 - e^k)^+ \leq c < 1$  we have*

$$\begin{aligned} Y_{BS}(k, c) &= \inf_{d_1 \in \mathbb{R}} [d_1 - \Phi^{-1}(e^{-k}(\Phi(d_1) - c))] \\ &= \inf_{d_2 \in \mathbb{R}} [\Phi^{-1}(c + e^k \Phi(d_2)) - d_2]. \end{aligned}$$

Furthermore, if  $c > (1 - e^k)^+$ , then the two infima are attained at

$$\begin{aligned} d_1^* &= -\frac{k}{y} + \frac{y}{2}, \\ d_2^* &= -\frac{k}{y} - \frac{y}{2}, \end{aligned}$$

where  $y = Y_{BS}(k, c)$ .

**Remark 3.2.** We are using the convention that  $\Phi^{-1}(u) = +\infty$  for  $u \geq 1$  and  $\Phi^{-1}(u) = -\infty$  for  $u \leq 0$ .

The following proof is due to De Smet [25], simplifying the proof in an early version of this paper. The idea is essentially that the inequality

$$(X - K)^+ \geq (X - K)\mathbf{1}_{[H, \infty)}(X)$$

holds for all  $X, K, H \geq 0$  with equality if and only if  $H = K$ .

**Proof.** Fix  $k \in \mathbb{R}$  and  $(1 - e^k)^+ \leq c < 1$  and let  $y \geq 0$  be such that  $C_{BS}(k, y) = c$ . Note that for any  $d_2 \in \mathbb{R}$  we have

$$\begin{aligned} c &= \int_{-\infty}^{\infty} (e^{yz-y^2/2} - e^k)^+ \phi(z) dz \\ &\geq \int_{-d_2}^{\infty} (e^{yz-y^2/2} - e^k)^+ \phi(z) dz \\ &\geq \int_{-d_2}^{\infty} (e^{yz-y^2/2} - e^k) \phi(z) dz \\ &= \Phi(d_2 + y) - e^k \Phi(d_2). \end{aligned}$$

There is equality from the first to the second line only if  $-d_2 \leq k/y + y/2$ , and there is equality from the second to the third line only if  $-d_2 \geq k/y + y/2$ . Rearranging then yields

$$y \leq \Phi^{-1}(c + e^k \Phi(d_2)) - d_2.$$

Let  $d_2 = \Phi^{-1}(e^{-k}(\Phi(d_1) - c))$  in the above inequality to obtain the first expression. ■

Let

$$(4) \quad H_1(d; k, c) = d - \Phi^{-1}(e^{-k}(\Phi(d) - c))$$

and

$$(5) \quad H_2(d; k, c) = \Phi^{-1}(c + e^k \Phi(d)) - d,$$

and note that

$$H_1(d; k, c) = H_2(-d; -k, e^{-k}c + 1 - e^{-k})$$

in line with put-call symmetry. We use this notation to compute  $Y_{\text{BS}}(k, c)$  in terms of a maximization problem. This representation can be used, in principle, to find lower bounds.

**Theorem 3.3.** *Let  $\mathcal{C}$  be the space of continuous functions on  $[0, 1]$ . For  $k > 0$  and  $0 < c < 1$ , we have*

$$Y_{\text{BS}}(k, c) = \sup_{D \in \mathcal{C}, d \in \mathbb{R}} \frac{2k}{H_i(d; k, 1 - \int_0^c \frac{2k}{H_j(D(u); k, u)^2} du)}$$

for any  $i, j \in \{1, 2\}$ .

*Proof.* By Theorem 3.1 we have  $Y_{\text{BS}}(k, u) \leq H_j(d; k, u)$  for all  $d$ , and since  $Y_{\text{BS}}(k, \cdot)$  is increasing, we have for any  $D \in \mathcal{C}$  that

$$\begin{aligned} Y_{\text{BS}}(k, \hat{C}(k, c)) &= Y_{\text{BS}}\left(k, 1 - \int_0^c \frac{2k}{Y_{\text{BS}}(k, u)^2} du\right) \\ &\leq Y_{\text{BS}}\left(k, 1 - \int_0^c \frac{2k}{H_j(D(u); k, u)^2} du\right) \\ &\leq H_i\left(d; k, 1 - \int_0^c \frac{2k}{H_j(D(u); k, u)^2} du\right). \end{aligned}$$

The conclusion follows from Proposition 2.2. ■

In light of Proposition 2.2 we now give a representation of  $\hat{C}$  in terms of a minimization problem. We restrict attention to  $k > 0$  with no real loss thanks to put-call symmetry.

**Proposition 3.4.** *For  $k > 0$  and  $0 < c < 1$  we have*

$$\hat{C}(k, c) = \sup_{y \geq 0} \left[ C_{\text{BS}}\left(k, \frac{2k}{y}\right) - \frac{2k}{y^2}(c - C_{\text{BS}}(k, y)) \right].$$

*Proof.* Recall by Proposition 2.4 that  $\hat{C}(k, \cdot)$  is convex. Hence,

$$\hat{C}(k, c) - \hat{C}(k, c^*) \geq -\frac{2k}{Y_{\text{BS}}(k, c^*)^2}(c - c^*)$$

for any  $c, c^* \in (0, 1)$ . Letting  $y = Y_{\text{BS}}(k, c^*)$  we have

$$\hat{C}(k, c) \geq C_{\text{BS}}\left(k, \frac{2k}{y}\right) - \frac{2k}{y^2}(c - C_{\text{BS}}(k, y))$$

as claimed. ■



Of course, there are other representations of  $Y_{BS}$  in terms of an optimization problem. For instance, we have

$$Y_{BS}(k, c) = \inf\{y \geq 0 : C_{BS}(k, y) \geq c\} \\ = \sup\{y \geq 0 : C_{BS}(k, y) \leq c\}.$$

Indeed, this simple observation underlies the bisection method discussed in the introduction.

We conclude this section with a slightly more interesting representation. It can be used to find upper and lower bounds of  $Y_{BS}(k, c)$ , at least in principle. However, in practice it is not clear how to choose candidate controls, so we do not explore this idea in what follows. This result is due to Manaster and Koehler [21], and is motivated by the Newton–Raphson method for computing implied volatility numerically.

**Proposition 3.5 (Manaster and Koehler).** *Fix  $k \geq 0$  and  $0 \leq c < 1$ . If  $c \leq C_{BS}(k, \sqrt{2k}) = 1/2 - e^k \Phi(-\sqrt{2k})$  then*

$$Y_{BS}(k, c) = \inf_{0 \leq y \leq \sqrt{2k}} \left[ y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)} \right].$$

*Otherwise, if  $c \geq 1/2 - e^k \Phi(-\sqrt{2k})$  then*

$$Y_{BS}(k, c) = \sup_{y \geq \sqrt{2k}} \left[ y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)} \right].$$

*Proof.* The restriction of  $C_{BS}(k, \cdot)$  to  $[0, \sqrt{2k}]$  is convex, as can be confirmed by differentiation. Hence, by the Black–Scholes vega formula, we have

$$C_{BS}(k, y^*) - C_{BS}(k, y) \geq \phi(-k/y + k/y)(y^* - y)$$

for any  $y, y^* \in [0, \sqrt{2k}]$ . Fixing  $y^*$  and letting  $c = C_{BS}(k, y^*)$  we have proven

$$y^* \leq y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)}$$

as desired. Similarly, since the restriction of  $C_{BS}(k, \cdot)$  to  $[\sqrt{2k}, \infty)$  is concave the second conclusion follows. ■

**4. Uniform bounds and asymptotics.** In this section, we will offer quick proofs of some asymptotic formulas for the function  $Y_{BS}$ . These formulas already appear in the literature, but the important novelty here is that we will derive bounds on the function  $Y_{BS}$  which hold uniformly, not just asymptotically. To obtain upper bounds in most cases, we simply choose a convenient  $d_1$  or  $d_2$  to plug into Theorem 3.1. Note that the proposed upper bound is close to the true value of  $Y_{BS}(c, k)$  when, for instance, the proposed value of  $d_1$  is close to the minimizer  $d_1^* = -k/y + y/2$ . In principle, lower bounds could be found by choosing convenient controls in Theorem 3.3. However, in practice, we have found other arguments which, while lacking the same unifying principle, do have the advantage of being simple. In the proofs that follow, we usually only consider the  $k \geq 0$  case, as the  $k < 0$  case follows directly from Proposition 2.1.

Before we begin, we need a lemma regarding the asymptotic behavior of the standard normal quantile function  $\Phi^{-1}$ .

**Lemma 4.1.** *As  $\varepsilon \downarrow 0$  we have*

$$[\Phi^{-1}(\varepsilon)]^2 = -2 \log \varepsilon + O(\log(-\log \varepsilon)).$$

*In particular, we have*

$$\Phi^{-1}(\varepsilon) = -\sqrt{-2 \log \varepsilon} + O\left(\frac{\log(-\log \varepsilon)}{\sqrt{-\log \varepsilon}}\right).$$

*Proof.* Let  $\varepsilon = \Phi(-x)$  for large  $x > 0$  and let

$$R(x) = \frac{\Phi(-x)x}{\phi(x)}.$$

In this notation we have the identity

$$\log \Phi(-x) = -x^2/2 - \log(\sqrt{2\pi}x) + \log R(x).$$

Since it is well known that  $R(x) \rightarrow 1$  as  $x \rightarrow \infty$  we have

$$\frac{\log \Phi(-x)}{x^2} \rightarrow -1/2$$

or, equivalently,

$$\frac{[\Phi^{-1}(\varepsilon)]^2}{\log \varepsilon} \rightarrow -2.$$

Plugging this limit into the identity yields the first conclusion, and Taylor's theorem yields the second. ■

The first example comes from [26]. This asymptotic formula considers the behavior of  $Y_{BS}$  when  $c$  is close to its upper bound of 1. This result is useful in studying implied volatility at very long maturities, when the strike is fixed.

**Theorem 4.2.** *For fixed  $k \in \mathbb{R}$ , we have*

$$Y_{BS}(k, c) = \sqrt{-8 \log(1-c)} + O\left(\frac{\log[-\log(1-c)]}{\sqrt{-\log(1-c)}}\right)$$

as  $c \uparrow 1$ .

The proof of the above theorem relies on the following simple bounds which hold uniformly in  $(c, k)$ .

**Proposition 4.3.** *Fix  $k \in \mathbb{R}$  and  $(1 - e^k)^+ \leq c < 1$ . For  $k \geq 0$  we have*

$$-2\Phi^{-1}\left(\frac{1-c}{2}\right) \leq Y_{BS}(k, c) \leq -2\Phi^{-1}\left(\frac{1-c}{1+e^k}\right)$$

and for  $k < 0$  we have

$$-2\Phi^{-1}\left(\frac{1-c}{2e^k}\right) \leq Y_{BS}(k, c) \leq -2\Phi^{-1}\left(\frac{1-c}{1+e^k}\right).$$

*Proof.* For the upper bound, let  $d_2 = \Phi^{-1}\left(\frac{1-c}{1+e^k}\right)$  in Theorem 3.1.

For the lower bound, let  $y = Y_{BS}(k, c)$ . Note that  $C_{BS}(\cdot, y)$  is decreasing and hence

$$\begin{aligned} 1 - 2\Phi(-y/2) &= C_{BS}(0, y) \\ &\geq C_{BS}(k, y) \\ &= c \end{aligned}$$

when  $k \geq 0$ . In the case when  $k < 0$ , note that

$$\frac{1 - e^{-k}p}{1 + e^{-k}} = \frac{1 - c}{1 + e^k}$$

and that

$$\frac{1 - e^{-k}p}{2} = \frac{1 - c}{2e^k}.$$

Now appeal to the put-call parity formula of Proposition 2.1. ■

*Proof of Theorem 4.2.* By Proposition 4.3 and Lemma 4.1, we have

$$\begin{aligned} Y_{BS}(k, c) &\leq -2\Phi^{-1}\left(\frac{1-c}{1+e^k}\right) \\ &= \sqrt{-8\log(1-c)} + O\left(\frac{\log[-\log(1-c)]}{\sqrt{-\log(1-c)}}\right), \end{aligned}$$

where we have used the fact that for fixed  $k$  we have

$$\sqrt{-2\log\left(\frac{1-c}{1+e^k}\right)} = \sqrt{-2\log(1-c)} + O\left(\frac{1}{\sqrt{-\log(1-c)}}\right)$$

as  $c \uparrow 1$ .

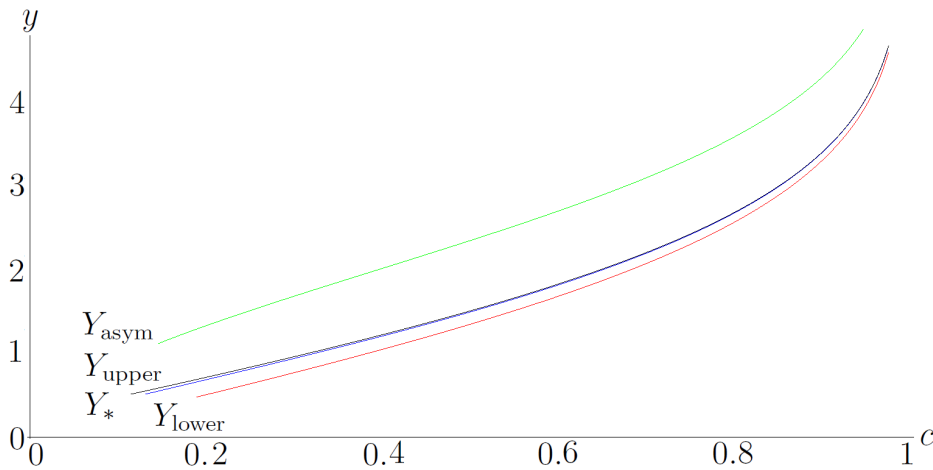
Similarly, by Proposition 4.3, we have for  $k \geq 0$  that

$$\begin{aligned} Y_{BS}(k, c) &\geq -2\Phi^{-1}\left(\frac{1-c}{2}\right) \\ &= \sqrt{-8\log(1-c)} + O\left(\frac{\log[-\log(1-c)]}{\sqrt{-\log(1-c)}}\right). \end{aligned}$$

The  $k < 0$  is identical. ■

Figure 2 illustrates the behavior of  $Y_{BS}(k, c)$  as  $c \uparrow 1$ , compared with the uniform upper and lower bounds of Proposition 4.3 and the asymptotic formula in Theorem 4.2. We fixed the log-moneyness  $k = 0.2$  and plotted four functions:

1.  $Y_{\text{upper}}(c) = -2\Phi^{-1}\left(\frac{1-c}{1+e^k}\right)$  is the upper bound from Proposition 4.3;
2.  $Y_*(c) = Y_{BS}(k, c)$  is the true function of our interest;
3.  $Y_{\text{lower}}(c) = -2\Phi^{-1}\left(\frac{1-c}{2}\right)$  is the lower bound from Proposition 4.3;
4.  $Y_{\text{asym}}(c) = \sqrt{-8\log(1-c)}$  is the asymptotic shape from Theorem 4.2.



**Figure 2.** Bounds and asymptotics of  $Y_{BS}(k, \cdot)$  as  $c \uparrow 1$ .

Note that  $Y_{upper} \geq Y_* \geq Y_{lower}$  as predicted. Also, it is interesting to see that  $Y_{lower}$  is a remarkably good approximation over a large range of  $c$ . Finally, note that  $Y_{asym} \geq Y_{upper}$  for this range of  $c$ . Indeed,  $Y_{asym}$  is a rather poor approximation of  $Y_*$  for realistic values of the normalized call price  $c$  due to the fact that the error term  $\log(-\log(1-c))/\sqrt{-\log(1-c)}$  is actually *increasing* for  $c < 1 - e^{-e^2} = 0.9994!$

The next example we consider in this section is due to Roper and Rutkowski [23] and deals with the case where  $c$  is close to its lower bound  $(1 - e^k)^+$ . In particular, this regime is useful for studying the implied volatility smile of options very near maturity.

**Theorem 4.4 (Roper and Rutkowski).** *If  $k > 0$  then*

$$Y_{BS}(k, c) = \frac{k}{\sqrt{-2 \log c}} + O\left(\frac{\log(-\log c)}{(-\log c)^{3/2}}\right)$$

as  $c \downarrow 0$ . *If  $k < 0$  then*

$$Y_{BS}(k, c) = \frac{-k}{\sqrt{-2 \log p}} + O\left(\frac{\log(-\log p)}{(-\log p)^{3/2}}\right)$$

as  $c \downarrow 1 - e^k$ , where  $p = c + e^k - 1$ .

As always, we will prove the asymptotic result by finding uniform bounds. As discussed in section 2, we can reuse the bounds which are tight when  $c$  is close to 1 by first bounding the function  $\hat{C}$ .

**Proposition 4.5.** *For  $k > 0$  and  $0 < c < 1$ , we have*

$$1 - c L(k, c) \leq \hat{C}(k, c) \leq 1 - c,$$

where

$$(6) \quad L(k, c) = \frac{2}{k} \left[ \Phi^{-1} \left( \frac{c}{1 + e^k} \right)^2 + 2 \right].$$

*Proof.* For the upper bound, simply note that

$$C_{BS}(k, y) + C_{BS}(k, 2k/y) = 1 - 2e^k \Phi(-k/y - y/2) \leq 1.$$

Now

$$\begin{aligned} \hat{C}(k, c) &= 1 - \int_0^c \frac{2k}{Y_{BS}(k, u)^2} du \\ &= 1 - \int_0^c \frac{Y_{BS}(k, \hat{C}(k, u))^2}{2k} du \\ &\geq 1 - \int_0^c \frac{Y_{BS}(k, 1-u)^2}{2k} du \end{aligned}$$

by two applications of Proposition 2.2 and the upper bound.

Now, we appeal to the upper bound in Proposition 4.3 to conclude that

$$\begin{aligned} \hat{C}(k, c) &\geq 1 - \frac{2}{k} \int_0^c \Phi^{-1} \left( \frac{u}{1+e^k} \right)^2 du \\ &= 1 - \frac{2(1+e^k)}{k} \int_{-\Phi^{-1}(\frac{c}{1+e^k})}^\infty x^2 \phi(x) dx. \end{aligned}$$

To complete the proof, note that the bound

$$\begin{aligned} \int_A^\infty x^2 \phi(x) dx &= A\phi(A) + \Phi(-A) \\ &\leq (A^2 + 2)\Phi(-A) \end{aligned}$$

which holds for all  $A \geq 0$ . ■

We now prove an inequality which provides an easy way to convert bounds which are good when  $c \uparrow 1$  into bounds which are good when  $c \downarrow 0$ .

**Proposition 4.6.** *Fix  $k > 0$  and  $0 < c < 1$ . Then*

$$\frac{2k}{Y_{BS}(k, 1-c)} \leq Y_{BS}(k, c) \leq \frac{2k}{Y_{BS}(k, 1-c L(k, c))},$$

where  $L(k, c)$  is defined by (6). In particular, we have

$$Y_{BS}(k, c) \geq \frac{k}{-\Phi^{-1} \left( \frac{c}{1+e^k} \right)}$$

and if  $c L(k, c) \leq 1$  we have

$$Y_{BS}(k, c) \leq \frac{k}{-\Phi^{-1} \left( \frac{c L(k, c)}{2} \right)}.$$

*Proof.* The first claim follows from the fact that  $Y_{BS}(k, \cdot)$  is increasing and from Proposition 2.2. The second set of claims follow from the bounds in Proposition 4.3. ■

*Remark 4.7.* The inequality

$$Y_{BS}(k, c)Y_{BS}(k, 1 - c) \geq 2k,$$

which holds for all  $k > 0$  and  $0 < c < 1$ , has an appealing symmetry!

*Proof of Theorem 4.4.* First fix  $k > 0$ . Using the second lower bound from Proposition 4.6, together with Lemma 4.1, we have that

$$Y_{BS}(k, c) \geq \frac{k}{-2 \log c} + O\left(\frac{\log(-\log c)}{(-\log c)^{3/2}}\right).$$

Similarly, since Lemma 4.1 implies that the quantity  $L(k, c)$  from Proposition 4.5 is of asymptotic order

$$L(k, c) = O(\log c)$$

as  $c \downarrow 0$  thanks to Proposition 4.1, the upper bound follows.

The case  $k < 0$  follows from the put-call symmetry of Proposition 2.1. ■

Figure 3 illustrates the behavior of  $Y_{BS}(k, c)$  as  $c \downarrow 0$ , compared with the uniform upper and lower bounds of Proposition 4.6 and the asymptotic formula in Theorem 4.4. We fixed the log-moneyness at  $k = 0.2$  and plotted four functions:

1.  $Y_{\text{upper}}(c) = \frac{k}{-\Phi^{-1}(c L(k, c)/2)}$  is the upper bound from Proposition 4.6;
2.  $Y_*(c) = Y_{BS}(k, c)$  is the true function of our interest;
3.  $Y_{\text{lower}}(c) = \frac{k}{-\Phi^{-1}(c/(1+e^k))}$  is the lower bound from Proposition 4.6;
4.  $Y_{\text{asym}}(c) = \frac{k}{\sqrt{-2 \log c}}$  is the asymptotic shape from Theorem 4.4.

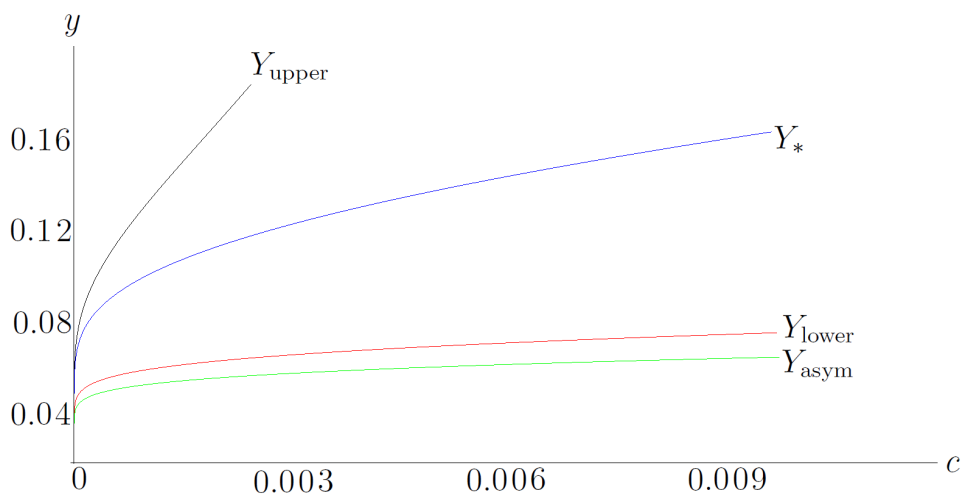


Figure 3. Bounds and asymptotics of  $Y_{BS}(k, \cdot)$  as  $c \downarrow 0$ .

Note again that  $Y_{\text{upper}} \geq Y_* \geq Y_{\text{lower}}$  as predicted. Finally, note that  $Y_{\text{asym}} \leq Y_{\text{lower}}$  for this range of  $c$ .

The next example is due to Gulisashvili [13]. This result is useful in studying the wings of the implied volatility surface for extreme strikes but fixed maturity date.

**Theorem 4.8 (Gulisashvili).** *If  $c(k) \downarrow 0$  as  $k \uparrow +\infty$  then*

$$Y_{\text{BS}}(k, c(k)) = \sqrt{-2 \log(e^{-k}c(k))} - \sqrt{-2 \log c(k)} + O\left(\frac{\log(-\log c(k))}{\sqrt{-\log c(k)}}\right).$$

*If  $e^{-k}p(k) \downarrow 0$  as  $k \downarrow -\infty$  then*

$$Y_{\text{BS}}(k, c(k)) = \sqrt{-2 \log p(k)} - \sqrt{-2 \log(e^{-k}p(k))} + O\left(\frac{\log(-\log(e^{-k}p(k)))}{\sqrt{-\log(e^{-k}p(k))}}\right),$$

where  $c(k) = 1 - e^k + p(k)$ .

As before, the proof will rely on appropriate uniform bounds.

**Proposition 4.9.** *Fix  $k \in \mathbb{R}$  and  $(1 - e^k)^+ \leq c < 1$ . If  $k \geq 0$  we have*

$$\Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k} \leq Y_{\text{BS}}(k, c) \leq \Phi^{-1}(2c) - \Phi^{-1}(e^{-k}c)$$

and for  $k < 0$  we have

$$\Phi^{-1}(e^{-k}p) + \sqrt{[\Phi^{-1}(e^{-k}p)]^2 - 2k} \leq Y_{\text{BS}}(k, c) \leq \Phi^{-1}(2e^{-k}p) - \Phi^{-1}(p),$$

where  $p = c + e^k - 1$ .

*Proof.* Consider the case  $k \geq 0$ . For the upper bound, let  $d_2 = \Phi^{-1}(e^{-k}c)$  in Theorem 3.1.

For the lower bound, let  $y = Y_{\text{BS}}(k, c)$ . Observe that

$$\begin{aligned} \Phi\left(-\frac{k}{y} + \frac{y}{2}\right) &= c + e^k \Phi\left(-\frac{k}{y} - \frac{y}{2}\right) \\ &\geq c. \end{aligned}$$

The conclusion follows from noting that the strictly increasing map

$$x \mapsto x + \sqrt{x^2 + 2k}$$

from  $\mathbb{R}$  to  $(0, \infty)$  is the inverse of the map

$$y \mapsto -\frac{k}{y} + \frac{y}{2}$$

from  $(0, \infty)$  to  $\mathbb{R}$ .

The case where  $k < 0$  is handled by put-call symmetry as always. ■

*Remark 4.10.* The idea behind the lower bound is the simple inequality

$$(X - K)^+ \leq X \mathbf{1}_{[K, \infty)}(X)$$

which holds for all  $X, K \geq 0$ .

*Proof of Theorem 4.8.* For  $k \geq 0$ , we apply Proposition 4.9 and Lemma 4.1 to get

$$\begin{aligned} Y_{\text{BS}}(k, c(k)) &\leq \Phi^{-1}(2c(k)) - \Phi^{-1}(e^{-k}c(k)) \\ &= -\sqrt{-2 \log c(k)} + \sqrt{-2 \log e^{-k}c(k)} + O\left(\frac{\log(-\log c(k))}{\sqrt{-\log c(k)}}\right), \end{aligned}$$

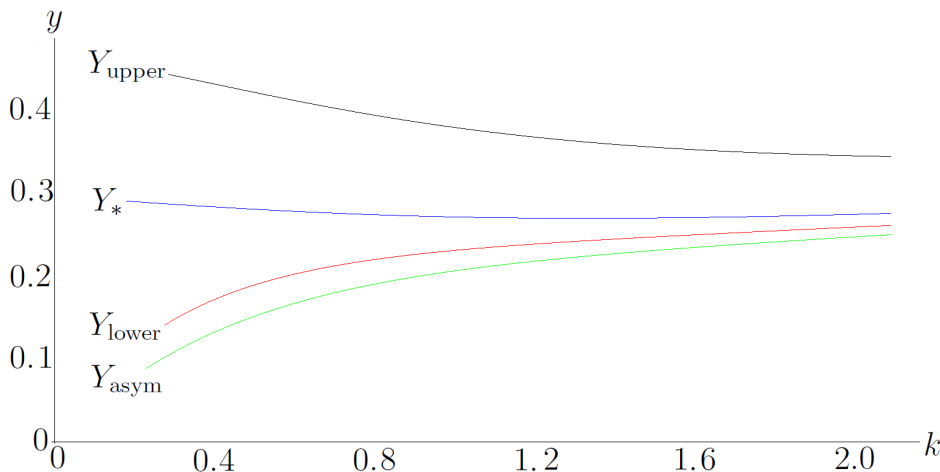
where we have used  $\sqrt{-2 \log(2c)} = \sqrt{-2 \log c} + O(\frac{1}{\sqrt{-\log c}})$  as  $c \downarrow 0$  to control the error from the first term, and the bound  $e^{-k}c(k) \leq c(k)$  to control the error from the second term. Similarly, for the upper bound, Proposition 4.9 and Lemma 4.1 yield

$$\begin{aligned} Y_{\text{BS}}(k, c(k)) &\geq \Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k} \\ &= -\sqrt{-2 \log c(k)} + \sqrt{-2 \log e^{-k}c(k)} + O\left(\frac{\log(-\log c(k))}{\sqrt{-\log c(k)}}\right). \end{aligned}$$

The  $k \downarrow -\infty$  case is similar. ■

Figure 4 illustrates the behavior of  $Y_{\text{BS}}(k, c(k))$  when  $c(k) \downarrow 0$  as  $k \uparrow \infty$ , compared with the uniform upper and lower bounds of Proposition 4.9 and the asymptotic formula in Theorem 4.8. We have chosen the function  $c(\cdot)$  according to the variance gamma model. That is, we fix a time horizon  $T > 0$  and let

$$c(k) = \mathbb{E}[(X - e^k)^+],$$



**Figure 4.** Bounds and asymptotics of  $Y_{\text{BS}}(\cdot, c(\cdot))$  as  $c(k) \downarrow 0$  as  $k \uparrow \infty$ .



where

$$X = e^{\sigma W(G_T) + \Theta G_T + mT}$$

and  $\sigma$  and  $\Theta$  are real constants, and the process  $W$  is a Brownian motion subordinated to the gamma process  $G$ , which is an independent Lévy process with jump measure  $\mu(dx) = \frac{1}{\nu x} e^{-x/\nu} dx$  for some constant  $\nu > 0$ . The constant  $m$  is chosen so that

$$\mathbb{E}[X] = 1.$$

It is well known that  $G_T$  has the gamma distribution with mean  $T$  and variance  $\nu T$ . By a routine calculation involving the moment generating functions of the normal and gamma distributions, we find the moment generating function  $M$  of  $\log X$  to be

$$M(r) = e^{rmT} (1 - \nu(\Theta r + \sigma^2 r^2/2))^{-T/\nu}.$$

Therefore, we must set

$$m = \frac{1}{\nu} \log(1 - \nu(\Theta + \sigma^2/2)).$$

Note that we must assume the parameters are such that

$$\Theta + \sigma^2/2 < 1/\nu$$

to ensure that  $m$  is real. Recall that the random variable  $X$  has the interpretation of the ratio  $X = S_T/F_{0,T}$  of the time- $T$  price  $S_T$  of some asset to its initial time- $T$  forward price. The expected value is computed under a fixed time- $T$  forward measure. Hence  $c(k)$  models the initial normalized price of a call option with log-moneyness  $k$  and maturity  $T$ . We use the parameters  $\sigma = 0.1213$ ,  $\nu = 0.1686$ , and  $\Theta = -0.1436$  as suggested by the calibration of Madan, Carr, and Chang [20] and set  $T = 5$ .

As before, we plotted four functions:

1.  $Y_{\text{upper}}(k) = \Phi^{-1}(2c(k)) - \Phi^{-1}(e^{-k}c(k))$  is the upper bound from Proposition 4.9;
2.  $Y_*(k) = Y_{\text{BS}}(k, c(k))$  is the true function of our interest;
3.  $Y_{\text{lower}}(k) = \Phi^{-1}(c(k)) + \sqrt{[\Phi^{-1}(c(k))]^2 + 2k}$  is the lower bound from Proposition 4.9;
4.  $Y_{\text{asym}}(k) = \sqrt{-2 \log(e^{-k}c(k))} - \sqrt{-2 \log c(k)}$  is the asymptotic shape from Theorem 4.8.

As always, note that  $Y_{\text{upper}} \geq Y_* \geq Y_{\text{lower}}$  as predicted. Finally, note that  $Y_{\text{asym}} \leq Y_{\text{lower}}$  for this example. It is worth remarking that for the points on the extreme right side of the graph of  $Y_*$ , the moneyness  $K/F_{0,T} \approx 10$  and normalized call price  $C_{0,T,K}/(F_{0,T}B_{0,T}) \approx 10^{-15}$  are outside the range of typical liquid market prices.

The recent paper [9] of De Marco, Hillairet, and Jacquier studies a similar asymptotic regime as the  $k \downarrow -\infty$  case of Theorem 4.8, except now the assumption is that  $e^{-k}p(k) \rightarrow u > 0$ . See also the paper of Gulisashvili [15] for further refinements. The motivation is to study the left-wing behavior of the implied volatility smile in the case where the price of the underlying stock may hit zero. The first two terms in the following expansion have been known for a few years; see, for instance, [27].

Even more recently, Jacquier and Keller-Ressel [16] have interpreted the corresponding (via Proposition 2.1) right-wing formula in terms of a market model with a price bubble. We will comment on this interpretation below.

**Theorem 4.11 (De Marco, Hillairet, and Jacquier).** *Suppose  $e^{-k}p(k) \rightarrow u$  as  $k \downarrow -\infty$ , where  $0 < u < 1$ . Then letting  $c(k) = p(k) + 1 - e^k$  we have*

$$Y_{\text{BS}}(k, c(k)) = \sqrt{-2k} + \Phi^{-1}(u) + O\left(\frac{1}{\sqrt{-k}} + \varepsilon(k)\right)$$

as  $k \downarrow -\infty$ , where

$$\varepsilon(k) = e^{-k}p(k) - u.$$

(Jacquier and Keller-Ressel). *Furthermore, suppose  $c(k) \rightarrow u$  as  $k \uparrow +\infty$ , where  $0 < u < 1$ . Then*

$$Y_{\text{BS}}(k, c(k)) = \sqrt{2k} + \Phi^{-1}(u) + O\left(\frac{1}{\sqrt{k}} + \varepsilon(k)\right),$$

where

$$\varepsilon(k) = c(k) - u.$$

Our proof of Theorem 4.11 reuses the uniform lower bound from Proposition 4.9. However, another upper bound is needed in this situation.

**Proposition 4.12.** *Fix  $k \in \mathbb{R}$  and  $(1 - e^k)^+ \leq c < 1$ . If  $k \geq 0$ , we have*

$$Y_{\text{BS}}(k, c) \leq \Phi^{-1}(c + e^k \Phi(-\sqrt{2k})) + \sqrt{2k}$$

and if  $k < 0$  we have

$$Y_{\text{BS}}(k, c) \leq \Phi^{-1}(e^{-k}p + e^{-k}\Phi(-\sqrt{-2k})) + \sqrt{-2k},$$

where  $p = c + e^k - 1$ .

*Proof.* In the statement of Theorem 3.1, let  $d_2 = -\sqrt{2k}$  if  $k \geq 0$ , or let  $d_1 = \sqrt{-2k}$  if  $k < 0$ . ■

*Proof of Theorem 4.11.* It is sufficient to prove only the  $k < 0$  case. Recall the standard bound on the normal Mills ratio

$$e^x \Phi(-\sqrt{2x}) \leq \frac{1}{\sqrt{4\pi x}} \rightarrow 0 \text{ as } x \uparrow \infty.$$

Hence by Proposition 4.12 we have

$$\begin{aligned} Y_{\text{BS}}(k, c) - \sqrt{-2k} &\leq \Phi^{-1}(u + \varepsilon(k) + (-4\pi k)^{-1/2}) \\ &= \Phi^{-1}(u) + O(\varepsilon(k) + (-k)^{-1/2}). \end{aligned}$$

Similarly by Proposition 4.8 we have

$$\begin{aligned} Y_{\text{BS}}(k, c(k)) - \sqrt{-2k} &\geq \Phi^{-1}(u + \varepsilon(k)) + \sqrt{[\Phi^{-1}(u + \varepsilon(k))]^2 - 2k} - \sqrt{-2k} \\ &= \Phi^{-1}(u) + O(\varepsilon(k) + (-k)^{-1/2}) \end{aligned}$$

completing the proof. ■

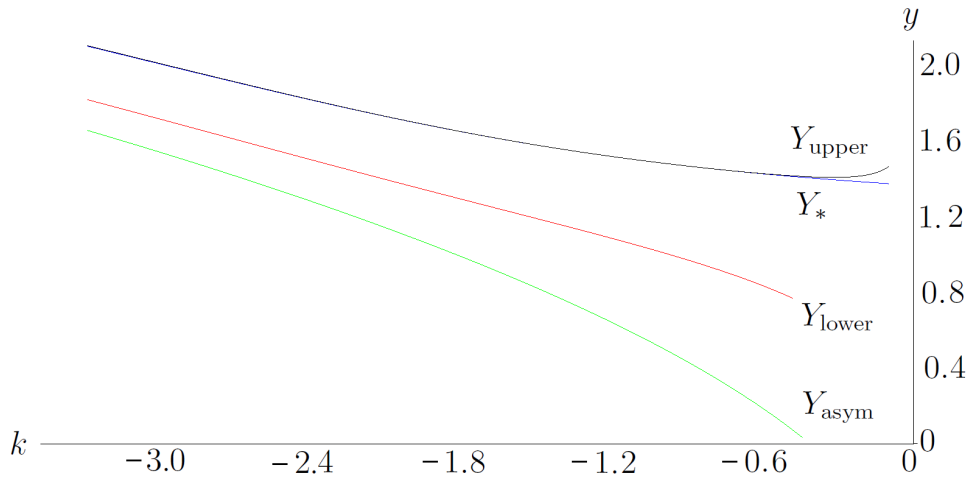


Figure 5. Bounds and asymptotics of  $Y_{BS}(\cdot, c(\cdot))$  as  $e^{-k}p(k) \downarrow u > 0$  as  $k \downarrow -\infty$ .

Figure 5 illustrates the behavior of  $Y_{BS}(k, c(k))$  when  $e^{-k}p(k) \rightarrow u > 0$  as  $k \downarrow -\infty$ , where  $p(k) - c(k) = e^k - 1$ , compared with the uniform upper bounds of Proposition 4.12, lower bounds of Proposition 4.9, and the asymptotic formula in Theorem 4.11. We have chosen the function  $c(\cdot)$  according to the Black–Scholes model with a jump to default. That is, we fix a horizon  $T > 0$  and let

$$c(k) = \mathbb{E}[(X - e^k)^+],$$

where

$$X = \mathbf{1}_{\{T < \tau\}} e^{\sigma W_T + (\lambda - \sigma^2/2)T}$$

and  $\sigma$  and  $\lambda$  are positive constants, the process  $W$  is a Brownian motion, and the random variable  $\tau$  is independent of  $W$  and exponentially distributed with rate  $\lambda$ , so that

$$\mathbb{E}[X] = 1.$$

Note that

$$\begin{aligned} e^{-k}p(k) &= \mathbb{E}[(1 - e^{-k}X)^+] \\ &\rightarrow \mathbb{P}(X = 0) \\ &= \mathbb{P}(\tau \leq T) \\ &= 1 - e^{-\lambda T}. \end{aligned}$$

On the other hand, it is straightforward to calculate

$$c(k) = C_{BS}(k - \lambda T, \sigma\sqrt{T}).$$

We use the parameters  $\sigma = 0.60$  and  $\lambda = 0.05$  with time horizon  $T = 4$ .

As before, we plotted four functions:

1.  $Y_{\text{upper}}(k) = \Phi^{-1}[e^{-k}p(k) + e^{-k}\Phi(-\sqrt{-2k})] + \sqrt{-2k}$  is the upper bound from Proposition 4.12;
2.  $Y_*(k) = Y_{\text{BS}}(k, c(k))$  is the true function of our interest;
3.  $Y_{\text{lower}}(k) = \Phi^{-1}(e^{-k}p(k)) + \sqrt{\Phi^{-1}(e^{-k}p(k))^2 - 2k}$  is the lower bound from Proposition 4.9;
4.  $Y_{\text{asym}}(c) = \sqrt{-2k} + \Phi^{-1}(u)$  is the asymptotic shape from Theorem 4.11.

As always, note that  $Y_{\text{upper}} \geq Y_* \geq Y_{\text{lower}}$  as predicted, that  $Y_{\text{upper}}$  is a surprisingly good approximation for  $Y_*$ , and that  $Y_{\text{asym}} \leq Y_{\text{lower}}$  for this example. For the left-hand points of the graph, the moneyness  $K/F_{0,T} \approx 0.04$  is somewhat outside the range of typical liquid market prices.

*Remark 4.13.* To compute the implied volatility for a given model, one generally needs three ingredients: the bond price  $B_{0,T}$ , the forward price  $F_{0,T}$ , and the call price  $C_{0,T,K}$ . Consider the case where the interest rate is zero and the underlying stock pays no dividends. In particular, for this discussion  $B_{0,T} = 1$ .

In the discrete time case, one typically models the stock price  $(S_t)_{t \geq 0}$  as a martingale so that there is no arbitrage. The call price is then calculated as

$$C_{0,T,K} = \mathbb{E}[(S_T - K)^+]$$

with the justification that the above price is consistent with no-arbitrage in general, and in the case of a complete market, the expected payout under the unique risk-neutral measure is the replication cost of the option and hence the unique no-arbitrage price. Similarly, we have for the forward price the following formula

$$F_{0,T} = \mathbb{E}[S_T] = S_0.$$

In the continuous time setting, things are more subtle because of the existence of doubling strategies. If one assumes the NFLVR notion of no-arbitrage, then by Delbaen and Schachermayer's fundamental theorem of asset pricing [10] the asset prices are sigma-martingales, but not necessarily true martingales. In particular, given a dynamic model of the underlying process  $(S_t)_{t \geq 0}$ , this no-arbitrage condition alone does not uniquely specify the call and forward prices, even in a complete market. See, for instance, the paper of Ruf [24] for a discussion of this issue. When the market is complete, a candidate call price is the minimal replication cost

$$C^{\text{repl}} = \mathbb{E}[(S_T - K)^+].$$

Another sensible way to price the call is to assume that the put price is its minimal replication cost and the call is priced by put-call parity:

$$\begin{aligned} C^{\text{parity}} &= S_0 - K + \mathbb{E}[(K - S_T)^+] \\ &= S_0 - \mathbb{E}[K \wedge S_T]. \end{aligned}$$

Similarly, the forward price can be either given by static replication

$$F^{\text{static}} = S_0$$

or by dynamic replication

$$F^{\text{dyn}} = \mathbb{E}[S_T].$$

Of course, if  $(S_t)_{t \geq 0}$  is a true martingale the corresponding candidate prices agree; however, there has been recent interest in models where, for instance, the process  $(S_t)_{t \geq 0}$  is a nonnegative strict local martingale, and hence a strict supermartingale. (Such price processes are often described as bubbles; see, for instance, the paper of Cox and Hobson [8].)

The result of Jacquier and Keller-Ressel quoted here as the second half of Theorem 4.11 corresponds to choosing  $C_{0,T,K} = C^{\text{parity}}$  and  $F_{0,T} = F^{\text{static}}$  so that the implied volatility is

$$\sigma^{\text{implied}} = \frac{1}{\sqrt{T}} Y(\log(K/S_0), 1 - \mathbb{E}[K \wedge S_T]/S_0).$$

We note here that this convention for defining implied volatility was also adopted in [26]. On the other hand, note that the convention  $C_{0,T,K} = C^{\text{repl}}$  and  $F_{0,T} = F^{\text{dyn}}$  is used in (2) of the introduction.

We conclude with some remarks on the bounds and asymptotic formulas in this section. The numerical results suggest that for at least some situations, one of the upper or lower bounds is a better approximation to the implied total standard deviation than the corresponding lowest order asymptotic formula. One could argue that with more terms in the asymptotic series, better accuracy could be attained with the asymptotics. Although such a claim is indeed plausible, there are a few reasons why it is beside the point.

First, the numerical results presented here should only be considered a proof of concept, rather than a head-to-head competition between state-of-the-art approximations. Nevertheless, it is worth noting both the given bounds and the asymptotic formulas are only approximations, and therefore have an error term. But unlike the error terms of an asymptotic formula, the error term for our bounds have a known sign.

Second, given one bound, the theorems of section 3 give a systematic way of finding a better bound. Indeed, fix  $(k, c)$  with  $k > 0$ , and let  $y_* = Y_{\text{BS}}(k, c)$ . Suppose it known that

$$y_* < y_1,$$

where  $y_1$  is some given approximation. Define  $F : (y_{\min}, \infty) \rightarrow (0, \infty)$  by

$$F(y) = H_1(-k/y + y/2; k, c),$$

where

$$y_{\min} = \Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k}$$

and  $H_1$  is the functions defined by (4) of section 3. Letting

$$y_2 = F(y_1)$$

we have by Theorem 3.1 that

$$y_* < y_2.$$

However, more is true. Note that the map  $F$  has a unique fixed point  $y_*$ . Since

$$\lim_{y \downarrow y_{\min}} F(y) = \infty,$$

we conclude by the continuity of  $F$  that  $F(y) > y$  for  $y_{\min} < y < y^*$  and, more importantly, that  $F(y) < y$  for  $y > y^*$ . In particular,

$$y_2 = F(y_1) < y_1.$$

That is,  $y_2$  is a *better* approximation of  $y_*$  and the error term has the same sign as the original approximation. Of course, this process can be iterated. Letting  $y_n = F(y_{n-1})$  we see that the sequence  $(y_n)_{n \geq 1}$  is decreasing and  $\inf_n y_n = y_*$ .

Notice that this sequence converges very rapidly. Indeed, by Taylor's theorem

$$\begin{aligned} y_n &= F(y_{n-1}) \\ &= F(y_*) + F'(y_*)(y_{n-1} - y_*) + \frac{1}{2}F''(\hat{y})(y_{n-1} - y_*)^2 \end{aligned}$$

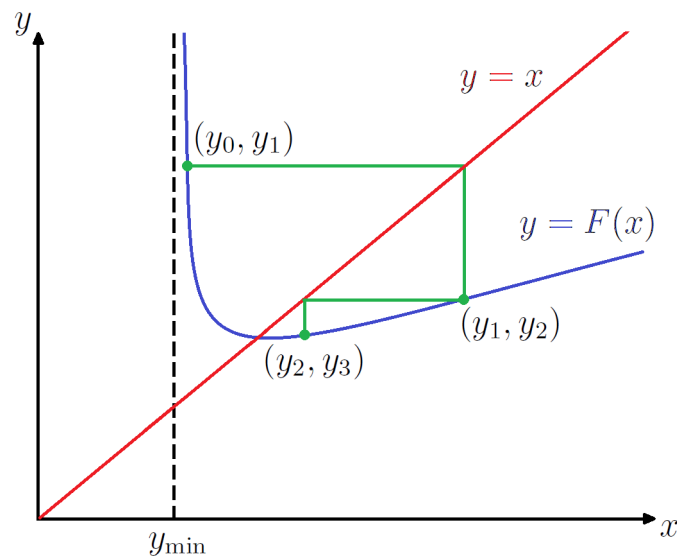
for some  $y_* < \hat{y} < y_{n-1}$ . Since  $y_*$  minimizes  $F$  we have

$$F'(y_*) = 0$$

and, hence, by the continuity of  $F''$ , we have

$$\frac{y_n - y_*}{(y_{n-1} - y_*)^2} \rightarrow \frac{1}{2}F''(y_*) = \frac{1}{2y_*} \left( \frac{k}{y_*} + \frac{y_*}{2} \right)^2$$

as  $n \rightarrow \infty$ .



**Figure 6.** The cobweb diagram illustrating the convergence of  $y_0, y_1, y_2, \dots$  to the fixed point  $y_* = F(y_*)$ .

Furthermore, we can find our initial upper bound  $y_1$  by choosing any  $y_0 > y_{\min}$  and letting  $y_1 = F(y_0)$ . This procedure is illustrated by the cobweb diagram of Figure 6. Of course, the convergence can be helped along by an inspired choice of  $y_0$  as discussed at the beginning of this section.

The above discussion of a rapidly converging sequence should be contrasted with the approach taken, for instance, in the paper of Gao and Lee [12]. There, a systematic method of computing terms in the asymptotic series for implied volatility is obtained. However, unlike the procedure discussed above, an asymptotic series may diverge as more terms are added.

A third and final point is that the approximations for the implied total standard deviation are not particularly interesting on their own. Indeed, to use the formulas in Proposition 4.9 one must already know the normalized bond price  $c(k)$ . If this quantity is to be calculated numerically from a certain model, one might as well compute the  $Y_{BS}(k, c(k))$  numerically also. The point of these bounds is to be used in conjunction with other, model dependent, bounds on  $c(k)$  to obtain useful bounds on the quantities of interest.

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