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The Impact of Jump Distributions on the Implied Volatility of Variance*

E. Nicolato†, C. Pisani†, and D. Sloth‡

Abstract. We consider a tractable affine stochastic volatility model that generalizes the seminal Heston model by augmenting it with jumps in the instantaneous variance process. In this framework, we consider both realized variance options and VIX options, and we examine the impact of the distribution of jumps on the associated implied volatility smile. We provide sufficient conditions for the asymptotic behavior of the implied volatility of variance for small and large strikes. In particular, by selecting alternative jump distributions, we show that one can obtain fundamentally different shapes of the implied volatility of variance smile—some clearly at odds with the upward-sloping volatility skew observed in variance markets.

Key words. jump distributions, stochastic volatility, realized variance, VIX options, moment formula, affine processes

AMS subject classifications. 91G20, 60G51, 62P05

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1. Introduction.

There is a vast and lively literature investigating the presence of jumps in the evolution of financial assets. In a seminal paper, [15] proposed the class of affine jump-diffusion processes, a flexible and tractable modeling framework allowing for jumps both in asset prices and in their stochastic variances. Since then, affine models have been applied empirically in a number of studies, including [8], [12], [17], and [16] among others. These studies generally find evidence for discontinuities both in the price level and its volatility. In particular, the consensus appears to be that jumps are needed to capture the steep and negative skew observed in the short end of the volatility surface implied by equity options. We refer to [15] and [19] for a detailed analysis of the effect of jumps when calibrating affine models to S&P500 option prices.

A substantial body of literature considers the pricing of derivatives written on the volatility or the variance of an asset such as realized variance options or options on the CBOE's VIX index. In contrast to the case of equity markets, the volatility surface implied by volatility options is characterized by an upward-sloping smile. As indicated by [34], this stylized feature reflects the fact that out-of-the-money call options on volatility provide protection against market crashes. To compensate for the insurance risk, the writer of a call on volatility will charge a premium accordingly, very much like the writer of a put on the stock or index

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†Department of Economics and Business Economics, Aarhus University, Denmark (eln@econ.au.dk, cpisani@econ.au.dk).
‡Danske Bank, Denmark (dap@danskebank.dk).

itself. Several authors suggest that the inclusion of jumps in the volatility process provides a parsimonious and empirically justifiable way to capture the positive skew associated with volatility derivatives. Among them, [33], [34], and [29] propose augmenting the popular square-root dynamics of [24] to include exponential jumps in the instantaneous variance process.

Despite the common notion that jumps are a useful modeling ingredient, the question of how to model the distribution of jumps and its financial implications seems to be a matter of lesser attention. A number of different jump specifications have been examined within the literature concerning equity derivatives. For example, [30] and [9] compare the performances of alternative affine models when calibrated to S&P500 option prices. These studies indicate that, ceteris paribus, the specific choice of jump distribution has a minor effect on the qualitative behavior of the skew and the term structure of the implied volatility surface of equity options. This might be the reason why the discussion about jump selection is basically absent from the literature on volatility derivatives. To the best of our knowledge, the above-mentioned exponential distribution appears to be the standard choice to model jumps in the instantaneous variance.

As a matter of fact, several authors have looked at alternative volatility *dynamics* based on diffusive or more general continuous paths processes. To name a few, [14] considers logarithmic Ornstein-Uhlenbeck (OU) stochastic volatility models, [20] proposes a double mean reverting volatility process, while [13] and [3] examine the performance of the 3/2 model. Instead, in this work we acknowledge the presence of jumps, we fix the underlying dynamics within the affine jump-diffusion framework, and we establish precisely how pricing of volatility options depends on the choice of jumps in volatility. We show that the specific distribution of jumps has a profound impact on the pricing of volatility derivatives, predicting completely different shapes and characteristics of the VIX and realized variance implied volatility surfaces.

To keep matters simple, we consider the class of stochastic volatility jump (SVJ)-v models, a particular case of affine models which accounts for jumps only in the dynamics of the variance process. The SVJ-v framework allows for an enormous variety of jump distributions, and it includes variance specifications of the OU type introduced by [4]. This latter subclass is particularly neat, as the instantaneous variance evolves solely by jumps, which allows us to isolate the unique impact of the jump distribution on volatility derivatives.

We start out by analyzing realized variance options. Derivatives on realized variance used to be the preferred choice for hedging volatility and tail risk and were traded actively in the OTC market. The main advantage of dealing with this kind of contracts in the SVJ-v framework is that the realized variance can be identified with the integrated variance and its Laplace transform is available in closed form. Thus, we can draw upon classical tauberian theorems which, in combination with the results of [28] and [23] on volatility smile asymptotics, allow us to investigate the intimate link between the distribution of jumps, the distribution of realized variance, and ultimately the impact on the implied volatility of realized variance options. However, following the recent financial crisis, demand has shifted to listed volatility derivatives such as VIX options, which now constitute a quarter of the total turnover in options on the S&P500 index. Hence, we extend our analysis to the case of VIX options, obtaining analogous results on the impact of the jump distribution on the VIX implied volatility smile.

Specifically, for both realized variance and VIX options, we provide simple and easily verifiable sufficient conditions relating the tail distribution of jumps with the asymptotic behavior of implied volatility for low as well as high strikes. We are particularly interested in the asymptotics of the wings, i.e., whether they are upward or downward sloping, as this gives an indication of the overall qualitative behavior of the volatility smile and skew. To support our results, we provide numerical illustrations for a number of positive distributions for jump specifications and we show how the commonly used exponential law might not be the optimal choice as it inherently leads to a downward-sloping implied volatility skew.

The relevance of the analysis is not purely constrained to the academic scene. In fact, it offers a number of applications of industrial interest. First, market makers tend to prefer quite simple models—for many good reasons—when managing their options books, often neglecting modeling jumps. Instead, to alleviate shortcomings of their model, they will make various more or less ad hoc adjustments when hedging their delta risk. By clarifying how jumps (and their distribution) impact the pricing of volatility options, our results give market makers a much better handling of volatility jump risk, allowing them to "set their delta" in accordance with the true volatility dynamics. Similarly, for alpha investors (whether hedge funds, asset managers, etc.) who trade volatility as an asset class, understanding the distributional properties of volatility jumps and their impact is essential for developing more accurate trading models and forecasting future volatility.

The rest of the paper is organized as follows: In section 2 we present the necessary background on wing asymptotics and we provide some preliminary results for a general distribution of the underlying. In section 3 we specialize the analysis to the case of realized variance options in the SVJ-v modeling framework and we derive sufficient conditions based on the jump component for the asymptotic behavior of the smile. In section 4 we describe a number of alternative jump distributions and present numerical illustrations for the selected cases. In section 5 we extend the analysis to the wing asymptotics of VIX options and in section 6 we summarize and conclude the paper. In the appendix we provide the details of some lengthy proofs.

2. Preliminary results on wing asymptotics.

We start out by outlining a few preliminary results relating the asymptotic behavior of the implied volatility at small or large strikes to the distribution of the underlying random quantity. To be more precise, fix a maturity T and denote by H_T the risk-neutral value of the underlying asset at maturity. Assuming a simplified economy with zero interest rates and dividend payments, the price of a European call with strike K is given by $C(K) = \mathbb{E}\,(H_T - K)^+$. The corresponding put price $P(K)$ can be obtained by the put-call parity relation

$$C(K) - P(K) = \mathbb{E}[H_T] - K.$$

At this stage, we do not specify further the nature of the underlying. We only require that H_T is a positive random variable with finite first moment which, without loss of generality, we normalize to one, $\mathbb{E}[H_T] = 1$. The distribution function, the tail function, and the Laplace transform of H_T are denoted by

$$F_H(x) = \mathbb{Q}(H_T \leq x), \qquad \overline{F}_H(x) = 1 - F_H(x), \qquad \mathcal{L}_H(x) = \mathbb{E}\left[e^{-xH_T}\right].$$

The implied volatility $I(K)$ associated with $C(K)$ is defined as the solution of the equation

$$C(K) = \Phi\left(\frac{\log(1/K)}{I(K)\sqrt{T}} + \frac{I(K)\sqrt{T}}{2}\right) - K\Phi\left(\frac{\log(1/K)}{I(K)\sqrt{T}} - \frac{I(K)\sqrt{T}}{2}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal law. If the underlying variable H_T satisfies the condition $F_H(x) < 1$ for all $x > 0$, then $I(K)$ is well defined for all $K \geq 1$ and one can study the asymptotic behavior as $K \to \infty$. Similarly, if $F_H(x) > 0$ for all $x > 0$, then $I(K)$ is well defined for all $K < 1$ and it can be analyzed as $K \to 0$.

The analysis of $I(K)$ at extreme strikes, referred to as *smile wings*, has attracted considerable attention during the last decade. In a groundbreaking paper, [28] relates the smile wings to the number of moments of the underlying distribution H_T. Since then, a large part of the literature has been devoted to providing refinements and extensions of Lee's moment formulas. See, for example, the work of [5], [6], [21], [23], and the monograph by [22]. The results relevant to this work are summarized in Theorems 2.1 and 2.3 below. The function ψ appearing in the formulations is given by

$$\psi(x) = 2 - 4(\sqrt{x^2 + x} - x),$$

and $g(x) \sim h(x)$ means that $g(x)/h(x) \to 1$ as either $x \to 0$ or $x \to \infty$ depending on the context. Also, recall that a positive, measurable function f on $\mathbb{R}_+$ is said to be *regularly varying* at ∞ with index $\rho \in \mathbb{R}$ if the following holds:

$$\lim_{x\to\infty} \frac{f(\xi x)}{f(x)} = \xi^\rho \tag{2.1}$$

for all $\xi > 0$. In this case we write $f \in R_\rho$. When $f \in R_0$, then we say that f is *slowly varying* at ∞. It can be shown that $f \in R_\rho$ if and only if it takes the following form

$$f(x) = x^\rho \ell(x), \tag{2.2}$$

where $\ell \in R_0$.
Let us start by considering the behavior of $I(K)$ at large strikes.

Theorem 2.1. *Assume $F_H(x) < 1$ for all $x > 0$. Then the following statements hold for the implied volatility $I(K)$ at large strikes.*

(i) *Let $p_H = \sup\,\{p : \mathbb{E}[H_T^{p+1}] < \infty\}$, then*

$$\limsup \frac{I^2(K)T}{\log(K)} = \psi(p_H) \quad \textit{as} \quad K \to \infty. \tag{2.3}$$

(ii) *If $p_H < \infty$, then we can replace* lim sup *with the limit and write*

$$I^2(K) \sim \psi(p_H)\frac{\log(K)}{T} \quad \textit{as} \quad K \to \infty \tag{2.4}$$

if and only if we can find $f_1, f_2 \in R_{-\rho}$ with $\rho = p_H$, such that $f_1(K) \leq C(K) \leq f_2(K)$ for all $K > K_0$ with K_0 large enough.

(iii) *If $p_H = \infty$, then*

$$I(K) \sim \frac{1}{\sqrt{2T}} \log(K) \left(\log \frac{1}{C(K)} \right)^{-1/2} \quad as \quad K \to \infty. \tag{2.5}$$

The large strike formula (2.3) is derived in [28], while statements (2.4) and (2.5) can be found in [23]. Simple applications of Theorem 2.1 yield the following results which will be pivotal to the analysis we carry out in sections 3–5.

Proposition 2.2. *Assume $F_H(x) < 1$ for all $x > 0$.*

(a) *Suppose that the tail function $\overline{F}_H \in R_{-\rho-1}$ for $\rho > 0$. Then the asymptotic equivalence (2.4) at large strikes holds with $p_H = \rho$.*

(b) *Suppose that the tail function $\overline{F}_H$ is dominated by a Weibull-type function, i.e., there exist positive constants α, β, $\gamma > 0$ and an $x_0 > 0$ such that*

$$\overline{F}_H(x) \le \gamma e^{-\alpha x^\beta} \quad for\ all \quad x > x_0. \tag{2.6}$$

Then $p_H = \infty$ and $I(K) \to 0$ as $K \to \infty$.

Proof. Recall that

$$\mathbb{E}\left[H_T^{p+1}\right] = \int_0^\infty u^{p+1} dF_H(u) = (p+1) \int_0^\infty u^p \overline{F}_H(u) du \tag{2.7}$$

and

$$C(K) = \int_K^\infty \overline{F}_H(u) du. \tag{2.8}$$

(a) Well–known results from regular variation theory (see, e.g., [18, VIII.9]), state that for a bounded $\ell \in R_0$ and for $x \to \infty$ the following holds:

$$\text{If } q < -1, \text{ then } \int_0^x u^q \ell(u) du \text{ converges, while it diverges if } q > -1. \tag{2.9}$$

$$\text{If } q < -1, \text{ then } \frac{x^{q+1}\ell(x)}{\int_x^\infty u^q \ell(u) du} \to -(q+1). \tag{2.10}$$

Hence, by virtue of (2.2) and (2.7), the first statement implies that $p_H = \rho$. The second statement applied to (2.8) shows that $C(K) \in R_{-\rho}$ and the conclusion follows immediately from (ii) in Theorem 2.1.

(b) Still from (2.7) we see that condition (2.6) implies that $p_H = \infty$. Furthermore, notice that for $q \in \mathbb{R}$ the following holds:

$$\lim_{x \to \infty} \frac{x^{q+1} e^{-\alpha x^\beta}}{\int_x^\infty u^q e^{-\alpha u^\beta} du} = \infty. \tag{2.11}$$

Hence, for K large enough, it holds that

$$C(K) \le \int_K^\infty \gamma e^{-\alpha u^\beta} du \le AKe^{-\alpha K^\beta},$$

where A is a positive constant, and the result now follows from Gulisashivili's criterion (2.5) for large strikes. ■

Direct application of Proposition 2.2 might be difficult as in many models the tail function of the underlying distribution is not known, while the Laplace transform is available in closed form. In some cases, conditions based on $\mathcal{L}_H$ can be obtained via tauberian theory, which offers a number of results relating the behavior of F_H near infinity/zero to that of $\mathcal{L}_H$ near zero/infinity. In particular, Theorem 8.1.6 in [7] states that if n is a nonnegative integer and $\rho = n + q$ with $\rho > 0$ and $0 < q < 1$, then

$$\overline{F}_H(x) \in R_{-\rho} \quad \text{if and only if} \quad (-1)^{n+1}\mathcal{L}_H^{(n+1)}(s) \sim s^{q-1}\ell(1/s) \qquad \text{as } s \to 0,$$

where we have used the notation $f^{(n)} = \frac{d^n f}{dx^n}$. In view of (2.2), the equivalence above can be translated into the regular variation of $\mathcal{L}_H^{(n+1)}(1/x)$ at ∞ as follows:

$$\overline{F}_H(x) \in R_{-\rho} \quad \text{if and only if} \quad (-1)^{n+1}\mathcal{L}_H^{(n+1)}(1/x) \in R_{1-q}. \tag{2.12}$$

So, for a noninteger ρ, the regular variation condition required in part (a) of Proposition 2.2 may be assessed via (2.12). In contrast, we cannot find an equivalent formulation of the tail condition (2.6) in terms of $\mathcal{L}_H$ unless $\beta = 1$. In this case, the domain $\mathcal{D}_H = \{x \in \mathbb{R} : \mathcal{L}_H(x) < \infty\}$ determines whether $\overline{F}_H(x)$ is exponentially dominated, since

$$\overline{F}_H(x) \le \gamma e^{-\alpha x} \quad \text{for all } x \ge x_0 \qquad \text{if and only if} \qquad 0 \in \mathring{\mathcal{D}}_H, \tag{2.13}$$

as one may show via Markov's inequality.

Let us now consider the behavior of $I(K)$ at small strikes. Once again, we start by listing the relevant results from [28] and [23].

Theorem 2.3. *Assume $F_H(x) > 0$ for all $x > 0$. Then the following statements hold for the implied volatility $I(K)$ at small strikes.*

(i) *Let $q_H = \sup\,\{q : \mathbb{E}[H_T^{-q}] < \infty\}$, then*

$$\limsup \frac{I^2(K)T}{\log(1/K)} = \psi(q_H) \quad \textit{as} \quad K \to 0. \tag{2.14}$$

(ii) *If $q_H < \infty$, then we can replace* lim sup *with the limit and write*

$$I^2(K) \sim \psi(q_H)\frac{\log(K)}{T} \quad \textit{as} \quad K \to 0 \tag{2.15}$$

if and only if we can find $f_1, f_2 \in R_{-\rho}$ with $\rho = q_H + 1$ such that $f_1(1/K) \le P(K) \le f_2(1/K)$ for all $K < K_0$ with K_0 small enough.

(iii) *If $q_H = \infty$, then*

$$I(K) \sim \frac{1}{\sqrt{2T}}\left(\log\frac{1}{K}\right)\left(\log\frac{K}{P(K)}\right)^{-1/2} \quad \textit{as} \quad K \to 0. \tag{2.16}$$

The small strikes analogue of Proposition 2.2 reads as follows.

Proposition 2.4. *Assume $F_H(x) > 0$ for all $x > 0$ and that F_H is continuous.*

(a) *Suppose that for some $\rho \geq 0$, $F_H(1/x) \in R_{-\rho}$. Then the asymptotic equivalence (2.15) holds with index $q_H = \rho$.*

(b) *Suppose that there exist positive constants α, β, $\gamma > 0$ and $x_0 > 0$ such that F_H satisfies*

$$F_H(x) \leq \gamma e^{-\alpha x^{-\beta}} \quad \text{for all} \quad x < x_0. \tag{2.17}$$

Then the left-wing index $q_H = \infty$ and $I(K) \to 0$ as $K \to 0$.

Proof. Since F_H is continuous, the moment $E[H_T^{-q}]$, $q > 0$ can be expressed as

$$\mathbb{E}[H_T^{-q}] = q \int_0^\infty u^{q-1} F_H(1/u) du,$$

while the price of a put is given by

$$P(K) = \int_0^K F_H(u) du = \int_{1/K}^\infty u^{-2} F_H(1/u) du.$$

The results now follow from Theorem 2.3, proceeding in complete analogy with the proof of Proposition 2.2. ∎

Similarly to the large strikes case, part (a) of Proposition 2.4 can be reformulated in terms of the Laplace transform as a result of Karamata's tauberian theorem. In fact, by [18, XIII.5, Theorem 3], we have that for $\rho > 0$, the following holds:

$$F_H(1/x) \in R_{-\rho} \quad \text{if and only if} \quad \mathcal{L}_H(x) \in R_{-\rho} \tag{2.18}$$

and, in this case, $\mathcal{L}_H(x) \sim \Gamma(1+\rho) F_H(1/x)$ as $x \to \infty$. As for condition (2.17) in part (b), one may use a tauberian result of the exponential type to verify the stronger requirement that $\log F_H(x) \sim -\alpha x^{-\beta}$. More precisely, from de Bruijn's tauberian theorem it follows that if $r \in (0,1)$ and $\beta > 0$ satisfy $\frac{1}{r} - \frac{1}{\beta} = 1$, and $\alpha, s > 0$, then

$$\lim_{x \to 0} x^\beta \log F_H(x) = -\alpha \quad \text{if and only if} \quad \lim_{x \to \infty} \frac{\log \mathcal{L}_H(x)}{x^r} = -s, \tag{2.19}$$

and in this case $(r\,s)^{1/r} = (\alpha\,\beta)^{1/\beta}$. See, e.g., [7, Theorem 4.12.9.],

3. Realized variance options in the SVJ-v model.

In this section, we specialize the analysis carried out above, to the case of options written on the realized variance of an asset. Recall that the continuously sampled realized variance, hereafter denoted by V_T, is defined as the annualized quadratic variation of the log-price process over the time interval $[0, T]$. More precisely, we set

$$V_T = \frac{1}{T}[X]_T, \tag{3.1}$$

where $X_t = \log S_t$ denotes the log-price and

$$[X]_T = \lim_{\Delta \to 0} \sum_{n=1}^N (X_{t_n} - X_{t_{n-1}})^2,$$

where $\Delta = t_{i+1} - t_i$ is the step size of the partition $0 < t_1 < \cdots < t_N = T$.

In this work, we assume that the underlying price process evolves according to an affine stochastic volatility model known in the literature as the SVJ-v model. Specifically, we maintain the assumption of zero dividends and interest rates, and we model the log-price X and its instantaneous variance v via the following risk-neutral dynamics

$$
\begin{aligned}
dX_t &= -\frac{1}{2} v_t\, dt + \sqrt{v_t} dW_t, \\
dv_t &= \lambda(\theta - v_t) dt + \sigma \sqrt{v_t} dB + dJ_t.
\end{aligned} \tag{3.2}
$$

The processes W and B are—possibly correlated—Brownian motions while J is an increasing and driftless Lévy process which is independent of (W, B). Thus, the unit-time Laplace transform $\mathcal{L}_J(u) = \mathbb{E}[e^{-uJ_1}]$ takes the form

$$
\mathcal{L}_J(u) = e^{\kappa_J(u)} \quad \text{with} \quad \kappa_J(u) = \int_0^\infty (e^{-ux} - 1)\nu_J(dx), \quad u \geq 0\,, \tag{3.3}
$$

where the Lévy measure ν_J is a measure on the positive real line such that $\int_0^1 x\nu_J(dx) < \infty$. Finally, the parameters λ, θ, and σ are nonnegative constants.

The stochastic volatility model (3.2) generalizes the seminal [24] model by augmenting the square-root process describing v to allow for jumps. Furthermore, by setting $\theta = \sigma = 0$ in (3.2), we obtain the *non-Gaussian* OU model class proposed by [4]:

$$
v_t = e^{-\lambda t} v_0 + \int_0^t e^{-\lambda(t-s)} dJ_s, \tag{3.4}
$$

where the instantaneous variance moves uniquely by jumps.

The name SVJ-v refers to the fact that, in (3.2), jumps are allowed only at the variance level, in contrast to more general affine specifications such as the SVJJ model of [15], where both v and X are affected by jumps. The main advantage of the SVJ-v framework is that the quadratic variation coincides with the integrated variance, and therefore the realized variance (3.1) is given by

$$
V_T = \frac{1}{T} \int_0^T v_t dt.
$$

In affine models, such a quantity is easy to handle as its Laplace transform is known in closed form. [15] shows that $\mathcal{L}_V(u, T) = \mathbb{E}[e^{-uV_T}]$ is given by

$$
\mathcal{L}_V(u, T) = e^{\kappa_V(u,T)} \quad \text{with} \quad \kappa_V(u, T) = \alpha(u, T) + v_0 \beta(u, T) + \delta(u, T), \tag{3.5}
$$

where the functions α, β, and δ satisfy the ODEs

$$
\frac{\partial}{\partial t} \alpha = \theta \lambda \beta, \tag{3.6}
$$

$$
\frac{\partial}{\partial t} \beta = -\lambda \beta + \frac{1}{2} \sigma^2 \beta^2 - \frac{u}{T}, \tag{3.7}
$$

$$
\frac{\partial}{\partial t} \delta = \kappa_J(-\beta) \tag{3.8}
$$

with initial conditions $\alpha(u,0)=\beta(u,0)=\delta(u,0)=0$. For the full SVJ-v model with $\theta,\sigma>0$, the explicit solutions read as follows:

$$\alpha(u,t)=-\frac{2\lambda\theta}{\sigma^2}\log\left(\frac{\gamma(u)+\lambda+(\gamma(u)-\lambda)e^{-\gamma(u)t}}{2\gamma(u)}\right)-\frac{u}{T}\frac{2\lambda\theta}{\gamma(u)+\lambda}t, \tag{3.9}$$

$$\beta(u,t)=-\frac{u}{T}\frac{2(1-e^{-\gamma(u)t})}{\gamma(u)+\lambda+(\gamma(u)-\lambda)e^{-\gamma(u)t}}, \tag{3.10}$$

$$\delta(u,t)=\int_0^t \kappa_J(-\beta(u,s))ds, \tag{3.11}$$

where $\gamma(u)=\sqrt{\lambda^2+2\sigma^2\frac{u}{T}}$. In the OU specification (3.4), the realized variance V_T can be explicitly written as

$$V_T=v_0\epsilon(T)+\int_0^T \epsilon(T-t)dJ_t, \tag{3.12}$$

where

$$\epsilon(t)=\frac{1-e^{-\lambda t}}{\lambda T}, \tag{3.13}$$

and the expression for κ_V simplifies to

$$\kappa_V(u,T)=-u\,\epsilon(T)v_0+\int_0^T \kappa_J(u\,\epsilon(t))dt. \tag{3.14}$$

Based on the explicit form of $\mathcal{L}_V$, the tauberian theory allows for a comfortable analysis of the distributional properties of the realized variance. In particular, we are interested in analyzing how alternative selections of the jump law J_1 affect V_T and, in turn, the asymptotic behavior of the volatility curve $I(K)$ implied by realized variance options. Consistent with the notation adopted so far, we write F_J, F_V for the distribution functions of J_1, V_T and $\overline{F}_J$, $\overline{F}_V$ for the corresponding tail functions.

We start with a preliminary lemma stating that moment finiteness and regularly varying tail are properties which J_1 passes on to V_T basically unchanged. The proof is based on a careful examination of the ODEs (3.6)–(3.8), and makes use of the tauberian equivalence (2.12). Although quite simple, the derivation is somewhat lengthy and therefore the details are reported in the appendix.

Lemma 3.1. *In the SVJ-v model* (3.2) *the following hold.*

(i) *If* $p\geq 0$, *then* $\mathbb{E}\left[J_1^p\right]<\infty$ *if and only if* $\mathbb{E}\left[V_T^p\right]<\infty$.

(ii) *If* $\overline{F}_J\in R_{-\rho}$ *with* $\rho>0$ *noninteger, then* $\overline{F}_V\in R_{-\rho}$. *In the OU subclass* (3.4), *the statement holds for arbitrary* $\rho>0$.

Statement (i) shows that to guarantee $\mathbb{E}\left[V_T\right]<\infty$ and enable a meaningful analysis of options written on V_T, we need to assume

$$\mathbb{E}[J_1]<\infty \quad \text{or, equivalently,} \quad \int_0^\infty x\nu_J(dx)<\infty. \tag{3.15}$$

In addition, we see that denoting

$$p_J = \sup \left\{ p : \mathbb{E}[J_1^{p+1}] < \infty \right\}, \tag{3.16}$$

Lee's formula (2.3) at large strikes holds with index $p_V = p_J$. Proposition 3.2 below illustrates further how the tail properties of the jump distribution determine the behavior of the implied volatility $I(K)$ at large strikes.

Proposition 3.2. *In the SVJ-v model (3.2) the following hold.*

(a) *Suppose that $\overline{F}_J \in R_{-\rho-1}$ with $\rho > 0$ noninteger. Then the large strikes asymptotic equivalence (2.4) holds with $p_V = \rho$. In the OU subclass (3.4), the statement holds for arbitrary $\rho > 0$.*

(b) *Suppose that $\overline{F}_J$ is exponentially dominated*

$$\overline{F}_J(x) \le \gamma e^{-\alpha x} \quad \textit{for all} \quad x \ge x_0$$

with α, $\gamma > 0$ and x_0 large enough. Then $p_V = \infty$ and $I(K) \to 0$ as $K \to \infty$.

Proof. Statement (a) follows directly from Proposition 2.2(a) and Lemma 3.1(ii). As for statement (b), (2.13) implies that $\kappa_J(u_0) < \infty$ for a $u_0 < 0$. A simple inspection of (3.9)–(3.11) shows that there exists $u_1 < 0$ such that $\kappa_V(u_1) < \infty$, and the conclusion follows from Proposition 2.2(b). ■

Let us now examine the impact of the jump distribution on the implied volatility at small strikes. It turns out that in SVJ-v specifications comprising a diffusion component, jumps have no effect on the asymptotic behavior of the left wing of $I(K)$, which always vanishes to zero.

Proposition 3.3. *Consider the SVJ-v model (3.2) with $\sigma > 0$. Then, for any choice of jump distribution J_1, it holds that $q_V = \infty$ and $I(K) \to 0$ as $K \to 0$.*

Proof. Let $s = \sqrt{\frac{2}{T\sigma^2}}$. From (3.9), (3.10), we see that $\lim_{u\to\infty} \frac{\alpha(u,T)+\beta(u,T)}{u^{1/2}} = -s$. Furthermore, $\beta(u,t) \ge -s\, u^{1/2}$ for all $u, t \ge 0$ and from (3.11) it follows that

$$T\kappa_J(s\, u^{1/2}) \le \delta(u,T) \le 0 \quad \text{for all } u \ge 0.$$

Since $\frac{1-e^{-u^{1/2}sx}}{u^{1/2}} \le \min(sx, 1)$ for all $u \ge 1$, we can use a dominated convergence argument to show that $\lim_{u\to\infty} \frac{\kappa_J(su^{1/2})}{u^{1/2}} = 0$, so that $\lim_{u\to\infty} \frac{\delta(u,T)}{u^{1/2}} = 0$. All in all, it holds that $\lim_{u\to\infty} \frac{\log \mathcal{L}_V(u)}{u^{1/2}} = -s$ and the result follows from de Bruijn's tauberian equivalence (2.19) and Proposition 2.4(b). ■

To obtain a more flexible behavior of $I(K)$ at small strikes, we need to stay within the OU model subclass. However, from expression (3.12) we see that V_T is bounded from below by $v_0\epsilon(T)$ and, therefore, the asymptotic analysis as $K \to 0$ is meaningful only if $v_0 = 0$.

Proposition 3.4. *Consider the OU model* (3.4) *with* $v_0 = 0$, *and assume* F_J *is continuous. Then, the following hold.*

(a) *Suppose that* $F_J(1/x) \in R_{-\rho}$ *with* $\rho > 0$. *Then the asymptotic equivalence* (2.15) *holds with small strikes index* $q_V = \rho T$.

(b) *Suppose that* $\log F_J(x) \sim -\alpha x^{-\beta}$ *as* $x \to 0$ *with* $\alpha, \beta > 0$. *Then* $q_V = \infty$ *and* $I(K) \to 0$ *as* $K \to 0$.

Proof. Part (a): By Karamata's equivalence (2.18), it holds that $\mathcal{L}_J(x) = x^{-\rho}\ell(x)$ with $\ell \in R_0$. Also, recall a result from [27, IV.2], stating that if $\ell \in R_0$, then for any $\delta > 0$, there are positive constants b and B such that

$$b\left(\frac{u+1}{v+1}\right)^{-\delta} \le \frac{\ell(u)}{\ell(v)} \le B\left(\frac{u+1}{v+1}\right)^{\delta} \quad \text{whenever } 0 \le v \le u < \infty. \tag{3.17}$$

Then, using the hypothesis $v_0 = 0$, we obtain from (3.14) that, for a $\xi > 0$, the following holds:

$$\lim_{x\to\infty}\frac{\mathcal{L}_V(\xi x)}{\mathcal{L}_V(x)} = \lim_{x\to\infty}\exp\int_0^T\left(\kappa_J\left(\xi\epsilon(t)x\right) - \kappa_J\left(\epsilon(t)x\right)\right)dt =$$
$$= \exp\int_0^T \log\left(\lim_{x\to\infty}\frac{\mathcal{L}_J\left(\xi\epsilon(t)x\right)}{\mathcal{L}_J\left(\epsilon(t)x\right)}\right)dt = \xi^{-\rho T},$$

where, to interchange limit and integral, we have applied a dominated convergence argument based on (3.17). Therefore, $\mathcal{L}_V \in R_{-\alpha T}$ and the statement follows from Proposition 2.4(a). As for assertion (b), set r and s as in (2.19) and use (3.14) to obtain

$$\lim_{x\to\infty}\frac{\log\mathcal{L}_V(x)}{x^r} = \int_0^T \lim_{x\to\infty}\frac{\kappa_J(x\epsilon(t))}{x^r}dt = -s\int_0^T \epsilon(t)^r dt,$$

where, once again, we have used a dominated convergence argument based on the monotonicity of κ_J. The conclusion follows from Proposition 2.4(b). ■

4. Numerical examples.

In this section we consider a selection of positive distributions of the jumps J and we illustrate how such a choice impacts the associated realized variance smile $I(K)$ for a fixed maturity T. The numerical examples we provide are based on the OU subclass (3.4) with $v_0 = 0$, as this specification, in contrast to the full SVJ-v model, allows for both an upward-sloping and a downward-sloping behavior of the left wing. For the unit-time jump distribution J_1, we consider the gamma, the inverse gamma, and the generalized inverse Gaussian laws and we refer to the corresponding model specifications as the OU-Γ, OU-$I\Gamma$, and OU-GIG models.

Recall that a gamma distribution $\Gamma(\alpha, \beta)$ has density function f_Γ and Laplace transform $\mathcal{L}_\Gamma$ given by

$$f_\Gamma(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\mathbf{1}_{x\ge 0}, \qquad \mathcal{L}_\Gamma(u) = \left(1 + \frac{u}{\beta}\right)^{-\alpha}.$$

Since $0 \in \mathring{\mathcal{D}}_J$, Proposition 3.2(b) predicts that in the OU-Γ model, the smile $I(K)$ is downward sloping to zero as $K \to \infty$. Also, we see that $\mathcal{L}_\Gamma$ is a regularly varying function of index $-\alpha$

at infinity. By Proposition 3.4(a) we can conclude that $I(K) \to \infty$ as $K \to 0$, in accordance with the asymptotic equivalence (2.15) with index $q_V = \alpha T$.

In the OU-$I\Gamma$ specification, we choose J_1 distributed according to an inverse gamma law $I\Gamma(\nu, \mu)$. The density function $f_{I\Gamma}$ and Laplace transform $\mathcal{L}_{I\Gamma}$ are given by

$$f_{I\Gamma}(x) = \frac{\mu^\nu}{\Gamma(\nu)} x^{-\nu-1} e^{-\mu/x} \mathbf{1}_{x \geq 0}\,, \qquad \mathcal{L}_{I\Gamma}(u) = \frac{2(\mu u)^{\nu/2}}{\Gamma(\nu)} K_\nu(\sqrt{4\mu u}),$$

where K_ν is the modified Bessel function of the second kind. Since $f_{I\Gamma} \in R_{-\nu-1}$, we see from (2.10) that the corresponding survival function $\overline{F}_{I\Gamma} \in R_{-\nu}$. Hence, to price realized variance options in the OU-$I\Gamma$ model, we need to impose that $\nu > 1$. Furthermore, Proposition 3.2(a) shows that for large strikes, the associated volatility curve $I(K)$ follows the asymptotic equivalence (2.4) with $p_V = \nu - 1$. For the small strikes behavior, we can use the fact that for large arguments $K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$ (see [1, p. 378]), and show that $\lim_{u\to\infty} \frac{\log \mathcal{L}_{I\Gamma}(u)}{u^{1/2}} = -\sqrt{4\mu} < 0$. Therefore, Proposition 3.4(b), combined with de Bruijn's tauberian result (2.19), implies that $I(K) \to 0$ as $K \to 0$.

Finally, we consider the OU-GIG specification, where the law of J_1 is given by generalized inverse Gaussian distribution $GIG(p, a, b)$ with density and Laplace transform given by

$$f_{GIG}(x) = \frac{(b/a)^p}{2K_p(ab)} x^{\lambda-1} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)}, \qquad \mathcal{L}_{GIG}(u) = \left(\frac{b^2}{b^2+2u}\right)^{p/2} \frac{K_p(\sqrt{a^2(b^2+2u)})}{K_p(ab)}.$$

As above, K_p denotes the modified Bessel function of the second kind and it is immediate to see that $\lim_{u\to\infty} \frac{\log \mathcal{L}_{GIG}(u)}{u^{\frac{1}{2}}} < 0$. Also, $0 \in \mathring{\mathcal{D}}_{GIG}$, and in virtue of Proposition 3.2(b) and Proposition 3.4(b) we can conclude that $I(K) \to 0$ both at small and at large strikes.

In all the model specifications introduced above, the realized variance call price $C(K)$ and the associated implied volatility $I(K)$ can be computed by means of Fourier transform methods. In fact, [10] shows that the Laplace transform $\mathcal{L}_C$ of the call price $C(K)$ can be expressed as

$$\mathcal{L}_C(u) = \int_0^\infty e^{-uK} C(K) dK = \frac{\mathcal{L}_V(u) - 1}{u^2} + \frac{\mathbb{E}[V_T]}{u}. \tag{4.1}$$

Applying a Laplace inversion algorithm to (4.1) allows us to obtain prices of options on realized variance for a sequence of variance strikes. Figure 1 plots implied volatilities against variance strikes for the three selected models. We consider a maturity of 3 months and we set the parameters of the different jump distributions so that the mean and the variance are the same across the alternative specifications. In particular, we take $\alpha = 18$ and $\beta = 22.8$ in the gamma case, $\nu = 20$ and $\mu = 15$ for the inverse gamma, and $p = -0.5$, $a = 3.7697$, and $b = 4.7749$ for the GIG distribution. The value of λ is the same in all cases and it is equal to 8. The plots confirm that in the OU-Γ case, the implied volatility of variance smile is downward sloping, clearly at odds with the upward-sloping smile observed in variance markets. In contrast, the OU-$I\Gamma$ model implies an upward-sloping smile, and finally, in the OU-GIG specification, we observe a frown.

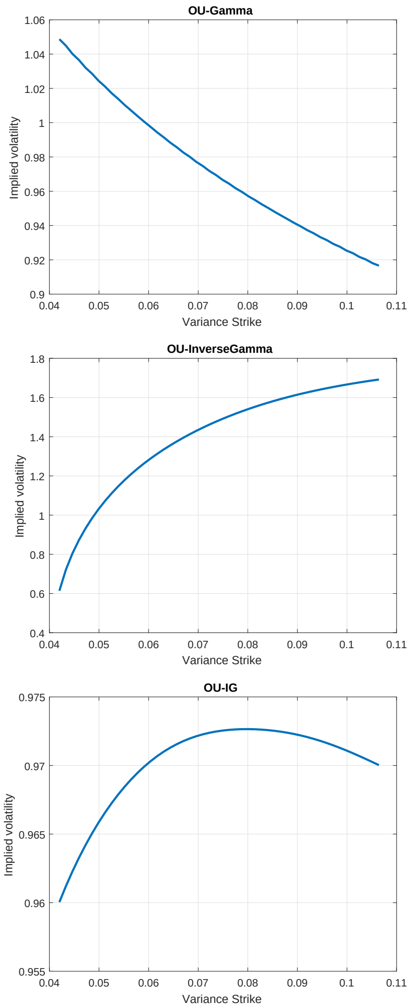
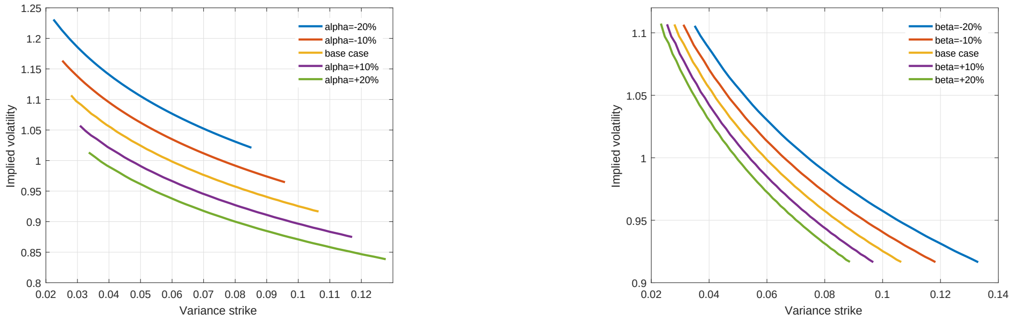


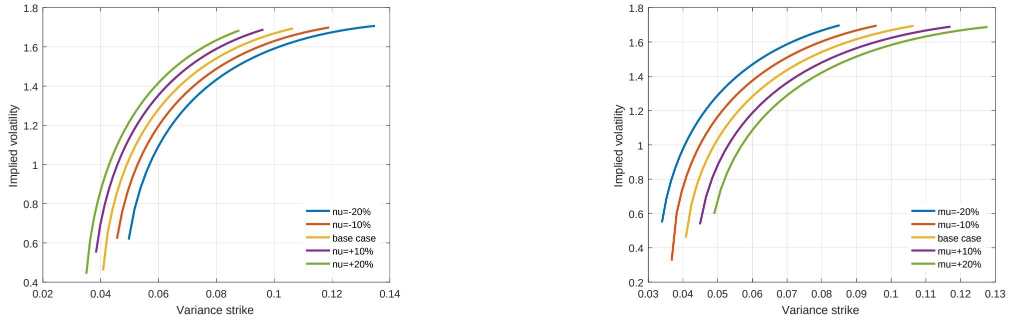
Figure 1. *Implied volatilities of variance for the OU-gamma with parameters $\alpha = 18$, $\beta = 22.8$ (top), OU-inverse gamma with $\nu = 20$, $\mu = 15$ (middle), and OU-IG with $a = 3.7697$, $b = 4.7749$ (bottom). In all cases we take $\lambda = 8$ and $v_0 = 0$ and we obtain the parameters of the different jump specifications by matching the mean and the variance.*

To further support these observations, we investigate the sensitivity of the implied volatility of variance with respect to the parameters of the jump distribution in Figures 2–4. To disclose the ceteris paribus effects on the implied volatility, we change one parameter while keeping any remaining parameters fixed. For each parameter set, we plot the implied volatility curve against variance strikes. For the selected jump distributions, we make the common observation that while altering the parameters of the distribution changes the levels of implied volatility and the wideness of the smile in variance strikes, the shapes of the implied volatility curves persist across different parameter values.



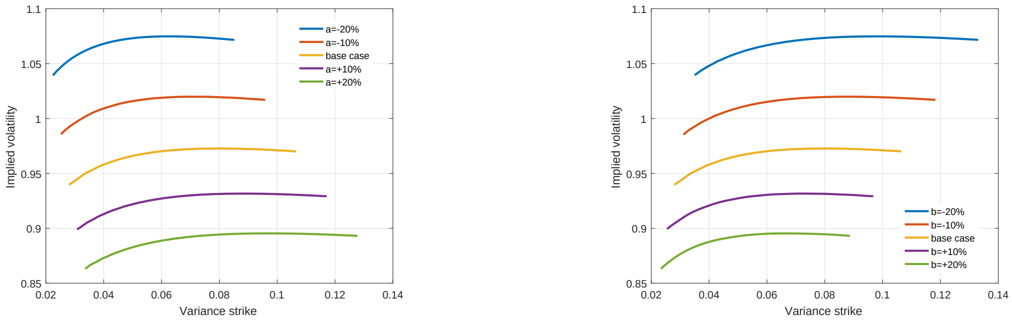
(a) Changes in α (b) Changes in β

Figure 2. *OU-gamma parameter sensitivities. Base case parameters:* $\alpha = 18$, $\beta = 22.8$, $\lambda = 8$, *and* $v_0 = 0$.



(a) Changes in ν (b) Changes in μ

Figure 3. *OU-inverse gamma parameter sensitivities. Base case parameters:* $\nu = 20$, $\mu = 15$, $\lambda = 8$, *and* $v_0 = 0$.



(a) Changes in a (b) Changes in b

Figure 4. *OU-IG parameter sensitivities. Base case parameters:* $a = 3.7697$, $b = 4.7749$, $\lambda = 12$, *and* $v_0 = 0$.

5. VIX options.

Since 2006 options have traded on CBOE's VIX index and constitute today a relatively liquid market of variance derivatives. The VIX index tracks the price of a portfolio of options on the S&P500 index (SPX index). As shown by [11], VIX squared approximates the conditional risk-neutral expectation of the realized variance of SPX over the next 30 calendar days. As such, it can be interpreted as the fair swap rate of a variance swap—an OTC contract in which one exchanges payments of realized variance against receiving a fixed variance swap rate.

It is immediate to show that under the general SVJ-v dynamics (3.2), the VIX squared—the price of future realized variance—is simply given by an affine transformation of the instantaneous variance

$$\mathrm{VIX}_T^2 = \mathbb{E}_T\left[\frac{1}{\tau}\int_T^{T+\tau} v_t dt\right] = a v_T + b, \tag{5.1}$$

where

$$a = \frac{1}{\lambda\tau}(1 - e^{-\lambda\tau}), \tag{5.2}$$

$$b = \left(\frac{\mathbb{E}[J_1]}{\lambda} + \theta\right)(1-a),$$

and $\tau = 30/365$.

Extending the previous analysis of realized variance options, we now examine the impact of the jump-distribution J on the price of options written on the VIX index. In the notation of section 2, we are therefore interested in the asymptotic behavior of the volatility curve $I(K)$ implied by options with underlying

$$H_T = \mathrm{VIX}_T = \sqrt{a v_T + b}.$$

We see from (5.2) that in the SVJ-v framework the VIX_T is bounded away from zero by the quantity b, for any parameter choice and any maturity T. Thus, we only consider the volatility curve at large strikes $K \to \infty$, as $I(K)$ is not defined for $K \to 0$.

Also, recall that the Laplace transform of the instantaneous variance $\mathcal{L}_v(u,T) = \mathbb{E}[e^{-u v_T}]$ is given by

$$\mathcal{L}_v(u,T) = \exp\left(\alpha(u,T) + v_0\beta(u,T) + \delta(u,T)\right), \tag{5.3}$$

where the functions α, β, and δ satisfy the ODEs

$$\frac{\partial}{\partial t}\alpha = \theta\lambda\beta, \tag{5.4}$$

$$\frac{\partial}{\partial t}\beta = -\lambda\beta + \frac{1}{2}\sigma^2\beta^2, \tag{5.5}$$

$$\frac{\partial}{\partial t}\delta = \kappa_J(-\beta), \tag{5.6}$$

with initial conditions $\alpha(u,0) = \delta(u,0) = 0$ and $\beta(u,0) = u$. Similarly to the realized variance case, a detailed analysis of the ODEs (5.4)–(5.6) reveals the close connection between the tail function $\overline{F}_J$ of the jump distribution J_1 and the tail function $\overline{F}_v$ of the instantaneous variance v_T. The main results are reported in Lemma 5.1 below. The derivations are omitted, as they follow closely the proof of Lemma 3.1.

Lemma 5.1. *In the SVJ-v model* (3.2) *the following hold.*
(i) *If $p \geq 0$, then $\mathbb{E}[J_1^p] < \infty$ if and only if $\mathbb{E}[v_T^p] < \infty$.*
(ii) *If $\overline{F}_J \in R_{-\rho}$ with $\rho > 0$ noninteger, then $\overline{F}_v \in R_{-\rho}$. In the OU subclass* (3.4)*, the statement holds for arbitrary $\rho > 0$.*

We see immediately that in order to price VIX options in the SVJ-v model, we need to assume that $\mathbb{E}[J_1^{1/2}] < \infty$. Furthermore, the large strikes Lee's formula (2.3) holds with index

$$p_{\mathrm{VIX}} = \sup\left\{p : \mathbb{E}\left[J_1^{\frac{p+1}{2}}\right] < \infty\right\}.$$

The analogue of Proposition 3.2 for VIX smiles at large strikes reads as follows.

Proposition 5.2. *In the SVJ-v model* (3.2) *the following hold.*
(a) *Suppose that $\overline{F}_J \in R_{-\rho}$ with $\rho > 1/2$ noninteger. Then the large strikes asymptotic equivalence* (2.4) *holds with $p_{VIX} = 2\rho - 1$. In the OU subclass* (3.4)*, the statement holds for arbitrary $\rho > 1/2$.*
(b) *Suppose that $\overline{F}_J$ is exponentially dominated. Then $p_{VIX} = \infty$ and $I(K) \to 0$ as $K \to \infty$.*

Proof. From (5.1), we see that

$$\overline{F}_{VIX}(x) = \overline{F}_v\left(\frac{x^2 - b}{a}\right).$$

From the definition of regular variation it follows that if $\overline{F}_v \in R_{-\rho}$, then $\overline{F}_{VIX} \in R_{-2\rho}$. Therefore part (a) follows immediately from Lemma 5.1 and Proposition 2.2(a). As for part (b), inspection of (5.4)–(5.6) shows that $\overline{F}_J$ is exponentially dominated if and only if $\overline{F}_v$ is exponentially dominated, which in turn implies that $\overline{F}_{VIX}$ is exponentially dominated. The conclusion follows from Proposition 2.2(b). ■

We conclude this section by providing some numerical examples illustrating the behavior of VIX smiles implied by different jump distributions. We consider the full SVJ-v specification obtained by augmenting the Heston model with compound Poisson jumps, i.e.,

$$J_t = \sum_{i=1}^{N(t)} Z_i, \quad Z_i \sim \text{independently and identically distributed } Z, \tag{5.7}$$

where $N(t)$ is a Poisson process with intensity ℓ while Z denotes the positive jump-size distribution. As mentioned in the introduction, this variance process, equipped with exponentially distributed jumps, has been used with the intent to capture the observed upward-sloping skew of VIX options. Here, besides this specification and the purely diffusive Heston model, we also

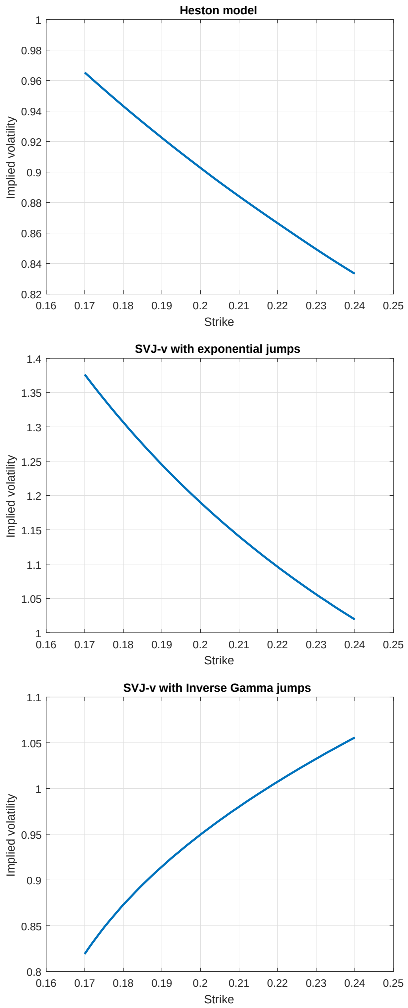


Figure 5. *Implied volatilities of VIX options for: the Heston model with* $v_0 = 0.0348$, $\lambda = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$ *(top), the SVJ-v model with exponential jumps with* $\ell = 1.5$ *and* $1/\beta = 0.3429$ *(middle), and the SVJ-v model with inverse gamma jumps with* $\nu = 4.5$ *and* $\mu = 1.2$ *(bottom). The jump parameters are chosen so that* $\mathbb{E}[Z]$ *is the same in both jump specifications.*

consider the case of inverse gamma jumps $Z \sim I\Gamma(\nu, \mu)$. Figure 5 plots the implied volatilities against strikes in the three different cases. We consider 3-month options and we use Heston parameters from [2], $v_0 = 0.0348$, $\lambda = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, obtained by calibration to out-of-the-money options on the $S\&P500$ index. One observes that, when using parameters fitted to equity option quotes, the VIX implied volatility skew is downward sloping. Next, we augment the Heston model with exponential jump sizes $Z \sim \Gamma(1, \beta)$ with mean $1/\beta = 0.3429$. The intensity is set to $\ell = 1.5$. Once again, we obtain a downward-sloping skew. However, if we maintain the same diffusive component, the same intensity level ℓ, but we substitute for the jump size with an inverse gamma law $Z \sim I\Gamma(\nu, \mu)$, we observe a dramatic change in the shape of the VIX implied volatility. Specifically, this leads to an upward-sloping skew as shown in Figure 5. The inverse gamma jump parameters are $\nu = 4.5$ and $\mu = 1.2$ and they have been obtained so that the first moment of Z is the same as in the exponential case.

6. Conclusion. We have considered options on the realized variance and the VIX index in the SVJ-v model, a tractable affine stochastic volatility model that generalizes the [24] model by augmenting it with jumps in the instantaneous variance. The model allowed us to isolate the unique impact of the jump distribution and we have shown that it has a profound effect on the characteristics and shape of the implied volatility of the variance smile. We provided sufficient conditions for the asymptotic behavior of the implied volatility of variance for small and large strikes. In particular, we showed that by selecting alternative jump distributions, one obtains fundamentally different shapes of the implied volatility smile. Some distributions of jumps predict implied volatilities of variance that are clearly at odds with the upward-sloping volatility skew observed in variance markets.

7. Appendix.

Proof of Lemma 3.1, part (i). We start by recalling a few basic facts relating the (possibly infinite) moments $\mathbb{E}[H^p]$, $p > 0$, of a nonnegative random variable H to its Laplace transform $\mathcal{L}_H$. Fix $p > 0$, $n \in \mathbb{Z}^+$, and r so that $n = p + r$ with $0 < r < 1$. Then the moments can be expressed as follows

$$\mathbb{E}[H^n] = (-1)^n \mathcal{L}_H^{(n)}(0+) \tag{7.1}$$

and

$$\mathbb{E}[H^p] = \frac{(-1)^n}{\Gamma(r)} \int_0^\infty u^{r-1} \mathcal{L}_H^{(n)}(u) du,$$

where we have employed the usual notation $f^{(n)}(u) = \frac{d^n}{du^n} f$. Here, as well as throughout the paper, we have implicitly used the fact that $\mathcal{L}_H$ is infinitely differentiable in the interior $\mathring{\mathcal{D}}$ of its domain $\mathcal{D}_H = \{u \in \mathbb{R} : \mathcal{L}_H(u) < \infty\}$, and

$$(-1)^n \mathcal{L}_H^{(n)}(u) = \mathbb{E}\left[H^n e^{-uH}\right] < \infty$$

for any $u \in \mathring{\mathcal{D}}$; see, e.g., [25, section 4.19.], Using the representation $\mathbb{E}[H^n e^{-uH}] = \mathbb{E}[1_{(H \le 1)} H^n e^{-uH}] + \mathbb{E}[1_{(H>1)} H^n e^{-uH}]$, and applying a dominated convergence argument to the two terms, we see that

$$(-1)^n \mathcal{L}_H^{(n)}(u) \to 0 \qquad \text{as} \qquad u \to \infty$$

for any nonnegative integer n. Therefore, for any fixed $u_0 > 0$, and $n > 0$, it holds that

$$(-1)^n \int_0^\infty u^{r-1} \mathcal{L}_H^{(n)}(u) du \le (-1)^n \int_0^{u_0} u^{r-1} \mathcal{L}_H^{(n)}(u) du + u_0^{r-1} (-1)^{(n-1)} \mathcal{L}_H^{(n-1)}(u_0),$$

and we can conclude that

$$\mathbb{E}[H^p] < \infty \iff (-1)^n \int_0^{u_0} u^{r-1} \mathcal{L}_H^{(n)}(u) du < +\infty. \tag{7.2}$$

Next, recall that applying the Faà di Bruno formula, we can express the nth derivative $\mathcal{L}_H^{(n)}$ in terms of Bell's polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ as follows:

$$\mathcal{L}_H^{(n)} = \mathcal{L}_H \cdot \sum_{k=1}^{n} B_{n,k}\left(\kappa_H^{(1)}, \kappa_H^{(2)}, \ldots, \kappa_H^{(n-k+1)}\right), \tag{7.3}$$

where $\kappa_H(u) = \log \mathcal{L}_H(u)$, and

$$B_{n,k}(x_1, x_2, \ldots) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{7.4}$$

and the sum is taken over all sequences $j_1, j_2, \ldots, j_{n-k+1}$ of nonnegative integers such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k,$$
$$j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

For the derivation of the Bell's polynomial version of the Faà di Bruno formula, we refer to [31] and [26]. For ease of reference, we collect the next few observations in the following lemma.

Lemma 7.1. *Fix $p > 0$, $n \in \mathbb{Z}^+$, and r so that $n = p + r$ with $0 < r < 1$. Assume that*

$$(-1)^m \kappa_H^{(m)}(u) \geq 0 \qquad \text{for all } u > 0 \text{ and } 1 \leq m \leq n. \tag{7.5}$$

Then the following hold.

(a) *For all $1 \leq k \leq n$, it holds that*

$$(-1)^n B_{n,k}\left(\kappa_H^{(1)}, \kappa_H^{(2)} \cdots \kappa_H^{(n-k+1)}\right) \geq 0. \tag{7.6}$$

(b) *For the integer nth moment it holds that*

$$\mathbb{E}[H^n] < \infty \iff (-1)^n \kappa_H^{(n)}(0+) < \infty \tag{7.7}$$

and, for the noninteger moment of order $p > 0$, it holds that

$$\mathbb{E}[H^p] < \infty \iff (-1)^n \int_0^{u_0} u^{r-1} \kappa_H^{(n)}(u) du < \infty, \tag{7.8}$$

where $u_0 > 0$ is arbitrary.

Proof. Notice that

$$(-1)^n B_{n,k}\left(\kappa_H^{(1)}, \kappa_H^{(2)}, \ldots\right) = \sum_{\substack{j_1+j_2+\cdots=k \\ j_1+2j_2+\cdots=n}} \frac{n!}{j_1! j_2! \cdots} (-1)^{(j_1+2j_2+\cdots)} \left(\frac{\kappa_H^{(1)}}{1!}\right)^{j_1} \left(\frac{\kappa_H^{(2)}}{2!}\right)^{j_2} \cdots$$
$$= \sum_{\substack{j_1+j_2+\cdots=k \\ j_1+2j_2+\cdots=n}} \frac{n!}{j_1! j_2! \cdots} \left(\frac{(-1)\kappa_H^{(1)}}{1!}\right)^{j_1} \left(\frac{(-1)^2\kappa_H^{(2)}}{2!}\right)^{j_2} \cdots$$

from which we see that statement (a) holds. As for statement (b), from (7.1), (7.3), and (7.6) it follows that

$$\begin{aligned}\mathbb{E}[H^n] < \infty \iff & \sum_{k=1}^{n} (-1)^n B_{n,k}\left(\kappa_H^{(1)}(0+), \kappa_H^{(2)}(0+), \dots\right) < \infty \\ \iff & (-1)^n B_{n,k}\left(\kappa_H^{(1)}(0+), \kappa_H^{(2)}(0+), \dots\right) < \infty \qquad \text{for all } 1 \le k \le n.\end{aligned}$$

The direct implication "$\Rightarrow$" in (7.7) follows from the fact that $B_{n,1}(x_1, x_2, \dots x_n) = x_n$. Since $(-1)^n \kappa_H^{(n)}(0+) < \infty \Rightarrow (-1)^m \kappa_H^{(m)}(0+) < \infty$ for all $1 \le m \le n-1$ the converse implication is immediate, and we can conclude that (7.7) holds. We now apply similar arguments to the case of a noninteger p. From (7.2), (7.3), and (7.6) it follows that

$$\begin{aligned}\mathbb{E}[H^p] < \infty \iff & \sum_{k=1}^{n} \int_0^{u_0} \mathcal{L}_H(u) u^{r-1} (-1)^n B_{n,k}\left(\kappa_H^{(1)}(u), \kappa_H^{(2)}(u), \dots\right) du < \infty \\ \iff & \int_0^{u_0} u^{r-1} (-1)^n B_{n,k}\left(\kappa_H^{(1)}(u), \kappa_H^{(2)}(u), \dots\right) du < \infty \qquad \text{for all } 1 \le k \le n.\end{aligned}$$

Once again, the direct implication $\Rightarrow$ in (7.8) is straightforward. For the converse, notice that

$$(-1)^n \int_0^{u_0} u^{r-1} \kappa_H^{(n)}(u) du < \infty \Rightarrow (-1)^n \int_0^{u_0} \kappa_H^{(n)}(u) du < \infty \Rightarrow (-1)^m \kappa_H^{(m)}(0+) < \infty$$

for all $1 \le m \le n-1$. Since $B_{n,k}(x_1, x_2, \dots)$, $k = 2, \dots, n$, do not depend on x_n, we see that

$$(-1)^n \int_0^{u_0} u^{r-1} \kappa_H^{(n)}(u) du < \infty \Rightarrow \int_0^{u_0} u^{r-1} (-1)^n B_{n,k}\left(\kappa_H^{(1)}(u), \kappa_H^{(2)}(u), \dots\right) du < \infty$$

for all $1 \le k \le n$, and we can conclude that (7.8) holds. ■

Consider now the unit-time law J_1 of the jump process. Differentiating expression (3.3) we obtain

$$(-1)^n \kappa_J^{(n)}(u) = \int_0^{\infty} x^n e^{-ux} \nu_J(dx) > 0 \quad \text{for all } u > 0, n \ge 1. \tag{7.9}$$

Thus, from Lemma 7.1 it follows that

$$\mathbb{E}[J_1^n] < \infty \iff \int_0^{\infty} x^n \nu_J(dx) < \infty \tag{7.10}$$

and, for a noninteger $p > 0$,

$$\begin{aligned}\mathbb{E}[J_1^p] < \infty \iff & (-1)^n \int_0^{u_0} u^{r-1} \kappa_J^{(n)}(u) du < \infty \\ \iff & \int_0^{\infty} x^p \gamma(x u_0; r) \nu_J(dx) < \infty, \end{aligned} \tag{7.11}$$

where $\gamma(x;r)$ denotes the incomplete gamma function $\gamma(x;r) = \int_0^x q^{r-1}e^{-q}dq$. In fact, both (7.10) and (7.11) can be obtained from well-known results linking the moments of an infinitely divisible distribution to those of its Lévy measure; see, e.g., [32, Corollary 25.8.], However, the structure of the integrated variance V_T is more complex, and we will resort to Lemma 7.1 to estimate its moments. First we show that κ_V fulfills the assumption (7.5). Differentiate expressions (3.5) and (3.6)–(3.8) to obtain that

$$\kappa_V^{(n)}(u,T) = \alpha^{(n)}(u,T) + v_0\beta^{(n)}(u,T) + \delta^{(n)}(u,T), \qquad u > 0, \tag{7.12}$$

where $\beta^{(1)}, \beta^{(2)}, \ldots$ solve the ODEs

$$\frac{\partial}{\partial t}\beta^{(1)} = -\lambda\beta^{(1)} + \sigma^2\beta\beta^{(1)} - \frac{1}{T}, \tag{7.13}$$

$$\frac{\partial}{\partial t}\beta^{(n)} = -\lambda\beta^{(n)} + \frac{1}{2}\sigma^2\sum_{i=0}^{n}\binom{n}{i}\beta^{(i)}\beta^{(n-i)}, \qquad n \geq 2, \tag{7.14}$$

with initial condition $\beta^{(1)}(u,0) = \beta^{(2)}(u,0) = \cdots = 0$, while $\alpha^{(n)}, \delta^{(n)}$ solve

$$\frac{\partial}{\partial t}\alpha^{(n)} = \theta\lambda\beta^{(n)}, \tag{7.15}$$

$$\frac{\partial}{\partial t}\delta^{(n)} = \frac{\partial^n}{\partial u^n}\left(\kappa_J(-\beta)\right) \tag{7.16}$$

with $\alpha^{(n)}(u,0) = \delta^{(n)}(u,0) = 0$ for all $n \geq 1$. Since $\beta \leq 0$, from (7.13)–(7.15) it follows that $(-1)^n\beta^{(n)} \geq 0$ and $(-1)^n\alpha^{(n)} \geq 0$ for any $n \geq 1$. As for $\delta^{(n)}$, set

$$L(u,t,x) = e^{\beta(u,t)x}$$

and integrate (7.16) to obtain

$$\begin{aligned}\delta^{(n)}(u,T) &= \int_0^T\int_0^\infty L^{(n)}(u,t,x)\nu_J(x)dxdt \\ &= \sum_{k=1}^{n}\int_0^T\int_0^\infty x^k L(u,t,x)B_{n,k}\left(\beta^{(1)}(u,t), \beta^{(2)}(u,t), \ldots\right)\nu_J(dx)dt, \end{aligned}\tag{7.17}$$

where, similar to (7.3), we have expressed the nth derivative $L^{(n)}$ in terms of Bell's polynomials. Since $(-1)^n\beta^{(n)} \geq 0$ for all $n \geq 1$, Lemma 7.1, part (a) implies that $(-1)^nL^{(n)}$ and $(-1)^n\delta^{(n)}$ are nonnegative functions. Therefore also $(-1)^n\kappa_V^{(n)}$ are all nonnegative functions. Next, we apply (7.7) in Lemma 7.1, to obtain

$$\mathbb{E}\left[V_T^n\right] < \infty \iff (-1)^n\kappa_V^{(n)}(0+,T) < \infty \iff (-1)^n\delta^{(n)}(0+,T) < \infty,$$

where the last equivalence follows from observing, e.g., from (3.9)–(3.10), that both $\alpha(u,T)$ and $\beta(u,T)$ are finite and infinitely differentiable in an open neighborhood of $u = 0$. Finally, since $\beta(0,t) = 0$, expression (7.17) shows that

$$(-1)^n\delta^{(n)}(0+,T) = \sum_{k=1}^{n}\int_0^\infty x^k\nu_J(dx)\cdot\int_0^T(-1)^nB_{n,k}\left(\beta^{(1)}(0,t), \beta^{(2)}(0,t), \ldots\right)dt,$$

and we can conclude that for n integer, statement (i) in Lemma 3.1 follows from (7.10).

Consider now $\mathbb{E}\big[V_T^p\big]$ for noninteger p. Based on similar arguments as above, Lemma 7.1 shows that

$$\mathbb{E}\big[V_T^p\big] < \infty \iff (-1)^n \int_0^{u_0} u^{r-1}\kappa_V^{(n)}(u,T)du < \infty$$

$$\iff (-1)^n \int_0^{u_0} u^{r-1}\delta^{(n)}(u,T)du < \infty. \tag{7.18}$$

Next, a simple inspection of the ODEs (3.7), (7.13), and (7.14) shows that for all $(u,t) \in \mathbb{R}_+^2$, the functions $\beta, \beta^{(1)}, \dots, \beta^{(n)}$ satisfy the following bounds

$$-u\,\epsilon(t) \le \beta(u,t) \le -u\left(1 - \frac{\sigma^2}{2\lambda^2 T}u\right)\epsilon(t)\,, \tag{7.19}$$

$$\left(1 - \frac{\sigma^2}{\lambda^2 T}u\right)\epsilon(t) \le (-1)\beta^{(1)}(u,t) \le \epsilon(t)\,, \tag{7.20}$$

$$0 \le (-1)^n\beta^{(n)}(u,t) \le b \qquad \text{for all } n \ge 1\,, \tag{7.21}$$

where $\epsilon(t)$ is the function given in (3.13), and $b > 0$ is a large enough constant.

In particular, the upper bounds in (7.21) imply that we can find a $\tilde{b} > 0$ large enough such that

$$(-1)^n B_{n,k}\left(\beta^{(1)}(u,t), \beta^{(2)}(u,t), \dots\right) \le \tilde{b} \qquad \text{for all } (u,t) \in \mathbb{R}_+^2$$

for all $1 \le k \le n$. Furthermore, choosing $u_0 < \frac{2\lambda^2 T}{\sigma^2}$ and setting $b_0 = u_0\frac{\epsilon(T)}{T}$, from (7.19) it follows that

$$\beta(u,t) \le -ub_0t \qquad \text{for all } (u,t) \in [0,u_0] \times [0,T].$$

Substituting these estimates into (7.17) we obtain that

$$\begin{aligned}
(-1)^n \int_0^{u_0} u^{r-1}\delta^{(n)}(u,T)du &\le \tilde{b}\sum_{k=1}^n \int_0^\infty \int_0^T \int_0^{u_0} x^k u^{r-1} e^{-xb_0tu}\,du\;dt\;\nu_J(x)dx \\
&= \frac{\tilde{b}}{b_0^r}\sum_{k=1}^n \int_0^\infty \int_0^T \int_0^{xb_0tu_0} x^{k-r}t^{-r}\zeta^{r-1}e^{-\zeta}d\zeta\,dt\,\nu_J(x)dx \\
&\le \frac{\tilde{b}}{b_0^r}\sum_{k=1}^n \left(\int_0^T t^{-r}dt\right)\int_0^\infty x^{k-r}\int_0^{xb_0Tu_0}\zeta^{r-1}e^{-\zeta}d\zeta\,\nu_J(x)dx \\
&= \frac{\tilde{b}T^{1-r}}{(1-r)b_0^r}\sum_{k=1}^n \int_0^\infty x^{k-r}\gamma(xb_0Tu_0;r)\,\nu_J(x)dx,
\end{aligned}$$

and in view of (7.11), (7.18) we can conclude that $\mathbb{E}[J_1^p] < \infty$ implies $\mathbb{E}\big[V_T^p\big] < \infty$. For the converse implication, notice that (7.19) implies

$$\beta(u,t) \ge -\frac{u}{\lambda T} \qquad \text{for all } (u,t) \in \mathbb{R}_+^2,$$

and choose $u_0 < \frac{\lambda^2 T}{\sigma^2}$ and set $a_0 = u_0 \frac{\epsilon(T)}{T}$, so (7.20) implies

$$(-1)\beta^{(1)}(u,t) \geq a_0 t \qquad \text{for all } (u,t) \in [0,u_0] \times [0,T].$$

Also recall that $B_{n,n}(x_1, x_2, \dots) = x_1^n$. Then, using the estimates above and expression (7.17), we obtain

$$\begin{aligned}(-1)^n \int_0^{u_0} u^{r-1}\delta^{(n)}(u,T)du &\geq \int_0^{u_0}\int_0^T\int_0^\infty u^{r-1}x^k e^{\beta(u,t)x}\left((-1)\beta^{(1)}(u,t)\right)^n \nu_J(dx)\,dt\,du \\ &\geq a_0^n \int_0^\infty\int_0^T\int_0^{u_0} x^n t^n u^{r-1} e^{-xu/\lambda T}\,du\;dt\;\nu_J(x)dx \\ &= \frac{a_0^n}{\lambda^r}\frac{T^{n+1-r}}{n+1}\int_0^\infty\int_0^{xu_0/T\lambda} x^{n-r}\zeta^{r-1}e^{-\zeta}d\zeta\,\nu_J(x)dx \\ &= \frac{a_0^n}{\lambda^r}\frac{T^{n+1-r}}{n+1}\int_0^\infty x^{n-r}\gamma(xu_0/T\lambda; r)\,\nu_J(x)dx,\end{aligned}$$

and from (7.11), (7.18) we see that $\mathbb{E}\left[V_T^p\right] < \infty$ implies $\mathbb{E}[J_1^p] < \infty$, which concludes the proof of part (i).

Proof of Lemma 3.1, part (ii). We start by proving the statement for the OU subclass (3.4). First, recall that for a positive and infinitely divisible distribution H with Lévy measure ν_H it holds that

$$\overline{F}_H(x) \in R_{-\rho} \quad \text{if and only if} \quad \nu_H(x,\infty) \in R_{-\rho} \tag{7.22}$$

and, in this case, $\overline{F}_H(x) \sim \nu_H(x,\infty)$; see, e.g., Remark 25.14 in [32]. Next, from (3.14) we see that

$$\begin{aligned}\kappa_V(u,T) &= -u\epsilon(T)v_0 + \int_0^T\int_0^\infty \left(e^{-u\epsilon(t)x} - 1\right)\nu_J(dx)\,dt \\ &= -u\epsilon(T)v_0 + \int_0^\infty \left(e^{-ux} - 1\right)\nu_V(dx),\end{aligned}$$

where ν_V is the measure defined by

$$\nu_V(x,\infty) = \int_0^T \nu_J(x/\epsilon(t),\infty)dt \qquad \text{for } x > 0. \tag{7.23}$$

Thus, the distribution of V_T is infinitely divisible with Lévy measure ν_V. In view of (7.22), we proceed to show that $\nu_J(x,\infty) \in R_{-\rho}$ implies $\nu_V(x,\infty) \in R_{-\rho}$. In (7.23), use (2.2) and apply the change of variable $z := x/\epsilon(t)$ in (7.23), to show that for $\xi > 0$, $\nu_V(\xi x,\infty)$ takes the form

$$\nu_V(\xi x,\infty) = \xi^{-\rho} x T \int_{x/\epsilon(T)}^\infty \ell(\xi z)\frac{z^{-\rho-1}}{z-\lambda T x}dz.$$

Since $\ell(\xi z)/\ell(z) \to 1$ as $z \to \infty$, for a given $\varepsilon > 0$ we can find z_* such that

$$(1-\varepsilon)\frac{z^{-\rho-1}}{z-\lambda T x}\ell(z) \leq \frac{z^{-\rho-1}}{z-\lambda T x}\ell(\xi z) \leq (1+\varepsilon)\frac{z^{-\rho-1}}{z-\lambda T x}\ell(z)$$

holds for all $z \geq x/\epsilon(T) \geq z_*$. So, integrating the chain of inequalities above, we see that

$$(1-\varepsilon)\xi^{-\rho}\nu_V(x,\infty) \;\leq\; \nu_V(\xi x,\infty) \;\leq\; (1+\varepsilon)\xi^{-\rho-1}\nu_V(x,\infty)$$

for any $x \geq x_* = z_*\epsilon(T)$, and we can conclude that $\nu_V(\xi x,\infty)/\nu_V(x,\infty) \to \xi^{-\rho}$ as $x \to \infty$.

Consider now the full SVJ-v specification (3.2) and assume $\overline{F}_J \in R_{-\rho}$ for a non integer $\rho > 0$. In view of the tauberian equivalence (2.12), albeit reparametrized, we aim to show that

$$\frac{\mathcal{L}_V^{(n)}(\xi u,T)}{\mathcal{L}_V^{(n)}(u,T)} \to \xi^{-r} \quad \text{as} \quad u \to 0 \tag{7.24}$$

for all $\xi > 0$, where $n \in \mathbb{Z}_+$, $0 < r < 1$, and $n = \rho + r$. Notice that (2.9) implies that $\int_0^\infty x^k \nu_J(dx) < \infty$ for $k = 1,\ldots,n-1$. Thus, using (7.3) and expressions (7.12), (7.17), we can represent $(-1)^n\mathcal{L}_V^{(n)}$ as follows:

$$(-1)^n\mathcal{L}_V^{(n)}(u,T) = \mathcal{L}_V(u,T)\,\mathcal{G}_V(u,T) + \ell_V(u,T), \tag{7.25}$$

where

$$\mathcal{G}_V(u,T) = \int_0^T \int_0^\infty x^n \left(-\beta^{(1)}(u,t)\right)^n e^{x\beta(u,t)}\nu_J(dx)dt, \tag{7.26}$$

while ℓ_V is such that $\ell_V(0+,T) < \infty$. Therefore, (7.24) is equivalent to

$$\frac{\mathcal{G}_V(\xi u,T)}{\mathcal{G}_V(u,T)} \to \xi^{-r} \quad \text{as} \quad u \to 0. \tag{7.27}$$

To show the above, fix $\xi > 0$, consider an arbitrary $0 < \varepsilon < 1$, and choose $u_\varepsilon < \min(1,\xi^{-1})\frac{\sigma^2}{\lambda^2 T}\varepsilon$. Then (7.19), (7.20) imply that, for all $u \leq u_\varepsilon$, the following holds:

$$\begin{aligned}(1-\varepsilon)\,\mathcal{G}_{\mathrm{OU}}(u,T) \leq{}& \mathcal{G}_V(u,T) \leq \mathcal{G}_{\mathrm{OU}}\left(u(1-\varepsilon),T\right),\\ (1-\varepsilon)\,\mathcal{G}_{\mathrm{OU}}(\xi u,T) \leq{}& \mathcal{G}_V(\xi u,T) \leq \mathcal{G}_{\mathrm{OU}}\left(\xi u(1-\varepsilon),T\right),\end{aligned}$$

where

$$\mathcal{G}_{\mathrm{OU}}(u,T) = \int_0^T \int_0^\infty x^n \epsilon(t)^n e^{-u\epsilon(t)x}\nu_J(dx)dt.$$

Notice that $\mathcal{G}_{\mathrm{OU}}$ corresponds to $\mathcal{G}_V$ defined in (7.25)–(7.26) when V_T is specified in the OU model. Therefore, $\mathcal{G}_{\mathrm{OU}}$ satisfies (7.27) for any $\xi > 0$, implying that

$$(1-\varepsilon)\,\xi^{-r} \leq \lim_{u\to 0}\frac{\mathcal{G}_V(\xi u,T)}{\mathcal{G}_V(u,T)} \leq \frac{1}{(1-\varepsilon)^{r+1}}\,\xi^{-r}.$$

Since ε is arbitrary, we see that $\mathcal{G}_V$ satisfies (7.27), which concludes the proof of part (ii).

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