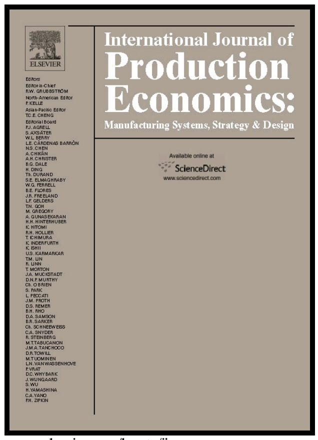

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An Integrated Production-Inventory Model for Deteriorating Items with Consideration of Optimal Production Rate and Deterioration during Delivery

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Abstract

Most of the literature of single-vendor single-buyer integrated production-inventory models for deteriorating items assumed a fixed production rate. Little attention has been paid to finding the optimal production rate for minimizing the total system cost. This paper investigates how production rate affects the total system cost, and develops a solution procedure for finding the optimal production rate for the traditional models. Based on the findings, this paper proposes an integrated single-vendor single-buyer model of an exponentially deteriorating item, in which non-stop production is considered and production rate is included as one of the decision variables. It has been shown, with numerical examples, that the proposed model can provide a lower cost solution than the traditional models which assume a fixed production rate. The proposed model also considers deterioration during deliveries, which is usually neglected in the literature of inventory models of deteriorating items. Furthermore, the proposed model is extended to relax the constant cost parameter assumption, which is prevalent even in non-constant production rate models, and optimize the cost for a system in which some of the cost parameters are production rate dependent.

Keywords: Supply Chain Management, Production-inventory Model, Coordination, Deteriorating items

1. Introduction

In the literature of inventory models of deteriorating items, production rate was usually fixed arbitrarily and much larger than the demand rate of the product. This paper investigates the effect of changing production rate on the system cost of a single-vendor single-buyer supply chain. It is found that in many cases, a lower production rate can result in a lower system cost. Hence, we propose a continuous production model in which the production rate is demand-driven. The results of our numerical experiments show that the model can achieve a lower system cost when compared with that of the traditional model which assumes an arbitrarily fixed production rate. The proposed model also considers deterioration during transportation which is usually ignored due to the general assumption of instantaneous shipments in most inventory models of deteriorating items. The model reduces the average

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inventory level and hence reduces the deterioration quantity. As deterioration results in wastage of resources that have adverse effects to environmental protection, this model also helps to address the issue of environmental concerns. Whereas cost parameters are usually assumed to be constant even in non-constant production rate models, the proposed model is extended to consider a scenario in which cost parameters increase when production rate decreases.

The remainder of this paper is organized as follows: Section 2 is a literature review of inventory models of deteriorating items from Economic Order Quantity models, Economic Production Quantity models, to integrated vendor-buyer models. Section 3 introduces the integrated vendor-buyer model. Section 4 considers the effect of changing production rate on the total system cost of the vendor-buyer model. From the findings, a solution procedure for finding the optimal production rate among a range of production rates, for minimizing the system cost is developed. In Section 5, an optimal production rate model, allowing deterioration during non-instantaneous transportation, is proposed for an integrated single-vendor single-buyer system. The solution procedure and an example of the proposed model are shown. A model in which deterioration and inventory holding costs increase when production rate decreases is then presented. Section 6 presents a procedure for choosing between the arbitrarily fixed production rate model and the proposed model for cost minimization. Section 7 is the conclusion of the study presented in this paper.

2. Literature Review

Ghare and Schrader (1963) used the term inventory decay for depletion of inventory by ‘other-than demand methods’, and developed an Economic Order Quantity (EOQ) model for an exponentially decaying item with a constant demand rate. Exponential deterioration is also referred as a constant rate (of the level of inventory) of deterioration in many models of deteriorating items. Covert and Philip (1973) developed an EOQ model for deteriorating items whose time to deterioration follows a two-parameter Weibull distribution. Later, Philip (1974) extended the model to deteriorating items following 3-parameter distribution. Misra (1975) developed the first Economic Production Quantity model and obtained an approximate relation between the length of the production time and that of the non-production time in a cycle for constant rate of deterioration.

After these pioneering works, researchers have presented EOQ and EPQ models of more complex scenarios. Shah (1977) developed an order-level lot-size model allowing shortages. Tadikamalla (1978) developed an EOQ model assuming the Gamma distribution for deterioration. Mak (1982) extended Misra’s EPQ model to one with backlogging for shortages. Park (1983) and Raafat (1985) presented EPQ models that include exponentially deteriorating raw materials and a non-deteriorating product. Goyal and Gunasekaran (1995) developed an EPQ model for maximizing the profit of a multi-stage production system. Balkhi and Benkherouf (1996) presented a production lot size inventory model for exponentially deteriorating items, in which the production rate and the demand rate are functions of time; whereas in Balkhi (1999)'s model, deterioration rates are also functions of time. In these non-constant production rate models, the cost parameters are assumed to be constant. Wee (1993), Chang and Dye (1999), Goyal and Giri (2003), Wu et al. (2006), Rajeswari and Vanjikkodi (2012), Chowdhury et al. (2014), Kumar and Rajput (2015) have proposed EOQ or EPQ models with partial backordering. Widyadana and Wee (2012) suggested an EPQ model for exponentially deteriorating items with multiple production setups followed by one rework setup in each cycle.

Jaggi et al. (2015a) and Jaggi et al. (2016a) presented fussy optimal ordering profit maximization models, for deteriorating items and non-deteriorating items, respectively, under conditions of permissible delay in payments. Jaggi et al. (2015b) proposed an EOQ model maximizing the retailer's profit for a two-warehouse system for a deteriorating item with imperfect quality. Jaggi et al. (2016b, 2016c) developed optimal ordering policies for non-instantaneous deteriorating items under credit financing and inflationary conditions, respectively, for systems with two storage facilities. Tiwari et al. (2016) considered both trade credit and inflation in their retailer's optimal replenishment policies for a two-warehouse system.

Researchers nowadays have put more emphasis on the optimization of the whole supply chain of supplier(s) and buyer(s). Yang and Wee (2000) developed an integrated vendor-buyer model for minimizing the total cost of a single-vendor single-buyer system for exponentially deteriorating items. In this model, the vendor starts production and makes the first shipment at the same instant. The authors applied Misra's approximate expression, for the relation between the length of the production period and the length of the non-production period for EPQ model, in their model in which shipments are delivered in lots. Jong and Wee (2008) presented a model in which production starts before the first shipment. Wee et al. (2008) provided an improved solution of the model. In addition to starting production at an appropriate instant before the first shipment, they also derived the formula for finding the length of the production period allowing multiple deliveries for the integrated single-vendor single-buyer lot-delivery model. In their solution procedure for finding the optimal solution, approximations to the exponential and the logarithm terms in the cost equation were made.

Some researchers assumed small deterioration rates and used an algebraic method in their production-inventory models for items with constant deterioration rates. Yan et al. (2011) proposed an algebraic method in deriving the inventory level and cost functions for their model, and presented a solution procedure for their model. Sarkar (2013) and Chang (2014) considered the same model with different solution procedures. These models assumed that the unit deterioration cost is the same for both the vendor and the buyer. However, it is more likely that the vendor and the buyer have different deterioration costs due to difference between the production cost and the purchase price, different scales of operations and some other factors.

Kim et al. (2014) developed a lot-for-lot delivery model for a supply chain using returnable transport items (RTIs) for shipments. In this model, empty RTIs were returned to the supplier with a stochastic return time approximated by an exponential distribution, and deterioration only occurred during stockouts of RTIs at the supplier due to late return. Deterioration during production and at the buyer's end was neglected.

3. Introduction to the Model

The assumptions for our single-vendor single-buyer coordinated model are:

(i) The item is deteriorating exponentially, that is, the rate of deterioration is a constant rate of the instantaneous inventory level of the item.

(ii) The demand rate and the cost parameters are constant. (The constant cost parameter assumption has been a general assumption in the literature of production-inventory models that do not consider quantity discounts, inflation, and time value of money.)

(iii) The production rate is constant, either fixed arbitrarily or determined by the procedure in the proposed model.

(iv) Shortages are not allowed.

(v) Shipments are made instantaneously (except in Section 5.4).

Notations for the parameters:

k: deterioration rate (fraction of the inventory level) of the item

D: demand rate of the buyer

A_b: ordering and other fixed cost per delivery of the buyer

C_b: unit deterioration cost of the buyer

H_b: inventory holding cost per unit per unit time of the buyer

P: production rate of the vendor

S: production set up cost of the vendor

A_v: vendor's order processing cost and shipment cost per delivery to the buyer

C_v: unit deterioration cost of the vendor

H_v: inventory holding cost per unit per unit time of the vendor

Other notations:

Q_0: delivery quantity

T: production cycle time of the arbitrarily fixed production rate model

T_p : production time in a production cycle of the arbitrarily fixed production rate model

T_c : delivery (ordering) cycle time of the buyer

n: number of deliveries in a production cycle of the buyer of the arbitrarily fixed production rate model

TC_b : total cost of the buyer per unit time

TC_v : total cost of the vendor per unit time

TC_s : total system cost per unit time

The inventory level of the buyer is shown in Fig. 1.

<< Insert Fig. 1 about here >>

The inventory level of the buyer is described by the following differential equation:

$$\frac{dI_b}{dt} = -kI_b - D. \tag{1}$$

Solving (1) and with the boundary condition $I_b = 0$ at $t = T_c$, the inventory level of the buyer is given by

$$I_b = \frac{D}{k}(e^{k(T_c - t)} - 1). \tag{2}$$

The order quantity is the inventory level at $t = 0$. Hence,

$$Q_0 = \frac{D}{k}(e^{kT_c} - 1). \tag{3}$$

Suppose the vendor has produced for a period of T_p without any shipment. The inventory level of the vendor, I_v, is described by the following equation:

$$\frac{dI_v}{dt} = -kI_v + P. \tag{4}$$

Solving (4) and with the initial condition $I_v = 0$ at $t = 0$, the inventory level of the vendor is given by

$$I_v = \frac{P}{k} - \frac{P}{k}e^{-kt}. \tag{5}$$

At $t = T_p$, $I_v = Q_v$. Hence,

$$Q_v = \frac{P}{k}(1 - e^{-kT_p}). \tag{6}$$

The inventory level of the vendor is shown in Fig. 2.

<< Insert Fig. 2 about here >>

If $T_p = T_c$ and $Q_v = Q_0$, equating (3) and (6) gives the corresponding production rate as

$$P = \frac{D(e^{kT_c} - 1)}{(1 - e^{-kT_c})} = De^{kT_c} . \tag{7}$$

At this production rate, the production time is as long as the delivery (ordering) cycle, and the quantity produced by the vendor is just sufficient to meet the buyer's requirement without shortages. In general, a production cycle includes a production period and a non-production period. The (arbitrarily fixed) production rate is set at a sufficiently large value satisfying the inequalities $P > De^{kT_c} = De^{kT/n}$ and $T_p < nT_c = T$. The decision variables are the number of deliveries in a production cycle and the production cycle time optimizing the total system cost per unit time. The effect of increasing/decreasing production rate on the total cost for the general case is analyzed in the following section.

4. How Production Rate Affects the Total System Cost

In the literature, there was little attention paid to the impacts of production rate on the total system cost. In this section, we will investigate how production rate affects the total system cost for a given production cycle time with production and non-production stages (i.e., $T_p < nT_c = T$) and a number of deliveries.

Wee et al. (2008) derived the following formula for the total relevant system cost per unit time for a single-vendor single-buyer system:

$$TC_s = \frac{S}{T} + \frac{n(A_v + A_b)}{T} + \frac{(H_b - H_v + kC_b - kC_v)nD}{kT}\left(\frac{1}{k}(e^{kT/n} - 1) - \frac{T}{n}\right) + \frac{(H_v + kC_v)(PT_p - DT)}{kT} \tag{8}$$

where the first term is the setup cost per unit time, the second term is the delivery related costs per unit time, the remaining two terms together are for inventory holding costs and deterioration costs per unit time, P is the arbitrarily fixed production rate, n is the number of deliveries in a production cycle of cycle time T, and T_p is the production time within a cycle given by

$$T_p = \frac{1}{k}\ln\left(1 + \frac{\frac{D}{P}(e^{kT} - 1)}{1 - \frac{D}{P}(e^{kT/n} - 1)}\right). \tag{9}$$

Wee et al. (2008) expanded the logarithmic term and obtained an approximate cost function in order to find the optimal solution for the problem. However, the condition for convergence of the expansion was not discussed. It can be shown (in Appendix A) that a sufficient condition for the convergence of the expansion of the logarithm function for the production time is given by $kT \le \ln 2 \approx 0.6931$ for any number of deliveries within a production cycle. In the literature of inventory models, production cycle time is usually not more than one year; and deterioration rates for exponentially deteriorating items in the numerical examples of most of these models are not more than 0.2 per year. Hence, the condition of $kT \le \ln 2$, where *T* is in years, is usually satisfied. In this paper, expansion for logarithmic terms is not needed as exact mathematical expressions are used. The assumption of $kT \le \ln 2$ is therefore not required. Instead, a less "stringent" requirement of $kT \le 0.863$ is assumed and will be explained.

Differentiate TC_s in (8) with respect to *P*, we obtain

$$\frac{\partial}{\partial P} TC_s = \frac{(H_v + kC_v)}{kT} \frac{\partial}{\partial P}(PT_p), \tag{10}$$

and

$$\frac{\partial}{\partial P}(PT_p) = \frac{1}{k}\left[\ln\left(1 + \frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT/n}-1)}\right) - \frac{\frac{D}{P}(e^{kT}-1)}{\left(1+\frac{D}{P}(e^{kT}-e^{kT/n})\right)\left(1-\frac{D}{P}(e^{kT/n}-1)\right)}\right]. \tag{11}$$

Hence, $\frac{\partial}{\partial P} TC_s$ has the same sign as $\frac{\partial}{\partial P}(PT_p)$ as k, H_v, C_v, and T are all positive.
Further differentiating (11) with respect to *P*,

$$\frac{\partial^2}{\partial P^2}(PT_p) = \frac{D}{P^2}(e^{kT}-1)\left(\frac{-\frac{D}{P}[e^{kT} - 2e^{kT/n} + 1 - 2\frac{D}{P}(e^{kT/n}-1)(e^{kT}-e^{kT/n})]}{\left(1+\frac{D}{P}(e^{kT}-e^{kT/n})\right)^2\left(1-\frac{D}{P}(e^{kT/n}-1)\right)^2}\right). \tag{12}$$

4.1 The Effect of One Delivery in a Production Cycle

Since $\frac{\partial}{\partial P}(PT_p) = \frac{1}{k}\left[\ln\left(1 + \frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT}-1)}\right) - \frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT}-1)}\right] < 0$ ($\because \ln(1+u) < u$), this implies that

$\frac{\partial}{\partial P} TC_s < 0$. Hence, the total system cost per unit time is a decreasing function of production rate for a given value of *T*.

4.2 The Effect of Two Deliveries in a Production Cycle

With given values for *T* and *n*, $P > De^{kT/n}$. At the limiting value of $P = De^{kT/n}$,

$$\frac{\partial}{\partial P}(PT_p) = \frac{1}{k}\left(\frac{kTe^{\left(1-\frac{1}{n}\right)kT} - e^{kT} + 1}{e^{\left(1-\frac{1}{n}\right)kT}} \right). \tag{13}$$

It can be shown that (in Appendix B):

(i) for $n = 2$, at $P = De^{\frac{kT}{2}}$, $\frac{\partial}{\partial P}(PT_p) < 0$;

(ii) for $n = 2$, $\frac{\partial}{\partial P}(PT_p)$ is maximized when $P = 2De^{\frac{kT}{2}}$;

(iii) for $n \geq 3$, at $P = De^{\frac{kT}{n}}$, $\frac{\partial}{\partial P}(PT_p) > 0$ for $0 < kT < 4.6223$;

(iv) for $n \geq 3$, $\frac{\partial^2}{\partial P^2}(PT_p) < 0$, i.e., $\frac{\partial}{\partial P}(PT_p)$ is a decreasing function of *P*, for $kT \leq 0.863$;

(v) for all $n \geq 1$, $\lim_{P \to \infty} \frac{\partial}{\partial P}(PT_p) = 0$.

Hence, from (i), (ii) and (v) above, for $n = 2$ and a given value of *T*, $\frac{\partial}{\partial P}(PT_p)$ behaves as follows:

(a) $\frac{\partial}{\partial P}(PT_p)$ is negative but increasing (i.e., getting less negative) in the interval $(De^{kT/2}, P^*)$ where P^* is the production rate at which $\frac{\partial}{\partial P}(PT_p) = 0$.

(b) $\frac{\partial}{\partial P}(PT_p)$ is positive and increasing in the interval $(P^*, 2De^{kT/2})$.

(c) $\frac{\partial}{\partial P}(PT_p)$ remains positive but is decreasing for production rates larger than $2De^{kT/2}$, and approaches 0 for very large production rates.

This means that for $n = 2$ and a given value of *T*, the total system cost per unit time is not a monotonic function of production rate. There is a particular production rate at which the total system cost per unit time is the minimum. This particular production rate, P^*, can be found by solving the equation $\frac{\partial}{\partial P}(PT_p) = 0$.

With the assumption of $T \le 1$, an upper bound for P^* can be found by substituting $n = 2$, $\rho = D/P$ and $T = 1$ in (11); and we have

$$\frac{\partial}{\partial P}(PT_p) = \frac{1}{k}\left[\ln\left(1 + \frac{\rho(e^k - 1)}{1 - \rho(e^{k/2} - 1)}\right) - \frac{\rho(e^k - 1)}{\left(1 + \rho(e^k - e^{k/2})\right)\left(1 - \rho(e^{k/2} - 1)\right)}\right]. \quad (14)$$

Assuming $\rho = \rho^*$ is the solution of (14), then for $T < 1$, $\frac{\partial}{\partial P}(PT_p) = 0$ is reached at a production rate smaller than D/ρ^*. Hence, for any $T \le 1$, the total system cost per unit time is an increasing function of production rate if $P > D/\rho^*$.

4.3 The Effect of Three or More Deliveries in a Production Cycle

Numerical experiments show that for $n \ge 3$, $\frac{\partial}{\partial P}(PT_p)$ is positive for any production rate larger than the minimum production rate $De^{kT/n}$ for a given *T*. This observation can be generalized and its validity can be established with the assumptions mentioned in Section 3 as follows:

From the results of (iii), (iv) and (v), for $n \ge 3$, $\frac{\partial}{\partial P}(PT_p)$ is positive at the smallest feasible production rate; it decreases when the production rate increases and remains positive and approaches 0 when the production rate is very large. Therefore, when $n \ge 3$, $k \le 0.863$ and $T \le 1$, (i.e., $kT \le 0.863$), $\frac{\partial}{\partial P}(PT_p)$ is positive and thus $\frac{\partial}{\partial P}TC_s$ is positive (see Equation 10), for any production rate larger than the minimum production rate. Hence, the total system cost per unit time is an increasing function of production rate for a given value of *T*.

4.4 Summary

For a given value of the production cycle time *T* and a given number of deliveries *n*, the effect of changing production rate on the total system cost per unit time is summarized as follows:

(#1) For $n = 1$, increasing production rate reduces the cost.

(#2) For $n = 2$, there exists a production rate that minimizes the cost.

(#3) For $n \ge 3$, increasing production rate increases the cost.

Based on these findings, the following proposition is developed to determine the optimal production rate for the concerned supply chain.

Proposition I (Highest/Lowest Production Rate for Cost Minimization)

If a single-vendor single-buyer system supplying an exponentially deteriorating item with a demand rate of *D* satisfies the following conditions:

(a) the rate of deterioration is not more than 0.863, i.e., $k \le 0.863$;

(b) the production cycle time is within one year, i.e., $T \le 1$; and

(c) the cost parameters are constant for production rates in the interval $[P_a, P_b]$ where $P_a \ge D/\rho^*$, ρ^* being the solution of equation (14);

then among all production rates in the interval $[P_a, P_b]$, the production rate giving the optimal total system cost per unit time is either P_a or P_b.

Proof:

For any production rate P_s in the interval $[P_a, P_b]$,

(i) Suppose the cost is optimal with $n = 1$ and $T = T_s^*$. Since $P_s < P_b$, we have $TC_s(P_s, 1, T_s^*) > TC_s(P_b, 1, T_s^*) \ge TC_s^*(P_b)$, where $TC_s^*(P_b)$ is the overall optimal cost at the production rate of P_b for all possible values of *n*.

The inequality $TC_s(P_s, 1, T_s^*) > TC_s(P_b, 1, T_s^*)$ is based on (#1).

(ii) Suppose the cost is optimal with $n = 2$ and $T = T_s^*$. Since $P_s > P_a \ge D/\rho^*$, we have $TC_s(P_s, 2, T_s^*) > TC_s(P_a, 2, T_s^*) \ge TC_s^*(P_a)$, where $TC_s{}^*(P_a)$ is the overall optimal cost at the production rate of P_a for all possible values of *n*.

The inequality $TC_s(P_s, 2, T_s^*) > TC_s(P_a, 2, T_s^*)$ is based on the discussion after equation (14) in Section 4.2.

(iii) Suppose the cost is optimal with $n \ge 3$ and $T = T_s^*$. Since $P_s > P_a$, we have $TC_s(P_s, n, T_s^*) > TC_s(P_a, n, T_s^*) \ge TC_s^*(P_a)$, where $TC_s^*(P_a)$ is the overall optimal cost at the production rate of P_a for all possible values of *n*.

The inequality $TC_s(P_s, n, T_s^*) > TC_s(P_a, n, T_s^*)$ is based on (#3).

Hence, if the cost parameters are constant for production rates in the interval $[P_a, P_b]$ for a given system satisfying the above conditions, the optimal production rate that minimizes the total system cost per unit time is either the lowest production P_a or the highest production rate P_b. It only requires finding the optimal costs for these two production rates. The production rate, P_a or P_b, that gives the smaller cost is the optimal production rate for the system. In the literature of production-inventory models, production rates are arbitrarily fixed and usually much higher than the demand rates. Hence, the minimum production rate requirement for P_a in condition (c) is usually satisfied, in addition to conditions (a) and (b). When a range of production rates is considered for a certain supply chain satisfying these three conditions, Proposition I indicates that the optimal production rate is either the smallest or the largest production rate of the concerned range. The overall optimal production rate is hence the one (largest or smallest) with smaller total system cost.

4.5 A Supplementary Heuristic

It has been assumed that $P_a \geq D/\rho^*$ in Proposition I. In case this is not valid, that is, $P^* = D/\rho^* > P_a$, then (ii) in the proof of Proposition I does not hold true. Therefore, to determine the optimal production rate, theoretically the optimal costs for all production rates between P_a and P^* have to be found for $n = 2$, and compared with the optimal costs for production rates P_a and P_b. This is intractable. A suggested heuristic is to consider "several" production rates within the concerned range. For example, with $D = 1000$ units per year and $k = 0.1$ per year, solving (14) gives $P^* \approx 1402$ units per year. Suppose $P_a = 1200$ units per year and $P_b = 3200$ units per year, the procedure for the proposed heuristic is as follows:

(a) Find $TC_s{}^*(P_a = 1200)$ and $TC_s^*(P_b = 3200)$, the overall optimal costs for the lowest and largest production rates being considered.

(b) Determine the "steps", *I*, of production rates. The "steps" depend on the magnitudes of production rates being considered and the precision needed.

  (i) Find $m = \left\lfloor \dfrac{P^* - P_a}{I} \right\rfloor$

  (ii) For $j = 1$ to *m*, set $P(j) = P_a + jI$ and find the optimal cost for the production rate of $P(j)$ for $n = 2$.

(iii) Find the smallest cost among the costs obtained in (ii) and the corresponding production rate. Suppose this production rate is $P(\ell)$. If the cost associated with $P(\ell)$ is not lower than the smaller cost obtained in (a), go to (d). Otherwise, go to (iv).

(iv) The difference between the costs for $P(\ell-1)$ and $P(\ell)$, and that between the costs for $P(\ell+1)$ and $P(\ell)$, serve as a measure of the precision.

(v) If the precision is acceptable, go to (c). Otherwise, set a smaller value for *I*, and repeat (i), (ii) (iii) and (iv), until the precision is acceptable, and go to (c).

(c) Compare the smallest cost obtained in (b) and the two costs obtained in (a). The production rate that gives the minimum of these three costs is taken as overall optimal production rate for the system. Stop.

(d) The production rate that gives the smaller cost in (a) is taken as the optimal production rate for the system. Stop.

5. A Vendor-buyer Continuous Production Model with Demand-driven Production Rate

5.1 The Model

The model discussed in Section 4 is a conventional production-inventory model having a non-production phase in a production cycle. However, there are situations that non-stop production for some periods of time is desired so as to meet an order urgently. The urgent order may be due to, for example, a tight shipment schedule, an urgent demand or special events, etc. Hence in this section, a model without non-production phase is proposed for this purpose. Furthermore, in the literature of inventory models for deteriorating items, deliveries are assumed to be instantaneous and therefore there is no deterioration during deliveries. The proposed model also considers deterioration during delivery for situations that this cannot be neglected.

The inventory levels of the vendor, the buyer and during transportation, respectively, of the proposed model are depicted in Fig. 3.

<< Insert Fig. 3 about here >>

It has been shown in Section 3 that by running at a production rate of De^{kT_c} and a delivery cycle of T_c, the supply chain is adopting a lot-delivery continuous production model with $T_p = T_c$. Suppose the delivery lead-time is T_T and the delivery occurs between $t=-T_T$ and $t=0$. The equation for the inventory change of the goods during delivery is given by $\frac{dI}{dt}=-kI$. The solution is $I=Q_0e^{-kt}$, and the delivery quantity required is $Q_T=Q_0e^{kT_T}$ at $t=-T_T$.

The vendor has to deliver a quantity of $Q_0 e^{kT_T}$ units and the buyer will receive Q_0 "good" units upon receiving the delivery. The production rate, P_2, required to result in an inventory level of $Q_0 e^{kT_T}$ units in a period of T_c can be found from the equation $\frac{P_2}{k}(1-e^{-kT_c}) = \frac{D}{k}(e^{kT_c}-1)e^{kT_T}$ which gives

$$P_2 = De^{k(T_c+T_T)} \tag{15}$$

The total cost function of the model is derived as follows:

From equation (2), the average inventory level of the buyer is given by

$$\frac{1}{T_c}\frac{D}{k}\int_0^{T_c}\left(e^{k(T_c-t)}-1\right)dt = \frac{D}{kT_c}\left(\frac{1}{k}(e^{kT_c}-1)-T_c\right). \tag{16}$$

The inventory holding cost per unit time is given by

$$\frac{H_b D}{kT_c}\left(\frac{1}{k}(e^{kT_c}-1)-T_c\right). \tag{17}$$

Quantity of deteriorated items per cycle is

$$Q_0 - DT_c = D\left(\frac{e^{kT_c}-1}{k}-T_c\right). \tag{18}$$

The cost of deteriorated items per unit time is

$$\frac{C_b D}{T_c}\left(\frac{e^{kT_c}-1}{k}-T_c\right). \tag{19}$$

The total relevant cost per unit time for the buyer, TC_b, is given by

$$TC_b = \frac{A_b}{T_c} + \left(\frac{H_b}{k}+C_b\right)\left(\frac{e^{kT_c}-1}{k}\right)\frac{D}{T_c} - \frac{H_b D}{k} - C_b D. \tag{20}$$

From equation (5), total inventory level of the vendor over a period of T_c is

$$\frac{P_2}{k}\int_0^{T_c}(1-e^{-kt})dt = \frac{P_2}{k}[T_c + \frac{1}{k}(e^{-kT_c}-1)]. \tag{21}$$

The quantity of deteriorated items is

$$P_2 T_c - Q_v = P_2\left(T_c - \frac{1-e^{-kT_c}}{k}\right). \tag{22}$$

Total relevant cost per unit time for the vendor, TC_v, is given by

$$TC_v = \frac{1}{T_c}\left[S + A_v + \left(\frac{H_v}{k} + C_v\right)\left(\frac{e^{-kT_c} - 1}{k}\right)P_2 + \left(\frac{H_v}{k} + C_v\right)P_2T_c \right]. \quad (23)$$

Theoretically the continuous production system can run with one production setup forever. In practice, maintenance of the production facilities is inevitable and manufacturing setup is required after the activity. In this model, the time unit is "year". It is assumed that there is one manufacturing setup every time unit and hence there is a setup cost of S per unit time.

With this assumption, (23) becomes

$$TC_v = S + \frac{1}{T_c}\left[A_v + \left(\frac{H_v}{k} + C_v\right)\left(\frac{e^{-kT_c} - 1}{k}\right)P_2 + \left(\frac{H_v}{k} + C_v\right)P_2T_c \right]. \quad (24)$$

Assuming the unit inventory holding cost and unit deterioration cost during delivery are the same as that for the vendor, i.e. borne by the vendor, the holding cost per unit time for deterioration during delivery is

$$\frac{H_v D}{k^2 T_c}(e^{kT_c} - 1)(e^{kT_T} - 1), \quad (25)$$

and the deterioration cost during delivery per unit time is

$$\frac{C_v}{T_c}(Q_0 e^{kT_T} - Q_0) = \frac{C_v D}{kT_c}(e^{kT_c} - 1)(e^{kT_T} - 1). \quad (26)$$

Adding the costs in (20), (24), (25) and (26), the total relevant system cost per unit time is

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)\left(\frac{e^{kT_c} - 1}{T_c}\right) + \frac{H_v D e^{kT_T} e^{kT_c}}{k} + C_v D e^{kT_T} e^{kT_c} - \frac{H_b D}{k} - C_b D + S. \quad (27)$$

Alternatively, if the unit inventory holding cost and unit deterioration cost during delivery are the same as those for the buyer, i.e. borne by the buyer, (27) becomes

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k} e^{kT_T}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)\left(\frac{e^{kT_c} - 1}{T_c}\right) + \frac{H_v D e^{kT_T} e^{kT_c}}{k} + C_v D e^{kT_T} e^{kT_c} - \frac{H_b D}{k} - C_b D + S \quad (28)$$

If deliveries are assumed to be instantaneous, substitute $T_T = 0$ into (27) or (28), the total relevant cost per unit time for the system, TC_s, is therefore

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)\left(\frac{e^{kT_c} - 1}{T_c}\right) + \frac{H_v D e^{kT_c}}{k} + C_v D e^{kT_c} - \frac{H_b D}{k} - C_b D + S \qquad (29)$$

The convexity of the cost function (27) for the special case of $C_b = C_v$ and $H_b = H_v$, and the general case of $C_b > C_v$ and $H_b > H_v$ are shown in Appendix C. As cost functions (28) and (29) are of the same form as (27), their convexity follows. Hence, the optimal cycle time can be found by solving the equation $\frac{d}{dT_c} TC_s = 0$.

5.2 Solution Procedure

For a single-vendor single-buyer supply chain with known demand rate *D* and cost parameters *S*, A_v, H_v, C_v, A_b, H_b, C_b, and deterioration rate *k* for the exponentially deteriorating product, the optimal solution for minimizing the total system cost per unit time can be found by the following steps:

Step 1: If $C_b = C_v$ and $H_b = H_v$, go to Step 4. Otherwise, go to Step 2.

Step 2: Set $m_1 = \frac{D}{k}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)$ and $m_2 = \frac{De^{kT_T}(H_v + C_v k)}{k^2}$, if cost function (27) prevails;

$m_1 = \frac{D}{k} e^{kT_T}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)$ and $m_2 = \frac{De^{kT_T}(H_v + C_v k)}{k^2}$, if cost function (28) prevails.

Let $x = kT_c$ and the expression on the left hand side of equation (A.3) in Appendix III can be rewritten as $f(x) = m_1[(x-1)e^x + 1] + m_2 x^2 e^x - (A_b + A_v)$.

Step 3: Solve $f(x) = 0$ for *x* and the optimal cycle time is given by $T_c^* = x / k$. Go to Step 6.

Step 4: Let $x = T_c$ and equation (A.2) in Appendix III can be written as

$f(x) = D(H_v + C_v k)e^{kT_T} x^2 e^{kx} - (A_b + A_v)$.

Step 5: Solve $f(x) = 0$ for *x* and the optimal cycle time is given by $T_c^* = x$

Step 6: Set the production rate at $P^* = De^{k(T_c^* + T_T)}$. Hence, the delivery quantity is set as $Q_0^* = \frac{D}{k} e^{kT_T}(e^{kT_c^*} - 1)$ to be shipped at intervals of T_c^* and The optimal total system cost per unit time can be found by substituting $T_c = T_c^*$ into (27), (28) or (29) as appropriate.

5.3 Numerical Example and Discussion of Results

Wee et al. (2008) provided a numerical example for their arbitrarily fixed production rate model. The following example adopts the same parameters used by the authors in order to compare the performances of their and our proposed models.

Example 1:

D = 1000 units per year

P = 3200 units per year (arbitrarily fixed production rate)

k = 0.1 per year

S = \$400

$A_b + A_v$ = \$25

C_b = \$50

C_v = \$40

H_b = \$5 per unit per year

H_v = \$4 per unit per year

The results of our proposed model with $T_T = 0.02$ are summarized in Table 1. If the unit inventory holding cost and unit deterioration cost during delivery are the same as those for the vendor, the optimal cycle time is 0.05253 year, the optimal production rate is 1007.28 units per year, and the optimal cost is \$1510.89 per year. If the unit inventory holding cost and unit deterioration cost during delivery are the same as those for the buyer, the optimal cycle time is 0.05252 year, the optimal production rate is 1007.28 units per year, and the optimal cost is \$1551.04 per year.

Table 1 also shows the results of our proposed model with a deterioration rate of $k = 0.2$ per year. The optimal cycle time decreases slightly when deterioration during transportation is considered; and the optimal cost increases as expected.

<< Insert Table 1 about here>>

A comparison of the performances of our proposed model and that of Wee et al. (2008) is shown in Table 2. Assuming no deterioration during delivery (i.e. instantaneous delivery), the optimal cost for $k = 0.1$ per year is \$1349.89 in our proposed model, whereas the optimal cost of Wee et al. (2008) is \$2695.69 for the production rate = 3200 units per year. With the same cost parameters and deterioration rates of $k = 0.1$ and $k = 0.2$, the optimal solutions for Wee et al. (2008) with production rates 3200, 2500 and 4000 units per year are also shown in Table 2.

<< Insert Table 2 about here>>

These results indicate that by having lower inventory related costs and lower set up costs, the proposed model results in a lower total system cost per unit time although the number of deliveries has been increased. The optimal costs for the production rates of 2500, 3200, and 4000 units per year illustrates (#3) in section 4.4: for $n \geq 3$, increasing production rate increases the cost. The proposed model uses a smaller production rate, and gives a lower optimal cost than these arbitrarily fixed production rates. (The proof will be shown in Proposition II.)

The model in Wee et al. (2008) follows the traditional approach that a production cycle includes a non-production phase and the production rate is arbitrarily fixed. Our proposed model presented in Section 5.1 does not have a non-production phase and its production rate is one of the decision variables for minimizing total system cost. It is found that the optimal production rate of our model is much lower than that of Wee et al. (2008). Our proposed model with non-stop production is applicable when a company has to meet urgent demands of a product. The concerned production facilities are basically fully utilized during the period of non-stop production. Production planning and labour allocation are therefore easier than those of the traditional model which adopts intermittent production, i.e. a non-production phase is considered.

5.4 An Extended Model of Production Rate Dependent Cost Parameters

In the literature of inventory models in which special discounts, time value of money/inflation are not considered, cost parameters are usually assumed to be constant. For example, Balkhi and Benkherouf (1996), Balkhi (1999), Goyal and Giri (2003), and Kumar and Rajput (2015) mentioned in Section 2 are non-constant production rate models with constant cost parameters.

Depending on the operation of the production system and accounting methods adopted, some cost elements may be related to production rate. Labour cost per unit is independent of production rate if the amount of labour is proportional to production rate, but is inversely proportional to production rate if the same amount of labour is involved regardless of production rate. Machine rates and overhead per unit are independent of production rate if they are set as fixed monetary amount per unit, but are inversely proportional to production rate if they are set as fixed amount per unit time.

In this extended model, production cost is assumed to be partly constant and partly inversely proportional to production rate. Deterioration costs and inventory holding costs are assumed to be proportional to the production cost. With these assumptions, deterioration costs and inventory holding costs are functions of production rates as follows:

$$C_b = C_{ba} + \frac{C_{bb}}{P} \qquad C_v = C_{va} + \frac{C_{vb}}{P}$$

$$H_b = H_{ba} + \frac{H_{bb}}{P} \qquad H_v = H_{va} + \frac{H_{vb}}{P}$$

$$C_{ba} \geq C_{va} \qquad C_{bb} \geq C_{vb}$$

$$H_{ba} \geq H_{va} \qquad H_{bb} \geq H_{vb}$$

Substituting the above cost functions into (29), the total system cost per unit time is given by

$$\begin{aligned} TC_s = {} & \frac{A_b + A_v}{T_c} + \frac{D}{k}\left(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) + \frac{1}{ke^{kT_c}}\left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) + \\ & \frac{H_{va}De^{kT_c}}{k} + C_{va}De^{kT_c} + \frac{H_{vb}}{k} + C_{vb} - \frac{H_{ba}D}{k} - C_{ba}D - \frac{H_{bb}}{ke^{kT_c}} - \frac{C_{bb}}{e^{kT_c}} + S \\ = {} & \frac{1}{T_c}\left(A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k}\right) + \frac{D}{k}\left(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) - \\ & \left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right)\frac{1}{kT_c e^{kT_c}} + \left(\frac{H_{va}}{k} + C_{va}\right)De^{kT_c} - \left(\frac{H_{bb}}{k} + C_{bb}\right)\frac{1}{e^{kT_c}} + \\ & \frac{H_{vb}}{k} + C_{vb} - \frac{H_{ba}D}{k} - C_{bb}D + S \end{aligned} \tag{30}$$

Differentiating the cost function (30) with respect to T_c, and setting $x = kT_c > 0$, the following is obtained:

$$\frac{d}{dT_c}TC_s = \frac{1}{T_c^{\,2}e^x}\{-p_1e^x + p_2[(x-1)e^x + 1]e^x + p_3(x+1) + p_4x^2e^{2x} + p_5x^2\}, \text{ where}$$

$$p_1 = A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k} > 0, \quad p_2 = \frac{D}{k}\left(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va}\right) > 0,$$

$$p_3 = \frac{1}{k}\left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right) > 0, \quad p_4 = \frac{D(H_{va} + kC_{va})}{k^2} > 0, \text{ and } p_5 = \frac{H_{bb} + kC_{bb}}{k^2} > 0.$$

For finding the optimal cycle time T_c that minimizes the cost function (30), we set:

$$\frac{d}{dT_c}TC_s = \frac{1}{T_c^{\,2}e^x}\{-p_1e^x + p_2[(x-1)e^x + 1]e^x + p_3(x+1) + p_4x^2e^{2x} + p_5x^2\} = 0 \tag{31}$$

It can be shown (in Appendix D) that a unique solution of (31) exists.

5.4.1 Sensitivity Analysis

The split of the fixed component and the variable component (e.g. 1:9, 2:8 and 3:7, etc.) of each cost of concern may vary among companies as the split depends on the scale of the company and its accounting methods. Hence, a sensitivity analysis is conducted with Example 2 to investigate whether the split would affect the optimal solution of our extended model.

Example 2:

The following parameters from Example 1 are used:

D = 1000 units per year P = 3200 units per year

S = \$400 $A_b + A_v$ = \$25

C_b = \$50 C_v = \$40

H_b = \$5 per unit per year H_v = \$4 per unit per year

These deterioration costs and inventory holding costs are based on a production rate of 3200 units per year. Our sensitivity analysis is conducted by assigning 0.1, 0.2, …, 0.9 as the proportions of the fixed components of the concerned costs. The resulting values of C_{ba}, C_{va}, H_{ba} and H_{va}, etc. are shown in Table 3.

<< Insert Table 3 about here>>

For $k = 0.1$, the optimal solutions are presented in Table 4.

<< Insert Table 4 about here>>

The optimal total system cost per year with the fixed production rate of 3200 units per year in Wee et al. (2008) is \$2695.69. The proposed model with deterioration costs and holding costs related to production rate gives a smaller optimal cost even when the proportion of fixed components of the costs is as low as 0.1. When the proportion of the fixed components is higher, the deterioration costs and holding costs per unit are increased by smaller amounts. This results in a lower optimal cost and hence achieves a higher saving when compared with the arbitrarily fixed production rate model.

6. Propositions related to the Two Models

6.1 Proposition II

If a single-vendor single-buyer system supplying an exponentially deteriorating item with a demand rate of D satisfies the following conditions:

(a) the rate of deterioration is not more than 0.863, i.e., $k \le 0.863$;

(b) the system cycle time is within one year, i.e., $T \le 1$;

(c) the cost parameters are constant for production rates greater than or equal to D;

then the proposed non-stop production model always gives a better optimal cost than production rate P_s of the arbitrarily fixed production rate model if the cost for production rate P_s is optimal with $n \ge 3$.

Proof:

Suppose the cost for production rate P_s is optimal with $n \ge 3$ and $T = T_s^*$.

Then $P_s > De^{kT_s^*/n}$ and hence $TC_s{}^*(P_s, n, T_s^*) > TC_s(De^{kT_s^*/n}, n, T_s^*)$ due to (#3) in Section 3.4.

Consider the following two scenarios:

(i) the arbitrarily fixed production rate model with production rate $De^{kT_s^*/n}$ having n deliveries over a production cycle of cycle time T_s^*, and

(ii) the proposed model with delivery cycle time $T_c = T_s^* / n$ and production rate $De^{kT_s^*/n}$.

Both of these two scenarios have the same production rate and the same delivery cycle time. Therefore, they have the same delivery related costs, inventory holding costs and deterioration costs. The production setup cost per unit time of scenario (i) is S / T_s^*, and that of scenario (ii) is S.

Since $T_s^* \le 1$, we have $S \le S / T_s^*$, $TC_s(De^{kT_s^*/n}, n, T_s^*) \ge TC_{sD}(De^{kT_s^*/n}, T_c = T_s^* / n) \ge TC_{sD}{}^*$, where $TC_{sD}(De^{kT_s^*/n}, T_c = T_s^* / n)$ is the cost of our proposed model with production rate $De^{kT_s^*/n}$, and $TC_{sD}{}^*$ is the overall optimal cost for the proposed model for that system. Hence, $TC_s^*(P_s, n, T_s^*) > TC_{sD}^*$.

In Example 1, there are 4 or 5 deliveries, i.e., $n \ge 3$, in a production cycle in the optimal solutions, using the model of Wee et al. (2008), for different deterioration rates and arbitrarily fixed production rates (Table 2). Therefore, the proposed non-stop production model gives a smaller optimal cost for all these cases.

6.2 Proposition III

Given that a single-vendor single-buyer system satisfies the conditions in Proposition II. Suppose for the arbitrarily fixed production rate model of production rate P_s, the cost is optimal with $n = 2$. A sufficient condition for our proposed non-stop production model giving a smaller optimal cost is that $P_s \ge 2D^{\frac{k}{2}}$.

Proof:

Suppose at production rate P_s, the cost is optimal with $n=2$ and $T=T_s^*$.

$$TC_s(P_s,2,T_s^*)=\frac{S}{T_s^*}+\frac{2(A_v+A_b)}{T_s^*}+\frac{(H_b-H_v+kC_b-kC_v)2D}{kT_s^*}\left(\frac{1}{k}\left(e^{kT_s^*/2}-1\right)-\frac{T_s^*}{2}\right)+\frac{(H_v+kC_v)(P_sT_p-DT_s^*)}{kT_s^*} \quad (32)$$

where $T_p=\frac{1}{k}\ln\left(1+\frac{\frac{D}{P_s}(e^{kT_s^*}-1)}{1-\frac{D}{P_s}(e^{kT_s^*/2}-1)}\right)$.

And we have

$$TC_{sD}(T_c=T_s^*/2)=\frac{2(A_b+A_v)}{T_s^*}+\frac{2D}{k}\left(\frac{H_b}{k}+C_b-\frac{H_v}{k}-C_v\right)\left(\frac{e^{kT_s^*/2}-1}{T_s^*}\right)+\frac{H_vDe^{\frac{kT_s^*}{2}}}{k}+C_vDe^{\frac{kT_s^*}{2}}-\frac{H_bD}{k}-C_bD+S \quad (33)$$

where $TC_{sD}(T_c=T_s^*/2)$ is the cost of the continuous production model with production rate $De^{\frac{kT_s^*}{2}}$ and delivery cycle time $\frac{T_s^*}{2}$.

From (32) and (33), the difference of the costs, $TC_s(P_s,2,T_s^*)-TC_{sD}(T_c=\frac{T_s^*}{2})$, is given by

$$S\left(\frac{1}{T_s^*}-1\right)+\left(\frac{H_v}{k}+C_v\right)\left(\frac{P_sT_p}{T_s^*}-De^{kT_s^*/2}\right)\geq\left(\frac{H_v}{k}+C_v\right)\left(\frac{P_sT_p}{T_s^*}-De^{kT_s^*/2}\right). \quad (34)$$

Let $P_s=aDe^{kT_s^*/2}$, then $D/P_s=1/(ae^{kT_s^*/2})$. Hence, the expression of T_p following (32) can be written as $T_p=\frac{1}{k}\ln\left(1+\frac{(e^{kT_s^*}-1)/(ae^{kT_s^*/2})}{1-(e^{kT_s^*/2}-1)/(ae^{kT_s^*/2})}\right)=\frac{1}{k}\ln\left(1+\frac{e^{kT_s^*}-1}{(a-1)e^{kT_s^*/2}+1}\right)$. Hence,

$$\frac{P_sT_p}{T_s^*}-De^{kT_s^*/2}=De^{kT_s^*/2}\left(\frac{a}{kT_s^*}\ln\left(1+\frac{e^{kT_s^*}-1}{(a-1)e^{kT_s^*/2}+1}\right)-1\right). \quad (35)$$

If $a=2$, $\frac{a}{kT_s^*}\ln\left(1+\frac{e^{kT_s^*}-1}{(a-1)e^{kT_s^*/2}+1}\right)=\frac{2}{kT_s^*}\ln\left(\frac{e^{kT_s^*/2}+e^{kT_s^*}}{e^{kT_s^*/2}+1}\right)=\frac{2}{kT_s^*}\ln(e^{kT_s^*/2})=1.$

Hence, $\frac{P_sT_p}{T_s^*}-De^{kT_s^*/2}=0$, i.e. $P_sT_p=T_s^*De^{kT_s^*/2}$.

It has been shown in Section 4.2 that for $n=2$, $\frac{\partial}{\partial P}(PT_p)>0$ for $P>2De^{kT/2}$.

Hence, for $P_s>2De^{kT_s^*/2}$, $\frac{P_sT_p}{T_s^*}-De^{kT_s^*/2}>\frac{T_s^*De^{kT_s^*/2}}{T_s^*}-De^{kT_s^*/2}=0$,

and therefore, $TC_s(P_s,2,T_s^*)-TC_{sD}(T_c=T_s^*/2)>0$.

Finally, $TC_{sD}{}^*\leq TC_{sD}(T_c=T_s^*/2)<TC_s(P_s,2,T_s^*)$ where TC_{sD}^* is the overall optimal cost for the continuous production demand-driven production rate model for that system.

Corollary:

Combining Proposition II and Proposition III, the proposed non-stop production model gives a lower cost than that of an arbitrarily fixed production rate model with $P_s\geq 2De^{\frac{k}{2}}$, provided that the optimal solution for P_s is having $n\geq 2$.

6.3 Procedure for Model Selection for Cost Minimization

If one has to choose between a certain production rate P_s and the proposed non-stop production model, subject to the conditions of $k\leq 0.863$ and $T\leq 1$, the following indicates how the overall optimal cost can be found with minimum steps:

(a) if $P_s\geq 2De^{\frac{k}{2}}$, the overall optimal cost is the smaller cost of

- (i) the optimal cost for the non-stop production model, and
- (ii) the optimal cost for the production rate P_s with $n=1$.

(b) if $P_s<2De^{\frac{k}{2}}$, the overall optimal cost is the smallest cost of

- (i) the optimal cost for the non-stop production model,
- (ii) the optimal cost for the production rate P_s with $n=1$,
- (iii) the optimal cost for the production rate P_s with $n=2$.

7. Conclusion

Most production-inventory research assumes a fixed production rate which may not be optimal for minimizing total system cost. In this paper, production rate is considered as a decision variable for an integrated vendor-buyer supply chain of an exponentially deteriorating item. How production rate affects the total cost of such a system has been studied in details in Section 4. It has been shown that if the concerned production rates are all larger than D/ρ^*, the optimal production rate can be found analytically (see Proposition I). On the other hand, if not all production rates being considered are larger than D/ρ^*, this paper proposes a heuristic for determining the "optimal" value. Therefore, for both cases, the optimal production rate can be found systematically and it could be considered by manufacturers if they wish to minimize total system cost.

In traditional production models of deteriorating items, production stops before a production cycle ends and therefore a non-production phase is included. In practice, there are situations that non-stop production for some periods of time is desired so as to meet an order urgently. The urgent order may be due to, for example, a tight shipment schedule, an urgent demand or special events, etc. As such, this paper also proposes, in Section 5, a model that considers non-stop production. In this model, the optimal production cycle (and hence the optimal production rate) can be found analytically. We have shown that this model, in many cases, can give a lower cost than the traditional model which has a non-production phase and adopts an arbitrarily fixed production rate. The non-production phase considered by traditional models may result in low utilization of production facilities and create manpower allocation problems due to intermittent production. The proposed model can help management tackle these problems. Our numerical results also show that the proposed model can achieve a lower inventory level and hence less deterioration than the traditional model. This implies that the proposed model can achieve a better environmental performance.

Furthermore, in the literature of inventory models for deteriorating items, deliveries are assumed to be instantaneous and therefore there is no deterioration during deliveries. However, in practice, this assumption may not be valid because transportation time of deteriorating items is often significant, e.g. cross-city deliveries. In view of this, our non-stop production model has been developed in a manner that it can also consider deterioration during transportation.

Cost parameters are usually assumed to be constant in the existing literature of production-inventory models. However, some cost parameters may be production rate dependent in practice. In order to cater for this practical concern, this paper presents an extended model in which deterioration cost and inventory holding cost consist of a fixed component and a variable component which is inversely proportional to production rate. It is found that our proposed model, with such production rate dependent costs, can also achieve a lower optimal cost than the traditional model that adopts an arbitrarily fixed production rate.

There are a few limitations in our proposed model. Firstly, it only considers a single product and a single buyer. Hence, one direction for future research is to extend the proposed single-vendor single-buyer model to a single-vendor multi-buyer supply chain for multi-product. Secondly, our model only considers instantaneous deterioration. So, another direction is to extend our proposed model to consider an integrated production-inventory model considering non-instantaneous deterioration on both the vendor's and the buyer's sides. Thirdly, our proposed model assumes a fixed and known demand rate. Extending the model to cater for stochastic demands can increase the applicability of our model. Lastly, the objective of our proposed model is to minimize total system cost without any constraint on the amount of deterioration. In view of increasing environmental concerns, there may be more stringent regulations in limiting the amount of deterioration generated in a supply chain. It would be amenable to extend our model to consider some constraints on deterioration.

Appendix A: Condition for Convergence in Section 4

A sufficient condition for the convergence of the expansion of the logarithmic term for the production time $T_p = \frac{1}{k}\ln\left(1 + \frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT/n}-1)}\right)$ can be found as follows:

For a delivery interval of $T_c = \frac{T}{n}$, the minimum production rate is $De^{\frac{kT}{n}}$ in order to satisfy the demand without shortages. For the arbitrarily fixed production rate model, the production rate must satisfy $P > De^{\frac{kT}{n}}$ due to non-continuous production.

Hence, $\frac{D}{P}(e^{kT}-1) > 0$, $1-\frac{D}{P}(e^{\frac{kT}{n}}-1) > 1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}-1) = e^{\frac{-kT}{n}} > 0$, and

$$1-\frac{D}{P}(e^{\frac{kT}{n}}-1)-\frac{D}{P}(e^{kT}-1) = 1-\frac{D}{P}(e^{\frac{kT}{n}}+e^{kT}-2) > 1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}+e^{kT}-2) = (2-e^{kT})e^{\frac{-kT}{n}}$$

If $e^{kT} \le 2$, or $kT \le \ln 2 \approx 0.6931$, then $1-\frac{D}{P}(e^{\frac{kT}{n}}-1)-\frac{D}{P}(e^{kT}-1) > 0$, and

$$0 < \frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)} < 1.$$

Hence, $kT \le \ln 2$ is a sufficient condition for the convergence of the expansion of the logarithmic term for any production rate larger than the demand-driven production rate for any number of deliveries.

Appendix B: Proof of the Results (i) to (v) in Section 4.2

With given values for T and n, $P > De^{\frac{kT}{n}}$ or $\frac{D}{P} < e^{\frac{-kT}{n}}$.

At the limiting value of $P = De^{\frac{kT}{n}}$,

$$\frac{\partial}{\partial P}(PT_p) = \frac{1}{k}\{\ln[1+\frac{e^{\frac{-kT}{n}}(e^{kT}-1)}{1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}-1)}] - \frac{e^{\frac{-kT}{n}}(e^{kT}-1)}{[1+e^{\frac{-kT}{n}}(e^{kT}-e^{\frac{kT}{n}})][1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}-1)]}\}$$

$$= \frac{1}{k}[\ln(1+e^{kT}-1) - \frac{e^{kT}-1}{e^{(1-\frac{1}{n})kT}}] = \frac{1}{k}(kT - \frac{e^{kT}-1}{e^{(1-\frac{1}{n})kT}}) = \frac{1}{k}(\frac{kTe^{(1-\frac{1}{n})kT}-e^{kT}+1}{e^{(1-\frac{1}{n})kT}}).$$

Let $y = kT$.

For $n = 2$,

$$kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 = ye^{\frac{y}{2}} - e^{y} + 1$$

At $y = 0$, $ye^{\frac{y}{2}} - e^{y} + 1 = 0$.

For $y > 0$, $\frac{d}{dy}(ye^{\frac{y}{2}} - e^{y} + 1) = e^{\frac{y}{2}}(1 + \frac{y}{2} - e^{\frac{y}{2}}) < 0$.

Hence, $kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 < 0$.

Therefore, for $n = 2$, at $P = De^{\frac{kT}{2}}$, $\frac{\partial}{\partial P}(PT_p) < 0$. (i)

For $n \geq 3$,

$$kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 \geq ye^{\frac{2y}{3}} - e^{y} + 1$$

Solving $ye^{\frac{2y}{3}} - e^{y} + 1 = 0$, $y = 0$ or $y = 4.6223$.

It can be easily verified that $ye^{\frac{2y}{3}} - e^{y} + 1 > 0$ for $0 < y < 4.6223$.

So for $0 < y < 4.6223$, $kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 > 0$.

Therefore, for $n \geq 3$, at $P = De^{\frac{kT}{n}}$, $\frac{\partial}{\partial P}(PT_p) > 0$. (iii)

The second derivative $\frac{\partial^2}{\partial P^2}(PT_p)$ can be found as follows:

$$\frac{\partial^2}{\partial P^2}(PT_p) = \frac{-\frac{D}{P^2}(e^{kT}-1)}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]}$$

$$-\frac{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)](-\frac{D}{P^2})(e^{kT}-1)\}}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}$$

$$+\frac{\frac{D}{P}(e^{kT}-1)\{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]\frac{D}{P^2}(e^{\frac{kT}{n}}-1)+[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)](-\frac{D}{P^2})(e^{kT}-e^{\frac{kT}{n}})\}}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-1}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]}+$$

$$\frac{1+(\frac{D}{P})^2(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]+1+(\frac{D}{P})^2(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}[e^{kT}-2e^{\frac{kT}{n}}+1-2\frac{D}{P}(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})]}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

When $n=2$,

$$\frac{\delta^2}{\delta P^2}(PT_p)=\frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}[e^{kT}-2e^{\frac{kT}{2}}+1-2\frac{D}{P}(e^{\frac{kT}{2}}-1)(e^{kT}-e^{\frac{kT}{2}})]}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{2}})]^2[1-\frac{D}{P}(e^{\frac{kT}{2}}-1)]^2}\}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}(e^{\frac{y}{2}}-1)^2(1-2\frac{D}{P}e^{\frac{y}{2}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{2}})]^2[1-\frac{D}{P}(e^{\frac{kT}{2}}-1)]^2}\}$$

So $\frac{\delta^2}{\delta P^2}(PT_p)$ and $(e^{\frac{y}{2}}-1)^2(1-2\frac{D}{P}e^{\frac{y}{2}})$ have opposite signs.

Let $\rho = \dfrac{D}{P}$, $y = kT$ and $h(y) = (e^{\frac{y}{2}} - 1)^2 (1 - 2\rho e^{\frac{y}{2}})$.

$h(y) = (e^{\frac{y}{2}} - 1)^2 (1 - 2\rho e^{\frac{y}{2}}) = 0 \Rightarrow y = 0$ **or** $2\rho e^{\frac{y}{2}} = 1$, that is, $P = 2De^{\frac{y}{2}}$

When $P < 2De^{\frac{y}{2}}$, $h(y) < 0 \Rightarrow \dfrac{\delta^2}{\delta P^2}(PT_p) > 0$

When $P > 2De^{\frac{y}{2}}$, $h(y) > 0 \Rightarrow \dfrac{\delta^2}{\delta P^2}(PT_p) < 0$

Hence, $\dfrac{\delta}{\delta P}(PT_p)$ is maximum when $P = 2De^{\frac{y}{2}}$. (ii)

When $n \geq 3$, as $\dfrac{D}{P} < e^{\frac{-kT}{n}}$,

$$e^{kT} - 2e^{\frac{kT}{n}} + 1 - 2\frac{D}{P}(e^{\frac{kT}{n}} - 1)(e^{kT} - e^{\frac{kT}{n}}) > e^{kT} - 2e^{\frac{kT}{n}} + 1 - 2(1 - e^{\frac{-kT}{n}})(e^{kT} - e^{\frac{kT}{n}})$$

$$= 2e^{(1-\frac{1}{n})kT} - e^{kT} - 1$$

Let $f_1(y) = 2e^{(1-\frac{1}{n})y} - e^y - 1$ where $n \geq 3$.

$f_1(0) = 2 - 1 - 1 = 0$

$f_1'(y) = 2(1 - \dfrac{1}{n})e^{(1-\frac{1}{n})y} - e^y = e^{(1-\frac{1}{n})y}[2(1 - \dfrac{1}{n}) - e^{\frac{y}{n}})$

$2(1 - \dfrac{1}{n}) - e^{\frac{y}{n}} \geq 2(1 - \dfrac{1}{3}) - e^{\frac{y}{3}} > 0$ for $y \leq 0.863$

So $f_1'(y) > 0$ and hence, for $y > 0$, $f_1(y) > 0$ and $\dfrac{\partial^2}{\partial P^2}(PT_p) < 0$.

So $\dfrac{\delta}{\delta P}(PT_p)$ is a decreasing function of P for $kT \leq 0.863$. (iv)

For all $n \geq 1$, $\lim\limits_{P \to \infty} \dfrac{\delta}{\delta P}(PT_p) = \dfrac{1}{k}\{\ln[1 + \dfrac{0}{1}] - \dfrac{0}{[1+0][1-0]}\} = 0$ (v)

Appendix C: Proof of Convexity of the Cost Function (27) in Section 5.1

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}\left(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v\right)\left(\frac{e^{kT_c} - 1}{T_c}\right) + \frac{H_v De^{kT_T} e^{kT_c}}{k} + C_v De^{kT_T} e^{kT_c} - \frac{H_b D}{k} - C_b D + S$$

The derivative of the total cost per unit time is given by

$$\frac{d}{dT_c}TC_s = -\frac{A_b + A_v}{T_c^2} + \frac{1}{T_c^2}\frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + H_v De^{kT_T}e^{kT_c} + C_v Dke^{kT_T}e^{kT_c}$$

$$= \frac{1}{T_c^2}\{-(A_b + A_v) + \frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + D(H_v + C_v k)e^{kT_T}T_c^2 e^{kT_c}\}$$

Setting the derivative to zero,

$$\frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + D(H_v + C_v k)T_c^2 e^{kT_T}e^{kT_c} - (A_b + A_v) = 0 \quad \text{(C1)}$$

Case (i): $C_b = C_v$ and $H_b = H_v$

This may happen when both the vendor and the buyer belong to the same company; the buyer gets the produced goods at cost and the same unit holding cost is applicable to both parties.

$$\frac{d}{dT_c}TC_s = 0 \Rightarrow \frac{D(H_v + C_v k)(kT_c)^2 e^{kT_T}e^{kT_c}}{k^2} = A_b + A_v \quad \text{(C2)}$$

Since all the quantities are positive, $\frac{D(H_v + C_v k)(kT_c)^2 e^{kT_T}e^{kT_c}}{k^2} > 0$, is an increasing function and has no finite limit as $T_c \to \infty$, equation (C2) has a unique solution for any $A_b + A_v > 0$. The unique solution, found by solving the equation by numerical methods, gives the optimum cycle time for production and ordering. Order quantity can be calculated accordingly.

Case (ii): $C_b > C_v$ and $H_b > H_v$

This is the general case when the vendor sells the goods to the buyer with profit and the buyer's unit holding cost is higher than the vendor's unit holding cost.

Setting $m_1 = \frac{D}{k}(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)$ and $m_2 = \frac{De^{kT_T}(H_v + C_v k)}{k^2}$ into equation (C1),

$$\frac{d}{dT_c}TC_s = 0 \Rightarrow m_1[(kT_c - 1)e^{kT_c} + 1] + m_2(kT_c)^2 e^{kT_c} - (A_b + A_v) = 0. \quad \text{(C3)}$$

and $\frac{d}{dT_c}TC_s = \frac{1}{T_c^2}\{m_1[(kT_c - 1)e^{kT_c} + 1] + m_2(kT_c)^2 e^{kT_c} - (A_b + A_v)\}$.

Let $y_3 = m_1[(kT_c - 1)e^{kT_c} + 1] + m_2(kT_c)^2 e^{kT_c}$ where $m_1 \geq 0$, $m_2 > 0$, $T_c > 0$.

Then $y_3(0) = 0$, $\frac{dy_3}{dT_c} = m_1 k^2 T_c e^{kT_c} + 2m_2 k^2 T_c e^{kT_c} + m_2 k(kT_c)^2 e^{kT_c} > 0$, and $\lim_{T_c \to \infty} y_3 = \infty$.

Hence, there is a unique solution to equation (C3) for any $A_b + A_v > 0$.

$$\frac{d^2}{dT_c^2}TC_s = \frac{1}{T_c^4}\left\{m_1T_c[k^2T_c^2e^{kT_c} - 2\left((kT_c-1)e^{kT_c}+1\right)] + m_2k^3T_c^4e^{kT_c} + 2(A_b + A_v)T_c\right\}$$

Let $y_4 = k^2T_c^2e^{kT_c} - 2\left((kT_c-1)e^{kT_c}+1\right)$.

Then $y_4(0) = 0$ and $\frac{dy_4}{dT_c} = k^3T_c^2e^{kT_c} > 0$.

Hence, $\frac{d^2}{dT_c^2}TC_s > 0$

The total cost function is therefore convex and there is a unique delivery cycle time that minimizes the total system cost per unit time for the proposed continuous production model.

Appendix D: Proof of Convexity of the Cost Function (30) in Section 5.4

$$\frac{d}{dT_c}TC_s = \frac{1}{T_c^2}\{-p_1 + p_2[(x-1)e^x+1] + p_3(x+1)e^{-x} + p_4x^2e^x + p_5x^2e^{-x}\} = 0.$$

Set $h(x) = p_2[(x-1)e^x+1] + p_3(x+1)e^{-x} + p_4x^2e^x + p_5x^2e^{-x} - p_1$.

$$\frac{d}{dT_c}TC_s = 0 \Rightarrow h(x) = p_2[(x-1)e^x+1] + p_3(x+1)e^{-x} + p_4x^2e^x + p_5x^2e^{-x} - p_1 = 0. \qquad \text{(D1)}$$

As $(x-1)e^x+1>0$ for $x>0$, $p_2[(x-1)e^x+1] + p_3(x+1)e^{-x} + p_4x^2e^x + p_5x^2e^{-x} > 0$ and it is an increasing function for $x \le 1$ as shown below:

For $x \le 1$, (In general, $k<1$, and $T_c \le 1$. Hence $x \le 1$ suffices.)

$$\frac{d}{dx}\{p_2[(x-1)e^x+1] + p_3(x+1)e^{-x} + p_4x^2e^x + p_5x^2e^{-x}\}$$
$$= p_2xe^x - p_3xe^{-x} + p_4(x+2)xe^x + p_5(2-x)xe^{-x}$$
$$= p_2xe^x + p_4(x+2)xe^x - \frac{1}{k}\left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right)xe^{-x} + \frac{H_{bb}+kC_{bb}}{k^2}(2-x)xe^{-x}$$
$$= p_2xe^x + p_4(x+2)xe^x + \frac{1}{k}\left(\frac{H_{vb}}{k} + C_{vb}\right)xe^{-x} + \frac{H_{bb}+kC_{bb}}{k^2}(1-x)xe^{-x} > 0$$

Hence, we also have $h'(x) > 0$.

Since $\lim_{x\to\infty} p_2[(x-1)e^x+1] + p_4x^2e^x = \infty$, and

$\lim_{x\to\infty} p_3(x+1)e^{-x} + p_5 x^2 e^{-x} = \lim_{x\to\infty} \frac{p_3(1)}{e^x} + \lim_{x\to\infty} \frac{p_5(2)}{e^x} = 0$. (by L'Hopital's Rule)

Hence, $p_2[(x-1)e^x + 1] + p_3(x+1)e^{-x} + p_4 x^2 e^x + p_5 x^2 e^{-x}$ is positive, increasing and has no finite limit. There is a unique solution x^* for equation (31) for any $p_1 > 0$.

As $\frac{d}{dT_c} TC_s = \frac{h(x)}{T_c^2}$, $\frac{d^2}{dT_c^2} TC_s = \frac{T_c^2 h'(x) - 2T_c h(x)}{T_c^4}$.

Since $h(x^*) = 0$ and $h'(x^*) > 0$ as it is an increasing function, $\frac{d^2}{dT_c^2} TC_s > 0$ at $x = x^*$. Therefore, the unique solution of equation (31) gives the minimum total cost per unit time for the cost function (30).

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Tables:

Deterioration rate $k = 0.1$	No deterioration during delivery	Deterioration during delivery (Case 1)	Deterioration during delivery (Case 2)
Optimal cycle time (year)	0.0527	0.05253	0.05252
Optimal prod. rate (units/year)	1005.27	1007.28	1007.28
Optimal total cost per unit time ($ per year)	1349.89	1510.89	1551.04
Deterioration rate $k = 0.2$	No deterioration during delivery	Deterioration during delivery (Case 1)	Deterioration during delivery (Case 2)
Optimal cycle time (year)	0.04286	0.04278	0.04277
Optimal prod. rate (units/year)	1008.61	1012.635	1013.633
Optimal total cost per unit time ($ per year)	1564.30	1806.85	1867.23

Case 1: the unit inventory holding cost and unit deterioration cost during delivery are same as that for the vendor

Case 2: the unit inventory holding cost and unit deterioration cost during delivery are same as that for the buyer

Table 1: Results for having/not having deterioration during delivery (Example 1).

Deterioration rate $k = 0.1$	Arbitrarily fixed production rate (Wee et al. (2008) model)			Optimal production rate of our proposed model: 1005.27/ year
	2500/year	3200/year	4000/year	
No. of setups	2.4897	2.5712	2.7484	1
No. of deliveries	12.4483	12.8558	10.9937	19.0223
No. of deliveries in a production cycle	5	5	4	------
Inventory related cost	1304.23	1345.83	1369.32	474.33
Total annual cost	2611.30	2695.69	2743.53	1349.89
Deterioration rate $k = 0.2$	Arbitrarily fixed production rate (Wee et al. (2008) model)			Optimal production rate of our proposed model: 1008.61/year
	2500/year	3200/year	4000/year	
No. of setups	3.0498	3.1498	3.3672	1
No. of deliveries	15.2489	15.7492	13.4687	23.3340
No. of deliveries in a production cycle	5	5	4	------
Inventory related cost	1597.35	1648.30	1677.06	580.95
Total annual cost	3198.48	3301.97	3360.65	1564.30

Table 2: Comparison of Wee et al. (2008) model with our proposed model for instantaneous delivery (Example 1).

Proportion of fixed component	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
C_{ba}	5	10	15	20	25	30	35	40	45
C_{va}	4	8	12	16	20	24	28	32	36
H_{ba}	0.5	1	1.5	2	2.5	3	3.5	4	4.5
H_{va}	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6
C_{bb}	144000	128000	112000	96000	80000	64000	48000	32000	16000
C_{vb}	115200	102400	89600	76800	64000	51200	38400	25600	12800
H_{bb}	14400	12800	11200	9600	8000	6400	4800	3200	1600
H_{vb}	11520	10240	8960	7680	6400	5120	3840	2560	1280

$$C_b = C_{ba} + \frac{C_{bb}}{P} \qquad C_v = C_{va} + \frac{C_{vb}}{P}$$

$$H_b = H_{ba} + \frac{H_{bb}}{P} \qquad H_v = H_{va} + \frac{H_{v}}{P}$$

Table 3: Fixed and variable components of cost parameters for Example 2.

Proportion of fixed component	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0306	0.0318	0.0331	0.0346	0.0364	0.0385	0.0409	0.0439	0.0477
$De^{kT_c^*}$	1003.1	1003.2	1003.3	1003.5	1003.7	1003.9	1004.1	1004.4	1004.8
C_b	148.56	137.59	126.63	115.67	104.71	93.75	82.80	71.86	60.92
C_v	118.85	110.08	101.30	92.53	83.77	75.00	66.24	57.49	48.74
H_b	14.86	13.76	12.66	11.57	10.47	9.38	8.28	7.19	6.09
H_v	11.89	11.01	10.13	9.25	8.38	7.50	6.62	5.75	4.874
$TC_s^* - S$	1636.5	1575.0	1511.0	1444.1	1374.1	1300.3	1222.1	1138.5	1048.4
TC_s^*	2036.5	1975.0	1911.0	1844.1	1774.1	1700.3	1622.1	1538.5	1448.4

Table 4: Optimal solutions for Example 2.

Figures:

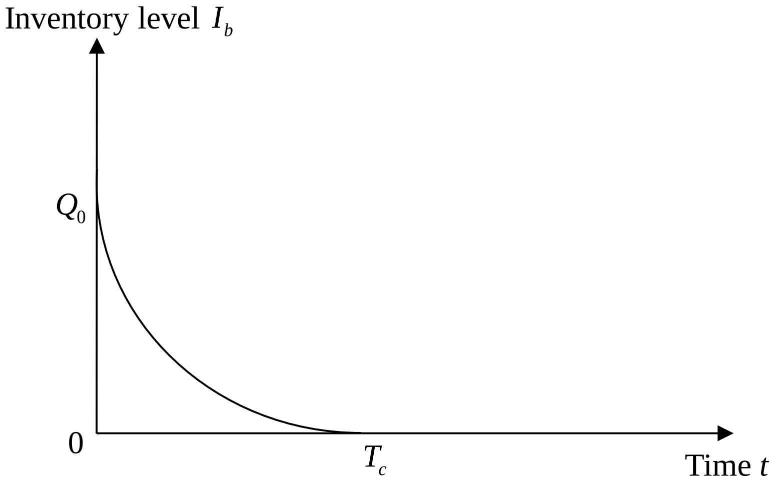

Fig. 1: Inventory level of the buyer.

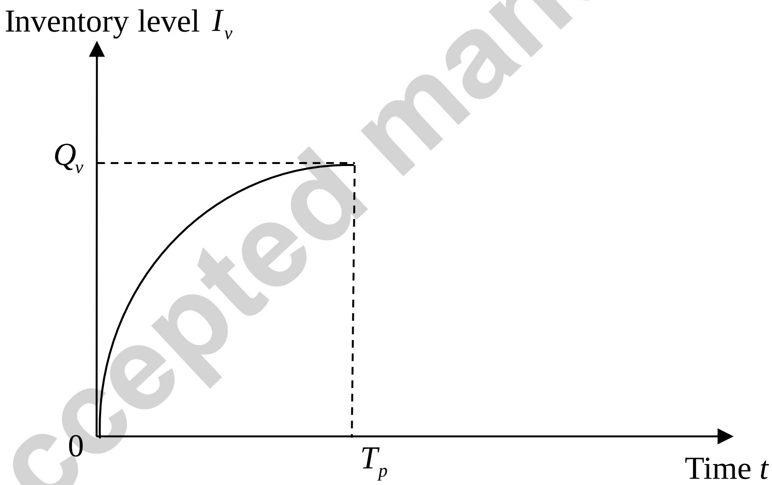

Fig. 2: Inventory level of the vendor.

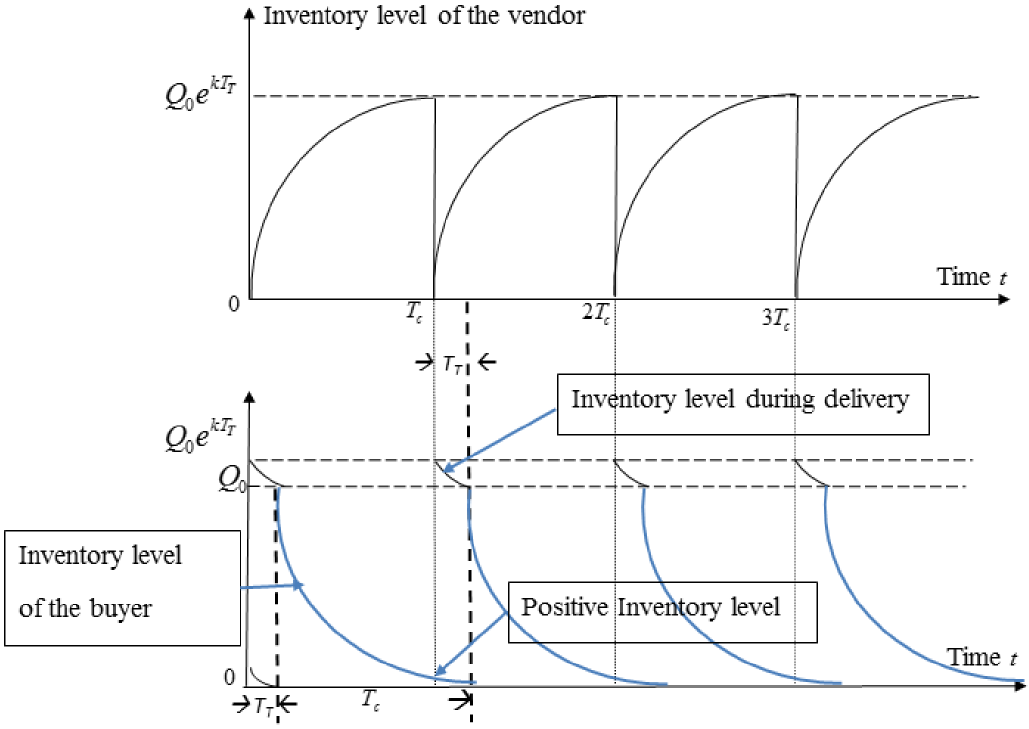

Fig. 3: The inventory levels of the vendor, that during delivery and that of the buyer.