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An easy method to derive EOQ and EPQ inventory models with backorders

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ABSTRACT

Recently, a cost minimization method to determine the lot size for the EOQ/EPQ models with backorders was published. This method is based on the well-known arithmetic–geometric mean inequality. Although the cost minimization method is correct and interesting, it does not focus on deriving the backorders level. This paper proposes another simple approach. The proposed method finds both the lot size and the backorders level.

1. Introduction

The lot size models have been studied extensively since the economic order quantity (EOQ) model was first introduced in 1913 by Harris [1]. The economic production quantity (EPQ) model was presented by Taft [2]. Later, the EOQ/EPQ models were extended to consider backorders. Harris [1] stated that in order to find the optimal solution for the EOQ it is necessary to know higher mathematics. Some scholars have attempted to develop the EOQ/EPQ models without using derivatives. Instead, they have utilized some approaches including tabular, graphical, marginal cost analysis, cost comparisons, arithmetic–geometric mean inequality (AGM), quasi-variational inequalities (QVI), and algebra.

Perhaps Grubbström [3] was the first researcher to develop the EOQ without derivatives. Instead, he employed algebraic optimization. Since then, Grubbström and Erdem [4], Cárdenas-Barrón [5–9], Yang and Wee [10], Chung and Wee [11], Wee et al. [12] and Lin et al. [13] have developed an EOQ model with backorders, an EPQ model without backorders, an EPQ model with backorders, a multi-stage multi-costumer supply chain model, an EPQ model with rework for a single-stage production system, an EOQ model with discount offer, an integrated vendor–buyer inventory system, an EOQ model with temporary sale price, a three-stage supply chain with backorders, and an imperfect quality EPQ with rework and backorders, respectively. All the previous works mentioned use algebraic optimization.

Recently, two optimization approaches appeared in the inventory literature: the cost comparisons and the arithmetic–geometric mean (AGM) inequality. These methods were developed by Minner [14] and Teng [15], respectively. However, the cost comparisons method does not focus on explicitly developing the mathematical expressions for the backorders

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level for the EOQ/EPQ models when backorders are allowed. Minner's method has been modified by Wee et al. [16]. The modified cost comparisons method is simpler than Minner's method. It is important to mention that the modified cost comparisons method does not develop the backorders level either.

Teng's AGM method is also very simple. However, it fails to solve the multi-variable problem. The AGM method can only determine the mathematical expression for the optimal lot size and, as with the cost comparisons methods, does not derive the mathematical expression for the backorders level.

It is important to note that Teng [15] is not the first researcher to apply the well-known AGM inequality in to optimization functions. It can be argued that Garver [17] and Niven [18] were the first researchers to use the AGM inequality in optimization functions. The application of the AGM inequality in optimizing an inventory function can be traced to [19–22]. The application of the Cauchy–Bunyakovsky–Schwarz (CBS) inequality in optimizing an inventory function can be traced to [23,24]. Beyer and Sethi [25] derive the EOQ model using quasi-variational inequalities (QVI).

To the best of our knowledge, the research papers by Minner [14], Wee et al. [16], and Teng [15] do not consider the optimization of the backorders level. This paper proposes another simple method which uses two well-known inequalities: the arithmetic–geometric mean inequality (AGM) and the Cauchy–Bunyakovsky–Schwarz (CBS) inequality.

2. Derivation of the lot size and the backorders level for the EOQ/EPQ models with backorders

This section provides a method to derive both the optimal lot size and the optimal backorders level for the EOQ/EPQ models with backorders taking into account the following two well-known inequalities: the arithmetic–geometric mean inequality (AGM) and the Cauchy–Bunyakovsky–Schwarz (CBS) inequality.

The AGM inequality. Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers, thus

$$\frac{\sum_{k=1}^{n} a_k}{n} \geq \sqrt[n]{\prod_{k=1}^{n} a_k}, \quad \text{with equality iff } a_1 = a_2 = a_3 = \cdots = a_n.$$

The CBS inequality. Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ be any two sets of real numbers, thus $\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2$, with equality iff the two sets of numbers are proportional: $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}$.

The following notation is used in both the EOQ/EPQ models with backorders presented in Sections 2.1 and 2.2:

d = demand rate per time unit,
A = ordering cost per order,
h = per unit holding cost per time unit,
v = per unit backorder cost per time unit,
p = production rate per time unit,
Q = order quantity,
B = backorders level.

2.1. The EOQ model with backorders

The total inventory cost function for the EOQ with backorders is given by:

$$TC(Q,B) = \frac{Ad}{Q} + \frac{h(Q-B)^2}{2Q} + \frac{vB^2}{2Q}. \tag{1}$$

Equivalently, the total inventory cost can also be written as:

$$TC(Q,B) = \frac{Ad}{Q} + \frac{Q}{2}\left\{h\left(1-\frac{B}{Q}\right)^2 + v\left(\frac{B}{Q}\right)^2\right\}, \tag{2}$$

and

$$TC(Q,B) = \frac{Ad}{Q} + \frac{Q}{2}\left\{\left[\sqrt{h}\left(1-\frac{B}{Q}\right)\right]^2 + \left[\sqrt{v}\left(\frac{B}{Q}\right)\right]^2\right\}\left\{\left[\frac{\sqrt{v}}{\sqrt{h+v}}\right]^2 + \left[\frac{\sqrt{h}}{\sqrt{h+v}}\right]^2\right\}. \tag{3}$$

Eq. (3) is required in order to apply the CBS inequality. By applying the CBS inequality to Eq. (3), one obtains:

$$TC(Q,B) \geq \frac{Ad}{Q} + \frac{Q}{2}\left\{\left(\frac{\sqrt{hv}}{\sqrt{h+v}}\right)\left(1-\frac{B}{Q}\right) + \left(\frac{\sqrt{hv}}{\sqrt{h+v}}\right)\left(\frac{B}{Q}\right)\right\}^2. \tag{4}$$

With equality iff

$$\frac{\sqrt{h}\left(1-\frac{B}{Q}\right)}{\frac{\sqrt{v}}{\sqrt{h+v}}}=\frac{\sqrt{v}\left(\frac{B}{Q}\right)}{\frac{\sqrt{h}}{\sqrt{h+v}}}. \tag{5}$$

Eq. (5) holds iff the two set of numbers are proportional according to the CBS inequality (for instance, see [26, pp. 5]).

Thus, Eq. (4) becomes:

$$TC(Q)=\frac{Ad}{Q}+\frac{Q}{2}\left\{\frac{hv}{h+v}\right\}. \tag{6}$$

In Eq. (6), there are actually two functions: $\frac{Ad}{Q}$ and $\frac{Q}{2}\left\{\frac{hv}{h+v}\right\}$. Thus, in order to apply the AGM inequality, the product of the functions has to be a constant. Fortunately, this is the case because the product is $\frac{Ad}{2}\left\{\frac{hv}{h+v}\right\}$. Therefore, the easy-to-use AGM inequality can be applied to solve the minimization problem in (6). By applying the AGM inequality to Eq. (6) yields:

$$TC(Q)\geq\sqrt{\frac{2Adhv}{h+v}} \tag{7}$$

with the equality iff

$$\frac{Ad}{Q}=\frac{Q}{2}\left\{\frac{hv}{h+v}\right\}. \tag{8}$$

Eq. (8) holds iff the product of the functions is a constant (for instance, see [17, pp. 435]).

Then it follows from Eqs. (5) and (8) that the optimal backorders level and the optimal lot size are given by:

$$B^{*}=\frac{hQ^{*}}{h+v} \tag{9}$$

and

$$Q^{*}=\sqrt{\frac{2Ad\,(h+v)}{hv}}. \tag{10}$$

It follows immediately from Eq. (7) that the optimal total inventory cost is:

$$TC^{*}=\sqrt{\frac{2Adhv}{h+v}}. \tag{11}$$

2.2. The EPQ model with backorders

Now, consider the total inventory cost function for the EPQ with backorders given by:

$$TC(Q,B)=\frac{Ad}{Q}+\frac{h\,[Q\,(1-d/p)-B]^{2}}{2Q\,(1-d/p)}+\frac{vB^{2}}{2Q\,(1-d/p)}. \tag{12}$$

Equivalently, the total inventory cost can also be written as:

$$TC(Q,B)=\frac{Ad}{Q}+\frac{Q\,(1-d/p)}{2}\left\{h\left(1-\frac{B}{Q\,(1-d/p)}\right)^{2}+v\left(\frac{B}{Q\,(1-d/p)}\right)^{2}\right\}, \tag{13}$$

and

$$TC(Q,B)=\frac{Ad}{Q}+\frac{Q\,(1-d/p)}{2}\left\{\left[\sqrt{h}\left(1-\frac{B}{Q\,(1-d/p)}\right)\right]^{2}+\left[\sqrt{v}\left(\frac{B}{Q\,(1-d/p)}\right)\right]^{2}\right\}\times\left\{\left[\frac{\sqrt{v}}{\sqrt{h+v}}\right]^{2}+\left[\frac{\sqrt{h}}{\sqrt{h+v}}\right]^{2}\right\}. \tag{14}$$

Eq. (14) is required in order to apply the CBS inequality. By applying the CBS inequality to Eq. (14), one obtains:

$$TC(Q,B)\geq\frac{Ad}{Q}+\frac{Q\,(1-d/p)}{2}\left\{\left(\frac{\sqrt{hv}}{\sqrt{h+v}}\right)\left(1-\frac{B}{Q\,(1-d/p)}\right)+\left(\frac{\sqrt{hv}}{\sqrt{h+v}}\right)\left(\frac{B}{Q\,(1-d/p)}\right)\right\}^{2} \tag{15}$$

with equality iff

$$\frac{\sqrt{h}\left(1-\frac{B}{Q(1-d/p)}\right)}{\frac{\sqrt{v}}{\sqrt{h+v}}}=\frac{\sqrt{v}\left(\frac{B}{Q(1-d/p)}\right)}{\frac{\sqrt{h}}{\sqrt{h+v}}}. \tag{16}$$

Eq. (16) holds iff the two set of numbers are proportional according to the CBS inequality (for instance, see [26, pp. 5]).

Thus, Eq. (14) becomes:

$$TC(Q)=\frac{Ad}{Q}+\frac{Q\,(1-d/p)}{2}\left\{\frac{hv}{h+v}\right\}. \tag{17}$$

Now, the functions are $\frac{Ad}{Q}$ and $\frac{Q(1-d/p)}{2}\left\{\frac{hv}{h+v}\right\}$. It is easy to see that the product is $\frac{ADQ(1-d/p)}{2}\left\{\frac{hv}{h+v}\right\}$, which is a constant. Again, one can apply the easy-to-use AGM inequality to optimize the minimization problem in (17). Applying the AGM inequality to Eq. (17) yields:

$$TC(Q)\geq\sqrt{\frac{2Adh\,(1-d/p)\,v}{h+v}} \tag{18}$$

with equality iff

$$\frac{Ad}{Q}=\frac{Q\,(1-d/p)}{2}\left\{\frac{hv}{h+v}\right\}. \tag{19}$$

Eq. (19) holds iff the product of the functions is a constant (for instance, see [17, pp. 435]).

It follows immediately from Eqs. (16) and (19) that the optimal backorders level and the optimal lot size are given by:

$$B^{*}=\frac{hQ^{*}\,(1-d/p)}{h+v} \tag{20}$$

and

$$Q^{*}=\sqrt{\frac{2Ad\,(h+v)}{h\,(1-d/p)\,v}}. \tag{21}$$

Then it follows from Eq. (18) that the optimal total inventory cost is:

$$TC^{*}=\sqrt{\frac{2Adh\,(1-d/p)\,v}{h+v}}. \tag{22}$$

3. Conclusion

Two optimization approaches have recently appeared in the inventory literature: the cost comparisons and the AGM inequality. These optimization methods were used to derive the EOQ/EPQ models with backorders. However, neither optimization method attempted to derive the optimal backordering level. The main contribution of this paper is to present an alternative method for deriving EOQ/EPQ models when backorders are allowed. In contrast to the methods of Minner [14], Wee et al. [16], and Teng [15], our method is simple and derives both the lot size and backorders level. Additionally, the proposed method is also simpler than the algebraic methods presented by Grubström and Erdem [4] and Cárdenas-Barrón [6]. Finally, this simple approach should be considered as a more accessible approach to ease the learning of inventory theory for students who lack knowledge of calculus.

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