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The time differential three-phase-lag heat conduction model: Thermodynamic compatibility and continuous dependence

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ABSTRACT

This paper deals with the time differential three-phase-lag heat transfer model aiming, at first, to identify the restrictions that make it thermodynamically consistent. The model is thus reformulated by means of the fading memory theory, in which the heat flux vector depends on the history of the thermal displacement gradient: the thermodynamic principles are then applied to obtain suitable restrictions involving the delay times. Consistently with the thermodynamic restrictions just obtained, a first result about the continuous dependence of the solutions with respect to the given initial data and to the supply term is established for the related initial boundary value problems. Subsequently, to provide a more comprehensive review of the problem, a further continuous dependence estimate is proved, this time conveniently relaxing the hypotheses about the above-said thermodynamic restrictions. This last estimate allows the solutions to grow exponentially in time and so to have asymptotic instability.

1. Introduction

Over the last decades, much attention has been devoted to the theory originally proposed by Tzou [1–3] about the so-called dual-phase-lag heat conduction model. Such a theory essentially replaces the classical Fourier law with the following constitutive equation

$$q_i(\mathbf{x}, t+\tau_q) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t+\tau_T), \quad \text{with} \quad \tau_q, \tau_T > 0 \qquad (1)$$

stating, synthesizing its meaning, that the temperature gradient $T_{,j}$ at a certain time $t+\tau_T$ results in a heat flux vector q_i at a different time $t+\tau_q$. In the above constitutive equation (1), besides the explicit dependence upon the spatial variable, we point out that q_i are the components of the heat flux vector, T represents the temperature variation from the constant reference temperature $T_0 > 0$ and k_{ij} are the components of the conductivity tensor; moreover, t is the time variable while τ_q and τ_T are the phase lags (or delay times) of the heat flux and of the temperature gradient, respectively. In particular, τ_q is a relaxation time connected to the fast-transient effects of thermal inertia, while τ_T is caused by microstructural interactions, such as phonon scattering or phonon–electron interactions [4].

We emphasize that the related time differential models (obtained considering the Taylor series expansions of both sides of Eq. (1) and retaining terms up to suitable orders in τ_q and τ_T) have been widely investigated with respect to their thermodynamic consistency as well as to interesting stability issues (see, for example, [5–7]).

A natural evolution of the dual-phase-lag heat conduction model by Tzou consisted in the addition, by Roy Choudhuri [8], of a third delay time τ_α, which has led to a three-phase-lag heat conduction theory. He took into account the model by Green and Naghdi [9–12] which includes, among the constitutive variables, not only the temperature gradient but also the thermal displacement gradient. Starting from the Green–Naghdi model, Roy Choudhuri [8] proposed the following constitutive equation for the heat flux vector

$$q_i(\mathbf{x}, t+\tau_q) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t+\tau_T) - K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x}, t+\tau_\alpha), \qquad (2)$$

where α is the thermal displacement variable, being T equal to the partial time derivative of α, K_{ij} is a thermal tensor characteristic of the considered theory and τ_α is a new phase lag related to the thermal displacement gradient $\alpha_{,i}$: we can suppose, for example, that $0 \leqslant \tau_\alpha < \tau_T < \tau_q$. Through this equation, that generalizes Eq. (1), once again a lagging behavior is described. In agreement with the

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Tzou's interpretation, Eq. (2) means that a temperature gradient and a thermal displacement gradient imposed across a volume element at times $t+\tau_T$ and $t+\tau_\alpha$, respectively, result in a heat flux flowing at a different time $t+\tau_q$.

Also in this case, exactly as for the constitutive equation by Tzou, time differential (three-phase-lag) models can be considered, obtained through the Taylor series expansions of both sides of Eq. (2) and retaining terms up to suitable orders in τ_q, τ_T and τ_α. At this regard, in Quintanilla [13] and Quintanilla and Racke [14,15] it is possible to find some interesting references to the Taylor expansion orders issue.

In the present work and with regard to Eq. (2), the terms up to the second order in τ_q and up to the first order in τ_T and τ_α are retained, leading to the following generalized heat conduction law valid at the position $\mathbf{x}$ and at the time instant t:

$$\frac{1}{2}\tau_q^2\ddot{q}_i(\mathbf{x},t)+\tau_q\dot{q}_i(\mathbf{x},t)+q_i(\mathbf{x},t)=-\tau_T k_{ij}(\mathbf{x})\dot{T}_{,j}(\mathbf{x},t) - \left[k_{ij}(\mathbf{x})+\tau_\alpha K_{ij}(\mathbf{x})\right]T_{,j}(\mathbf{x},t)-K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x},t). \tag{3}$$

The purpose of this paper is twofold: on one side, following [6], we want to reformulate the constitutive equation (3) in such a way that the heat flux vector q_i depends on the history of the thermal displacement gradient, in order to evaluate the thermodynamic consistency of the considered time differential three-phase-lag model. To this end, we rewrite Eq. (3) in the framework of Gurtin–Pipkin [16] and Coleman–Gurtin [17] fading memory theory, and on this basis we analyze the compatibility of the model with the thermodynamical principles. Subsequently, precisely under the thermodynamic compatibility hypotheses just found, we prove the continuous dependence of the solutions from the initial data and from the external heat supply. A uniqueness theorem is also readily obtained as a direct consequence of these results. Finally, to provide a more complete overview about the issue in question, a further continuous dependence estimate is established under a suitable assumption which relaxes the previous thermodynamic compatibility hypotheses. This last estimate allows the solutions to grow exponentially in time and so one can be led to an unstable system.

2. The basic mathematical model

In this paper, referring to a fixed system of rectangular Cartesian axes Ox_k, $(k=1,2,3)$, we employ the usual summation and differentiation conventions. For the components of tensors of order $p \geqslant 1$, the Latin subscripts range over the set $\{1,2,3\}$, while a superposed dot or a subscript preceded by a comma denote partial differentiation with respect to the time variable t or to the corresponding Cartesian coordinate x_i, respectively; summation is understood over the repeated subscripts. Moreover, with an overlying bar we want to denote the closure of the corresponding set indicated below it.We suppose to deal with a regular region B, whose boundary is denoted by ∂B, and consider the linear theory of the time differential three-phase-lag heat conduction model as formulated through the following set of equations: the energy equation

$$-q_{i,i}+\rho r = c\ddot{\alpha}, \quad \text{in} \quad B\times(0,\infty), \tag{4}$$

the constitutive equation

$$q_i+\tau_q\dot{q}_i+\frac{1}{2}\tau_q^2\ddot{q}_i=-\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\beta}_j-\tau_T k_{ij}\ddot{\beta}_j-K_{ij}\beta_j, \quad \text{in} \quad \overline{B}\times[0,\infty), \tag{5}$$

the geometrical equation

$$\beta_j=\alpha_{,j}, \quad \text{in} \quad \overline{B}\times[0,\infty). \tag{6}$$

For a greater clarity, let us repeat some concepts already shown in the above Introduction, representing here all the notations used: q_i are the components of the heat flux vector, ρ is the mass density of the considered medium, r is the external heat supply per unit mass, c is the specific heat and α is the thermal displacement, being $T=\dot{\alpha}$ the temperature variation from the constant reference temperature $T_0>0$. The components of the thermal displacement gradient vector are denoted by β_i and we also recall that k_{ij} are the components of the conductivity tensor and K_{ij} are the components of a thermal tensor characteristic of the considered theory.

Further, we define the initial boundary value problem $\mathcal{P}$ by the basic equations (4)–(6), the following initial conditions

$$\begin{aligned}&\alpha(\mathbf{x},0)=0, \quad \dot{\alpha}(\mathbf{x},0)=T^0(\mathbf{x}),\\ &q_i(\mathbf{x},0)=q_i^0(\mathbf{x}), \quad \dot{q}_i(\mathbf{x},0)=\dot{q}_i^0(\mathbf{x}), \quad \text{in } \overline{B},\end{aligned} \tag{7}$$

recalling that $\alpha(\mathbf{x},0)=0$ because

$$\alpha(\mathbf{x},t)=\int_0^t T(\mathbf{x},s)ds,$$

as well as the following boundary conditions

$$\alpha(\mathbf{x},t)=\omega(\mathbf{x},t), \quad \text{on} \quad \overline{\Sigma}_1\times[0,\infty),$$

$$q_i(\mathbf{x},t)n_i=q(\mathbf{x},t), \quad \text{on} \quad \Sigma_2\times[0,\infty),$$

where $\overline{\Sigma}_1\cup\Sigma_2=\partial B$ and $\Sigma_1\cap\Sigma_2=\emptyset$ and having denoted by q_in_i the heat flux at any regular point of ∂B. We assume that the initial data $T^0(\mathbf{x}), q_i^0(\mathbf{x}), \dot{q}_i^0(\mathbf{x})$ and the boundary data $\omega(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are continuous prescribed functions selected in such a way to guarantee the existence of reciprocal compatibility conditions in $t=0$ and on ∂B.

Let us call $S=\{\alpha,q_i\}$, with $\alpha(\mathbf{x},t)\in C^{1,2}(B\times(0,\infty))$ and $q_i(\mathbf{x},t)\in C^{1,2}(B\times(0,\infty))$, a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $D=\left\{r;T^0,q_i^0,\dot{q}_i^0;\omega,q\right\}$.

3. Thermodynamic consistency of the model

Following the example of Fabrizio and Lazzari [6], we want to rewrite Eq. (5) as a memory constitutive equation of the type described in Gurtin–Pipkin [16] and Coleman–Gurtin [17]. In order to do this, let us rewrite it in terms of the thermal displacement variable α $(T=\dot{\alpha})$:

$$\frac{1}{2}\tau_q^2\ddot{q}_i(t)+\tau_q\dot{q}_i(t)+q_i(t)=-\tau_T k_{ij}\ddot{\alpha}_{,j}(t)-\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)-K_{ij}\alpha_{,j}(t) \tag{8}$$

and then solve it as a linear non-homogeneous second-order differential (in time) equation. We immediately see that the homogeneous (complementary) solution has the form

$$q_i^0(t)=C_i^c\left(\exp\frac{-t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)+C_i^s\left(\exp\frac{-t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right). \tag{9}$$

Through the application of the method of variation of constants, we aim to find a couple of functions $K_i^c(t)$ and $K_i^s(t)$ so that

$$q_i^*(t)=K_i^c(t)\left(\exp\frac{-t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)+K_i^s(t)\left(\exp\frac{-t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right) \tag{10}$$

is a solution of Eq. (8). After appropriate differentiations and straightforward calculations, the problem is reduced to the study of a system in the variables $\dot{K}_i^c(t)$ and $\dot{K}_i^s(t)$, providing

$$\begin{cases}\dot{K}_i^c(t)=\frac{2}{\tau_q}\left(\exp\frac{t}{\tau_q}\right)\left(\sin\frac{t}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(t)+\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)+K_{ij}\alpha_{,j}(t)\right]\\ \dot{K}_i^s(t)=-\frac{2}{\tau_q}\left(\exp\frac{t}{\tau_q}\right)\left(\cos\frac{t}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(t)+\left(k_{ij}+\tau_\alpha K_{ij}\right)\dot{\alpha}_{,j}(t)+K_{ij}\alpha_{,j}(t)\right]\end{cases}$$

which, through a suitable integration between $-\infty$ and t, gives

$$K_i^c(t)=\int_{-\infty}^{t}\frac{2}{\tau_q}\left(\exp\frac{s}{\tau_q}\right)\left(\sin\frac{s}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(s)+(k_{ij}+\tau_\alpha K_{ij})\dot{\alpha}_{,j}(s)+K_{ij}\alpha_{,j}(s)\right]ds$$

and

$$K_i^s(t)=\int_{-\infty}^{t}-\frac{2}{\tau_q}\left(\exp\frac{s}{\tau_q}\right)\left(\cos\frac{s}{\tau_q}\right)\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(s)+(k_{ij}+\tau_\alpha K_{ij})\dot{\alpha}_{,j}(s)+K_{ij}\alpha_{,j}(s)\right]ds.$$

After replacing such expressions in (10), it is sufficient to add to the homogeneous solution (9) the particular solution just found through the method of variation of constants. Finally, assuming that $\lim_{t\to-\infty}q_i(t)=0$, we are readily led to the following expression for the heat flux vector $q_i(t)$:

$$q_i(t)=-\frac{2}{\tau_q}\int_{-\infty}^{t}\left(\exp\frac{s-t}{\tau_q}\right)\left(\sin\frac{t-s}{\tau_q}\right)\times\left[\tau_T k_{ij}\ddot{\alpha}_{,j}(s)+(k_{ij}+\tau_\alpha K_{ij})\dot{\alpha}_{,j}(s)+K_{ij}\alpha_{,j}(s)\right]ds.$$

Now, integrating twice by parts, we get

$$q_i(t)=-\frac{2\tau_T}{\tau_q^2}k_{ij}\alpha_{,j}(t)+\frac{2}{\tau_q^2}\int_{-\infty}^{t}\left(\exp\frac{s-t}{\tau_q}\right)\alpha_{,j}(s)\times\left\{\left[\left(\frac{2\tau_T}{\tau_q}-1\right)k_{ij}-\tau_\alpha K_{ij}\right]\left(\cos\frac{t-s}{\tau_q}\right)+\left[k_{ij}+(\tau_\alpha-\tau_q)K_{ij}\right]\left(\sin\frac{t-s}{\tau_q}\right)\right\}ds$$

and hence, with the change of variable $u=t-s$, i.e. $s=t-u$, we have

$$q_i(t)=-\frac{2\tau_T}{\tau_q^2}k_{ij}\alpha_{,j}(t)+\frac{2}{\tau_q^2}\int_0^{+\infty}v_{ij}(s)\alpha_{,j}^t(s)ds, \tag{11}$$

where

$$v_{ij}(s)=\left(\exp\frac{-s}{\tau_q}\right)\left\{\left[k_{ij}+(\tau_\alpha-\tau_q)K_{ij}\right]\left(\sin\frac{s}{\tau_q}\right)+\frac{2}{\tau_q}\left[\tau_T k_{ij}-\frac{\tau_q}{2}(k_{ij}+\tau_\alpha K_{ij})\right]\left(\cos\frac{s}{\tau_q}\right)\right\} \tag{12}$$

and

$$\alpha^t(s)=\alpha(t-s),$$

having omitted everywhere the explicit dependence upon the space variable.

In order to identify the restrictions that make the time differential three-phase-lag model thermodynamically consistent, we consider a closed cycle characterized by the following history

$$\alpha_{,j}^t(s)=G_j\cos\omega(t-s)+H_j\sin\omega(t-s),\quad \text{with}\quad \omega>0,\quad G_jG_j+H_jH_j>0, \tag{13}$$

so that

$$\dot{\alpha}_{,j}(t)=-\omega G_j\sin\omega t+\omega H_j\cos\omega t. \tag{14}$$

In view of Eqs. (11), (13) and (14), let us evaluate the scalar product $q_i(t)\dot{\alpha}_{,i}(t)$ as follows

$$q_i(t)\dot{\alpha}_{,i}(t)=-\frac{2\tau_T}{\tau_q^2}k_{ij}(G_j\cos\omega t+H_j\sin\omega t)(-\omega G_i\sin\omega t+\omega H_i\cos\omega t)+\frac{2}{\tau_q^2}\int_0^{+\infty}v_{ij}(s)\left[G_j\cos\omega(t-s)+H_j\sin\omega(t-s)\right]\times(\omega H_i\cos\omega t-\omega G_i\sin\omega t)ds,$$

whose integral on a closed cycle with respect to the time variable, postulating the Second Law of Thermodynamics in terms of the Clausius–Duhem inequality, has to be non positive. After simple calculations, and assuming that the constitutive tensors k_{ij} and K_{ij} satisfy the following symmetries

$$k_{ij}=k_{ji},\quad K_{ij}=K_{ji},$$

we obtain

$$\int_0^{2\pi/\omega}q_i(t)\dot{\alpha}_{,i}(t)dt=-\frac{2\pi}{\tau_q^2}\int_0^{+\infty}v_{ij}(s)(G_iG_j+H_iH_j)\sin\omega s\,ds\le 0$$

and thus

$$\int_0^{+\infty}v_{ij}(s)(G_iG_j+H_iH_j)\sin\omega s\,ds\geqslant 0. \tag{15}$$

In view of Eq. (12) and through a series of suitable integration by parts, we come to an expression equivalent to the condition (15) and valid for each $\omega\in\mathbb{R}$:

$$\frac{\tau_q^2\omega}{4+\tau_q^4\omega^4}\left\{2\left[k_{ij}+(\tau_\alpha-\tau_q)K_{ij}\right]+\tau_q^2\omega^2\frac{2}{\tau_q}\left[\tau_T k_{ij}-\frac{\tau_q}{2}(k_{ij}+\tau_\alpha K_{ij})\right]\right\}\times(G_iG_j+H_iH_j)\geqslant 0,$$

from which it follows that the time differential three-phase-lag model is consistent from the thermodynamical point of view if the following conditions are satisfied:

$$\left[k_{ij}+(\tau_\alpha-\tau_q)K_{ij}\right](G_iG_j+H_iH_j)\geqslant 0,\quad \text{for all}\quad G_i,H_i, \tag{16}$$

$$\left[\tau_T k_{ij}-\frac{\tau_q}{2}(k_{ij}+\tau_\alpha K_{ij})\right](G_iG_j+H_iH_j)\geqslant 0,\quad \text{for all}\quad G_i,H_i. \tag{17}$$

Concluding, we can see that the requirement of compatibility of the time differential three-phase-lag model described by the constitutive equation (8) with thermodynamics implies that the following tensors

$$\varkappa_{ij}=k_{ij}+(\tau_\alpha-\tau_q)K_{ij},$$

$$\kappa_{ij}=\tau_T k_{ij}-\frac{\tau_q}{2}(k_{ij}+\tau_\alpha K_{ij}),$$

are positive semi-definite.

Remark 1. A direct consequence of the two thermodynamic restrictions TR1, given in (16), and TR2, given in (17), is the following inequality of future usefulness

$$\left[(\tau_T+\tau_q)k_{ij}+\tau_q\left(\tau_\alpha-\frac{3}{2}\tau_q\right)K_{ij}\right]\xi_i\xi_j\geqslant 0\quad \text{for all}\quad \xi_i, \tag{18}$$

that is the tensor

$$\mathcal{K}_{ij}=(\tau_T+\tau_q)k_{ij}+\tau_q\left(\tau_\alpha-\frac{3}{2}\tau_q\right)K_{ij}$$

is positive semi-definite.

4. The initial boundary value problem $\mathcal{P}^{tr}$

For the analysis at issue, it is convenient to introduce the following notations. For any continuous function of the time variable $f(t)$, we denote by $f'(t)$ the integral over $[0,t]$ of that function, i.e.

$$f'(t)=\int_0^t f(s)ds,\quad f''(t)=\int_0^t\int_0^s f(z)dzds,\ \ldots;$$

in addition, for any continuous function of time $g(t)$ we denote by $\hat{g}(t)$ the following function

$$\hat{g}(t)=g''(t)+\tau_q g'(t)+\frac{1}{2}\tau_q^2 g(t)=\int_0^t\int_0^s g(z)dzds+\tau_q\int_0^t g(s)ds+\frac{1}{2}\tau_q^2 g(t). \tag{19}$$

In view of the above definitions, and taking into account the constitutive relation (5) and the initial conditions (7), one can easily prove that

$$\hat{q}_i(t) = q_i''(t) + \tau_q q_i'(t) + \frac{1}{2}\tau_q^2 q_i(t) = -\left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_j'(t) - \tau_T k_{ij}\beta_j(t) - K_{ij}\beta_j''(t) + \Psi_i^0(t),$$

where

$$\Psi_i^0(t) = \frac{1}{2}\tau_q^2 q_i^0 + t\left(\tau_q q_i^0 + \frac{1}{2}\tau_q^2 \dot{q}_i^0 + \tau_T k_{ij} T_{,j}^0\right). \tag{20}$$

For a subsequent convenience, we will shorten $\Psi_i^0(t)$ as follows:

$$\Psi_i^0(t) = a_i + t b_i, \tag{21}$$

where

$$a_i = \frac{1}{2}\tau_q^2 q_i^0, \quad b_i = \tau_q q_i^0 + \frac{1}{2}\tau_q^2 \dot{q}_i^0 + \tau_T k_{ij} T_{,j}^0; \tag{22}$$

moreover, by replacing the function $g(t)$ with $r(t)$ into Eq. (19), we have

$$\hat{r}(t) = r''(t) + \tau_q r'(t) + \frac{1}{2}\tau_q^2 r(t) = \int_0^t \int_0^s r(z)dzds + \tau_q \int_0^t r(s)ds + \frac{1}{2}\tau_q^2 r(t).$$

Now, after a direct comparison between $\hat{\ddot{\alpha}}$ and $\partial^2\hat{\alpha}/\partial t^2$ and denoting by

$$R = \rho\hat{r} + cT^0(t + \tau_q), \tag{23}$$

we are able to define, through the application of the hat operator, the new initial boundary value problem $\mathcal{P}^{tr}$ as follows:

The equation of energy

$$-\hat{q}_{i,i} + R = c\frac{\partial^2\hat{\alpha}}{\partial t^2}, \quad \text{in} \quad B \times (0, \infty), \tag{24}$$

The constitutive equation

$$\hat{q}_i = -\left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_j' - \tau_T k_{ij}\beta_j - K_{ij}\beta_j'' + \Psi_i^0, \quad \text{in} \quad \overline{B} \times [0, \infty), \tag{25}$$

The geometrical equation

$$\hat{\beta}_j = \hat{\alpha}_{,j}, \quad \text{in} \quad \overline{B} \times [0, \infty), \tag{26}$$

The initial conditions

$$\hat{\alpha}(\mathbf{x}, 0) = 0, \quad \frac{\partial\hat{\alpha}}{\partial t}(\mathbf{x}, 0) = \frac{1}{2}\tau_q^2 T^0(\mathbf{x}),$$
$$\hat{q}_i(\mathbf{x}, 0) = \frac{1}{2}\tau_q^2 q_i^0(\mathbf{x}), \quad \frac{\partial\hat{q}_i}{\partial t}(\mathbf{x}, 0) = \tau_q q_i^0(\mathbf{x}) + \frac{1}{2}\tau_q^2 \dot{q}_i^0(\mathbf{x}), \quad \text{in} \quad \overline{B}, \tag{27}$$

The boundary conditions

$$\hat{\alpha}(\mathbf{x}, t) = \hat{\omega}(\mathbf{x}, t), \quad \text{on} \quad \overline{\Sigma}_1 \times [0, \infty),$$
$$\hat{q}_i(\mathbf{x}, t)n_i = \hat{q}(\mathbf{x}, t), \quad \text{on} \quad \Sigma_2 \times [0, \infty). \tag{28}$$

Coherently with what we did above, it follows that if $S = \{\alpha, q_i\}$ is a solution of the initial boundary value problem $\mathcal{P}$, then $\widehat{S} = \{\hat{\alpha}, \hat{q}_i\}$ is a solution of the new initial boundary value problem $\mathcal{P}^{tr}$ (24)–(28), where Ψ_i^0 and R are defined through Eqs. (20) and (23), respectively.

5. The continuous dependence result based upon the thermodynamic restrictions TR1 and TR2

In this section, we want to evaluate the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$, with respect to the initial data as well as to the external heat supply, assuming the validity of the above restrictions TR1 and TR2: we remark that, precisely because we are interested in such a kind of continuous dependence, the boundary data are assumed null here. Starting from Eq. (24), suitably multiplied by $\partial\hat{\alpha}/\partial t$ and integrated over B, we apply the divergence theorem and consider Eqs. (25) and (28) to write

$$\frac{1}{2}\frac{d}{dt}\int_B c\left(\frac{\partial\hat{\alpha}}{\partial t}\right)^2 dv = \int_B R\frac{\partial\hat{\alpha}}{\partial t}dv + \int_B \frac{\partial\hat{\beta}_i}{\partial t}\left[-K_{ij}\beta_j'' - \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_j' - \tau_T k_{ij}\beta_j\right]dv + \int_B \frac{\partial\hat{\beta}_i}{\partial t}\Psi_i^0 dv. \tag{29}$$

Using the definition of the hat operator (19), we represent one of the integral terms of Eq. (29) in this way:

$$\begin{aligned}
\int_B \frac{\partial\hat{\beta}_i}{\partial t}&\left[-K_{ij}\beta_j'' - \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_j' - \tau_T k_{ij}\beta_j\right]dv \\
&= -\int_B K_{ij}\beta_i'\beta_j''dv - \int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i'\beta_j'dv - \tau_T\int_B k_{ij}\beta_i'\beta_j dv \\
&\quad - \tau_q\int_B K_{ij}\beta_i\beta_j''dv - \tau_q\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i\beta_j'dv \\
&\quad - \tau_T\tau_q\int_B k_{ij}\beta_i\beta_j dv - \frac{1}{2}\tau_q^2\int_B K_{ij}\dot{\beta}_i\beta_j''dv \\
&\quad - \frac{1}{2}\tau_q^2\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\dot{\beta}_i\beta_j'dv - \frac{1}{2}\tau_T\tau_q^2\int_B k_{ij}\dot{\beta}_i\beta_j dv.
\end{aligned} \tag{30}$$

A series of simple calculations leads to rewrite in the following way some selected integral terms of the right-hand side of Eq. (30):

$$-\int_B K_{ij}\beta_i'\beta_j''dv = -\frac{1}{2}\frac{d}{dt}\int_B K_{ij}\beta_i''\beta_j''dv;$$

$$-\tau_T\int_B k_{ij}\beta_i'\beta_j dv = -\frac{\tau_T}{2}\frac{d}{dt}\int_B k_{ij}\beta_i'\beta_j'dv;$$

$$-\tau_q\int_B K_{ij}\beta_i\beta_j''dv = -\frac{\tau_q}{2}\frac{d^2}{dt^2}\int_B K_{ij}\beta_i''\beta_j''dv + \tau_q\int_B K_{ij}\beta_i'\beta_j'dv;$$

$$-\tau_q\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i\beta_j'dv = -\frac{\tau_q}{2}\frac{d}{dt}\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i'\beta_j'dv;$$

$$-\frac{1}{2}\tau_q^2\int_B K_{ij}\dot{\beta}_i\beta_j''dv = -\frac{\tau_q^2}{4}\frac{d^3}{dt^3}\int_B K_{ij}\beta_i''\beta_j''dv + \frac{3}{4}\tau_q^2\frac{d}{dt}\int_B K_{ij}\beta_i'\beta_j'dv;$$

$$-\frac{1}{2}\tau_q^2\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\dot{\beta}_i\beta_j'dv = -\frac{\tau_q^2}{4}\frac{d^2}{dt^2}\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i'\beta_j'dv + \frac{\tau_q^2}{2}\int_B \left(k_{ij} + \tau_\alpha K_{ij}\right)\beta_i\beta_j dv;$$

$$-\frac{1}{2}\tau_T\tau_q^2\int_B k_{ij}\dot{\beta}_i\beta_j dv = -\frac{\tau_T\tau_q^2}{4}\frac{d}{dt}\int_B k_{ij}\beta_i\beta_j dv.$$

Coming back to Eq. (30) and, backward, to Eq. (29), integrating three times with respect to the time variable and taking into account the initial conditions (27), we observe the validity of the following relation:

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^s\int_B c\left(\frac{\partial\hat{\alpha}}{\partial z}\right)^2 dv dz ds + \frac{1}{2}\int_0^t\int_0^s\int_B K_{ij}\beta_i''\beta_j'' dv dz ds\\
&+\int_0^t\int_0^s\int_0^z\int_B\left[k_{ij}+(\tau_\alpha-\tau_q)K_{ij}\right]\beta_i'\beta_j' dv dr dz ds\\
&+\frac{\tau_q}{2}\int_0^t\int_B K_{ij}\beta_i''\beta_j'' dv ds\\
&+\frac{1}{2}\int_0^t\int_0^s\int_B\left[(\tau_T+\tau_q)k_{ij}+\tau_q\left(\tau_\alpha-\frac{3}{2}\tau_q\right)K_{ij}\right]\beta_i'\beta_j' dv dz ds\\
&+\int_0^t\int_0^s\int_0^z\int_B \tau_q\left[\tau_T k_{ij}-\frac{\tau_q}{2}(k_{ij}+\tau_\alpha K_{ij})\right]\beta_i\beta_j dv dr dz ds\\
&+\frac{\tau_T\tau_q^2}{4}\int_0^t\int_0^s\int_B k_{ij}\beta_i\beta_j dv dz ds+\frac{\tau_q^2}{4}\int_B K_{ij}\beta_i''\beta_j'' dv\\
&+\frac{\tau_q^2}{4}\int_0^t\int_B (k_{ij}+\tau_\alpha K_{ij})\beta_i'\beta_j' dv ds\\
&=\int_0^t\int_0^s\int_0^z\int_B\left(R\frac{\partial\hat{\alpha}}{\partial r}+\Psi_i^0\beta_i'+\tau_q\Psi_i^0\beta_i+\frac{1}{2}\tau_q^2\Psi_i^0\dot{\beta}_i\right)dv dr dz ds\\
&+t^2\frac{\tau_q^4}{16}\int_B c\left(T^0\right)^2 dv. \qquad (31)
\end{aligned}
$$

At this point, we must perform an appropriate evaluation, through an integration by parts, of the only integral term of Eq. (31) containing the vector $\dot{\beta}_i$:

$$
\begin{aligned}
&\frac{1}{2}\tau_q^2\int_0^t\int_0^s\int_0^z\int_B\Psi_i^0\dot{\beta}_i dv dr dz ds=\frac{1}{2}\tau_q^2\int_0^t\int_0^s\int_B\Psi_i^0\beta_i dv dz ds\\
&-\frac{1}{2}\tau_q^2\int_0^t\int_0^s\int_0^z\int_B\dot{\Psi}_i^0\beta_i dv dr dz ds \qquad (32)
\end{aligned}
$$

which, in turn, makes essential a further estimate of the term

$$\frac{1}{2}\tau_q^2\int_0^t\int_0^s\int_B\Psi_i^0\beta_i dv dz ds.$$

To this aim, we remind the validity of the subsequent relation for the conductivity tensor k_{ij}, being k_m related to the smallest eigenvalue of k_{ij}

$$k_m\xi_i\xi_i \leqslant k_{ij}\xi_i\xi_j, \quad \text{for all} \quad \xi_i. \qquad (33)$$

An application of the arithmetic–geometric mean inequality gives

$$
\begin{aligned}
&\frac{1}{2}\tau_q^2\int_0^t\int_0^s\int_B\Psi_i^0\beta_i dv dz ds\leqslant\frac{1}{4\varepsilon\tau_T}\tau_q^2\int_0^t\int_0^s\int_B\frac{1}{k_m}\Psi_i^0\Psi_i^0 dv dz ds\\
&+\frac{\varepsilon}{4}\tau_q^2\tau_T\int_0^t\int_0^s\int_B k_{ij}\beta_i\beta_j dv dz ds, \quad \forall\varepsilon>0. \qquad (34)
\end{aligned}
$$

From Eqs. (31), (32) and (34) for $\varepsilon = 1/2$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^s\int_B c\left(\frac{\partial\hat{\alpha}}{\partial z}\right)^2 dv dz ds+\frac{1}{2}\int_0^t\int_0^s\int_B K_{ij}\beta_i''\beta_j'' dv dz ds\\
&+\int_0^t\int_0^s\int_0^z\int_B \varkappa_{ij}\beta_i'\beta_j' dv dr dz ds+\frac{\tau_q}{2}\int_0^t\int_B K_{ij}\beta_i''\beta_j'' dv ds\\
&+\frac{1}{2}\int_0^t\int_0^s\int_B \mathcal{K}_{ij}\beta_i'\beta_j' dv dz ds+\int_0^t\int_0^s\int_0^z\int_B\tau_q\kappa_{ij}\beta_i\beta_j dv dr dz ds\\
&+\frac{\tau_q^2\tau_T}{8}\int_0^t\int_0^s\int_B k_{ij}\beta_i\beta_j dv dz ds+\frac{\tau_q^2}{4}\int_B K_{ij}\beta_i''\beta_j'' dv\\
&+\frac{\tau_q^2}{4}\int_0^t\int_B(k_{ij}+\tau_\alpha K_{ij})\beta_i'\beta_j' dv ds\\
&\leqslant\int_0^t\int_0^s\int_0^z\int_B\left[R\frac{\partial\hat{\alpha}}{\partial r}+\Psi_i^0\beta_i'+\tau_q\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\beta_i\right]dv dr dz ds\\
&+\frac{\tau_q^2}{2\tau_T}\int_0^t\int_0^s\int_B\frac{1}{k_m}\Psi_i^0\Psi_i^0 dv dz ds+t^2\frac{\tau_q^4}{16}\int_B c\left(T^0\right)^2 dv. \qquad (35)
\end{aligned}
$$

Remark 2. At this point, it would be possible to apply again the arithmetic–geometric mean inequality to the right-hand side of Eq. (35) in order to treat the terms containing β_i' and β_i as we previously did for $\dot{\beta}_i$ (Eq. (32)). Nevertheless, a specification is appropriate: in what immediately follows we will suppose, among the other assumptions, that the tensor $\varkappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q)K_{ij}$ is positive definite, while the tensor $\kappa_{ij} = \tau_T k_{ij} - \tau_q(k_{ij} + \tau_\alpha K_{ij})/2$ is positive semi-definite. To proceed as described in this Remark would force instead both the tensors to be positive definite, with a clear strengthening of our hypotheses.

Now, similarly to what has been said for Eq. (33) and having in mind the inequality (18), let $\mathcal{K}_m$ be a scalar related to the smallest eigenvalue of $\mathcal{K}_{ij} = (\tau_T + \tau_q)k_{ij} + \tau_q(\tau_\alpha - 3\tau_q/2)K_{ij}$. Again referring to the right-hand side of Eq. (35), the following further estimate is obtainable, invoking the Cauchy–Schwarz inequality:

$$
\begin{aligned}
&\int_0^t\int_0^s\int_0^z\int_B\left[R\frac{\partial\hat{\alpha}}{\partial r}+\Psi_i^0\beta_i'+\tau_q\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\beta_i\right]dv dr dz ds\\
&\leqslant\int_0^t\left\{\int_0^s\int_0^z\int_B\left[\frac{R^2}{c}+\frac{\Psi_i^0\Psi_i^0}{\mathcal{K}_m}+\frac{4}{k_m\tau_T}\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\right]dv dr dz\right\}^{1/2}\\
&\times\left\{\int_0^s\int_0^z\int_B\left[c\left(\frac{\partial\hat{\alpha}}{\partial r}\right)^2+\mathcal{K}_m\beta_i'\beta_i'+\frac{\tau_q^2\tau_T}{4}k_m\beta_i\beta_i\right]dv dr dz\right\}^{1/2}ds\\
&\leqslant\int_0^t\left\{\int_0^s\int_0^z\int_B\left[\frac{R^2}{c}+\frac{\Psi_i^0\Psi_i^0}{\mathcal{K}_m}+\frac{4}{k_m\tau_T}\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\left(\Psi_i^0-\frac{1}{2}\tau_q\dot{\Psi}_i^0\right)\right]dv dr dz\right\}^{1/2}\\
&\times\left\{\int_0^s\int_0^z\int_B\left[c\left(\frac{\partial\hat{\alpha}}{\partial r}\right)^2+\mathcal{K}_{ij}\beta_i'\beta_j'+\frac{\tau_q^2\tau_T}{4}k_{ij}\beta_i\beta_j\right]dv dr dz\right\}^{1/2}ds.
\end{aligned}
\qquad (36)
$$

Moreover, we simply have to recall the definitions (21) and (22) of Ψ_i^0 to write:

$$
\begin{aligned}
&\frac{\tau_q^2}{2\tau_T}\int_0^t\int_0^s\int_B\frac{1}{k_m}\Psi_i^0\Psi_i^0 dv dz ds\\
&=\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{t^2}{2}a_i a_i+\frac{t^3}{3}a_i b_i+\frac{t^4}{12}b_i b_i\right)dv. \qquad (37)
\end{aligned}
$$

Theorem 1 (*Continuous dependence*). *Let $S = \{\alpha, q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $D = \left\{r; T^0, q_i^0, \dot{q}_i^0; 0, 0\right\}$. Under the hypotheses:*

i. k_{ij} positive definite, K_{ij} positive semi-definite, $c > 0$,
ii. $\varkappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q)K_{ij}$ positive definite, $\kappa_{ij} = \tau_T k_{ij} - \tau_q(k_{ij} + \tau_\alpha K_{ij})/2$ positive semi-definite, and for any fixed time $S \in (0, +\infty)$, the following inequality holds true

$$\sqrt{\mathcal{E}(t)} \leqslant \phi(0) + \frac{1}{\sqrt{2}}\int_0^t g(s)ds, \quad t \in [0, S], \qquad (38)$$

where

$$\mathcal{E}(t)=\frac{1}{2}\int_0^t\int_0^s\int_B\left\{c\left(\frac{\partial\hat{\alpha}}{\partial z}\right)^2+\mathcal{K}_{ij}\beta_i'\beta_j'+\frac{\tau_q^2\tau_T}{4}k_{ij}\beta_i\beta_j\right\}dv dz ds, \qquad (39)$$

$$
\begin{aligned}
\phi(t)=&\left\{\int_0^t g(s)\sqrt{2\mathcal{E}(s)}ds+\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^3}{3}|a_i b_i|+\frac{S^4}{12}b_i b_i\right)dv\right.\\
&\left.+S^2\frac{\tau_q^2}{16}\int_B\left[\frac{4}{k_m\tau_T}a_i a_i+\tau_q^2 c\left(T^0\right)^2\right]dv\right\}^{1/2}, \qquad (40)
\end{aligned}
$$

$$g(t)=\sqrt{\int_0^t\int_0^s\int_B\left[\frac{R^2}{c}+\frac{\Psi_i^0\Psi_i^0}{\mathcal{K}_m}+\frac{4}{k_m\tau_T}\left(\Psi_i^0-\frac{\tau_q}{2}\dot{\Psi}_i^0\right)\left(\Psi_i^0-\frac{\tau_q}{2}\dot{\Psi}_i^0\right)\right]dv dz ds}. \qquad (41)$$

Proof. From a direct comparison of Eqs. (35)–(37) we obtain, for t belonging to the bounded interval $[0, S]$,

$$\frac{1}{2}\int_0^t\int_0^s\int_B\left\{c\left(\frac{\partial\hat{\alpha}}{\partial z}\right)^2+\mathcal{K}_{ij}\beta_i'\beta_j'+\frac{\tau_q^2\tau_T}{4}k_{ij}\beta_i\beta_j\right\}dvdzds$$
$$\leqslant\int_0^t g(s)\left\{\int_0^s\int_0^z\int_B\left[c\left(\frac{\partial\hat{\alpha}}{\partial r}\right)^2+\mathcal{K}_{ij}\beta_i'\beta_j'+\frac{\tau_q^2\tau_T}{4}k_{ij}\beta_i\beta_j\right]dvdrdz\right\}^{1/2}ds$$
$$+\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^2}{2}a_ia_i+\frac{S^3}{3}|a_ib_i|+\frac{S^4}{12}b_ib_i\right)dv+S^2\frac{\tau_q^4}{16}\int_B c\left(T^0\right)^2dv$$

and so we are directly led to the Gronwall-type inequality

$$\mathcal{E}(t)\leqslant\int_0^t g(s)\sqrt{2\mathcal{E}(s)}ds+\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^3}{3}|a_ib_i|+\frac{S^4}{12}b_ib_i\right)dv$$
$$+S^2\frac{\tau_q^2}{16}\int_B\left[\frac{4}{k_m\tau_T}a_ia_i+\tau_q^2c\left(T^0\right)^2\right]dv,\quad\text{for}\quad t\in[0,S],$$

from which it is evident that

$$\mathcal{E}(t)\leqslant\phi^2(t)\Rightarrow\sqrt{\mathcal{E}(t)}\leqslant\phi(t). \tag{42}$$

On the other side, from Eq. (40),

$$2\phi(t)\dot{\phi}(t)=g(t)\sqrt{2\mathcal{E}(t)} \tag{43}$$

and so, from Eqs. (42) and (43), we obtain

$$\dot{\phi}(t)\leqslant\frac{1}{\sqrt{2}}g(t)$$

which integrated with respect to the time variable between 0 and t provides, for $t\in[0,S]$,

$$\phi(t)\leqslant\phi(0)+\frac{1}{\sqrt{2}}\int_0^t g(s)ds. \tag{44}$$

Finally, from Eqs. (40), (42) and (44) we arrive to the inequality

$$\sqrt{\mathcal{E}(t)}\leqslant\frac{1}{\sqrt{2}}\int_0^t g(s)ds+\left\{\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^3}{3}|a_ib_i|+\frac{S^4}{12}b_ib_i\right)dv\right.$$
$$\left.+S^2\frac{\tau_q^2}{16}\int_B\left[\frac{4}{k_m\tau_T}a_ia_i+\tau_q^2c\left(T^0\right)^2\right]dv\right\}^{1/2},\quad\text{for}\quad t\in[0,S]$$

that proves the theorem at issue, showing in fact the desired continuous dependence with respect to the initial data and to the external heat supply. □

A direct consequence of the above theorem is the following uniqueness result.

Theorem 2 (*Uniqueness*). *Under the hypotheses of Theorem 1, it follows that the initial boundary value problem $\mathcal{P}$ has at most one solution.*

Remark 3. The above theorems remain provable when we replace the hypothesis *ii.* by the following one: *iii.* $\varkappa_{ij}=k_{ij}+(\tau_\alpha-\tau_q)K_{ij}$ positive semi-definite, $\kappa_{ij}=\tau_Tk_{ij}-\tau_q(k_{ij}+\tau_\alpha K_{ij})/2$ positive definite.

6. Other continuous dependence results

In order to provide a more complete description about the continuous dependence of the solutions of the initial boundary value problem $\mathcal{P}$ with respect to the initial data as well as to the external heat supply, in this section we will abandon the hypothesis of validity of the thermodynamic restrictions TR1 and TR2. That means that at least one of the tensors $\varkappa_{ij}=k_{ij}+(\tau_\alpha-\tau_q)K_{ij}$ and $\kappa_{ij}=\tau_Tk_{ij}-\tau_q(k_{ij}+\tau_\alpha K_{ij})/2$ is assumed to be of indefinite sign. As an alternative, the tensor $\mathcal{K}_{ij}=(\tau_T+\tau_q)k_{ij}+\tau_q(\tau_\alpha-3\tau_q/2)K_{ij}$ is here assumed to be positive definite (we remind that, previously, the inequality (18) was a direct consequence of TR1 and TR2). For our subsequent analysis we will assume, for convenience, that both the tensors $\varkappa_{ij}$ and κ_{ij} are of indefinite sign. On this basis we can establish the following theorem.

Theorem 3 (*Continuous dependence*). *Let $S=\{\alpha,q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $D=\left\{r;T^0,q_i^0,\dot{q}_i^0;0,0\right\}$. Under the hypotheses:*

i. k_{ij} positive definite, K_{ij} positive semi-definite, $c>0$,
ii. $\mathcal{K}_{ij}=(\tau_T+\tau_q)k_{ij}+\tau_q(\tau_\alpha-3\tau_q/2)K_{ij}$ positive definite,

and for any fixed time $S\in(0,+\infty)$, the following inequality holds true

$$\sqrt{\mathcal{E}(t)}\leqslant\varphi(0)\exp\left(\frac{t}{\delta}\right)+\frac{1}{\sqrt{2}}\int_0^t\exp\left(\frac{t-s}{\delta}\right)g(s)ds,\quad t\in[0,S], \tag{45}$$

where $\mathcal{E}(t)$ is defined through Eq. (39), $g(t)$ through (41), $\varphi(t)$ is defined by

$$\varphi(t)=\left\{\frac{2}{\delta}\int_0^t\mathcal{E}(s)ds+\int_0^t g(s)\sqrt{2\mathcal{E}(s)}ds\right.$$
$$+\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^3}{3}|a_ib_i|+\frac{S^4}{12}b_ib_i\right)dv$$
$$\left.+S^2\frac{\tau_q^2}{16}\int_B\left[\frac{4}{k_m\tau_T}a_ia_i+\tau_q^2c\left(T^0\right)^2\right]dv\right\}^{1/2}, \tag{46}$$

and

$$\frac{1}{\delta}=\max\left[\max_{\bar{B}}\frac{(\varkappa_{rs}\varkappa_{rs})^{1/2}}{\mathcal{K}_m},\ \frac{4}{\tau_T\tau_q}\max_{\bar{B}}\frac{(\kappa_{rs}\kappa_{rs})^{1/2}}{k_m}\right].$$

Proof. First of all we note that

$$\left|\int_0^t\int_0^s\int_0^z\int_B\varkappa_{ij}\beta_i'\beta_j'dvdrdzds\right|$$
$$\leqslant\int_0^t\int_0^s\int_0^z\int_B\frac{(\varkappa_{rs}\varkappa_{rs})^{1/2}}{\mathcal{K}_m}\left[\mathcal{K}_m\beta_i'\beta_i'\right]dvdrdzds$$
$$\leqslant\max_{\bar{B}}\frac{2(\varkappa_{rs}\varkappa_{rs})^{1/2}}{\mathcal{K}_m}\int_0^t\int_0^s\int_0^z\int_B\frac{1}{2}\mathcal{K}_{ij}\beta_i'\beta_j'dvdrdzds,$$

and

$$\left|\tau_q\int_0^t\int_0^s\int_0^z\int_B\kappa_{ij}\beta_i\beta_jdvdrdzds\right|$$
$$\leqslant\int_0^t\int_0^s\int_0^z\int_B\frac{\tau_q(\kappa_{rs}\kappa_{rs})^{1/2}}{k_m}\left[k_m\beta_i\beta_i\right]dvdrdzds$$
$$\leqslant\frac{8}{\tau_T\tau_q}\max_{\bar{B}}\frac{(\kappa_{rs}\kappa_{rs})^{1/2}}{k_m}\int_0^t\int_0^s\int_0^z\int_B\frac{\tau_T\tau_q^2}{8}k_{ij}\beta_i\beta_jdvdrdzds.$$

Further, we use these estimates into the inequality (35) and then use the notations (39), (41) and (46) and the estimate (36) in order to obtain

$$\mathcal{E}(t)\leqslant\frac{2}{\delta}\int_0^t\mathcal{E}(s)ds+\sqrt{2}\int_0^t g(s)\sqrt{\mathcal{E}(s)}ds$$
$$+\frac{\tau_q^2}{2\tau_T}\int_B\frac{1}{k_m}\left(\frac{S^3}{3}|a_ib_i|+\frac{S^4}{12}b_ib_i\right)dv$$
$$+S^2\frac{\tau_q^2}{16}\int_B\left[\frac{4}{k_m\tau_T}a_ia_i+\tau_q^2c\left(T^0\right)^2\right]dv,\quad t\in[0,S],$$

that is equivalent to

$$\mathcal{E}(t) \leqslant \varphi^2(t) \Rightarrow \sqrt{\mathcal{E}(t)} \leqslant \varphi(t), \quad t \in [0, S], \tag{47}$$

considering the definition (46). On the other side, again from Eq. (46), we have

$$2\varphi(t)\dot{\varphi}(t) = \frac{2}{\delta}\,\mathcal{E}(t) + \sqrt{2}g(t)\sqrt{\mathcal{E}(t)}$$

and from Eq. (47)

$$\dot{\varphi}(t) - \frac{1}{\delta}\,\varphi(t) \leqslant \frac{1}{\sqrt{2}}\,g(t).$$

Multiplying both sides of the above result by $\exp(-t/\delta)$, we have

$$\frac{d}{dt}\left[\varphi(t)\exp\left(-\frac{t}{\delta}\right)\right] \leqslant \frac{1}{\sqrt{2}}\,g(t)\exp\left(-\frac{t}{\delta}\right)$$

which integrated with respect to the time variable between 0 and t provides, for $t \in [0, S]$,

$$\sqrt{\mathcal{E}(t)} \leqslant \varphi(0)\exp\left(\frac{t}{\delta}\right) + \frac{1}{\sqrt{2}}\int_0^t g(s)\exp\left(\frac{t-s}{\delta}\right)ds,$$

where we have suitably multiplied both the members by $\exp(t/\alpha)$, and then recalled Eq. (47). The theorem at issue is then proved, expressing once again the continuous dependence with respect to the initial data and to the external heat supply. □

We conclude this Section observing that, unlike the case treated in the Theorem 1, the estimate (45) is representative of a continuous dependence with respect to which it is possible that also solutions exponentially growing in time exist.

7. Concluding remarks

We treated the thermodynamic compatibility of the time differential three-phase-lag model of heat conduction by means of the fading memory theory, and so we obtained the thermodynamic restrictions (16) and (17) upon the delay times and the thermal constitutive coefficients. With this appropriate approach, we studied the compatibility of the model in concern with the thermodynamical principles. We further used these restrictions in order to establish the estimate (38), describing the continuous dependence of the solution of the considered initial boundary value problem with respect to the given initial data and with respect to the supply term. The same estimate also implies the uniqueness result described in Theorem 2. In order to provide a more complete overview about the problem at issue, we also proved an additional continuous dependence theorem, this time conveniently relaxing the thermodynamic compatibility hypotheses TR1 and TR2, replacing them by the inequality (18) and thus obtaining the estimate (45). We underline here that the uniqueness result continues to hold true in this new considered case. These results were achieved thanks to a suitably formulated initial boundary value problem $\mathcal{P}^{tr}$ defined by the relations (24)–(28). In fact, it can be seen that $\mathcal{E}(t)$, appearing in both the continuous dependence theorems, represents a measure of the solution $S = \{\alpha, q_i\}$ of the initial boundary value problem $\mathcal{P}$ defined by means of the solution $\widehat{S} = \{\hat{\alpha}, \hat{q}_i\}$ of the problem $\mathcal{P}^{tr}$.

We finally remark that a natural agreement exists between the thermodynamic constrains for the model in concern and the hypotheses required to prove the well-posedness of the related initial boundary value problems.

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