Optimal design of a novel tuned mass-damper-inerter (TMDI) passive vibration control configuration for stochastically support-excited structural systems

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ABSTRACT

This paper proposes a novel passive vibration control configuration, namely the tuned mass-damper-inerter (TMDI), introduced as a generalization of the classical tuned mass-damper (TMD), to suppress the oscillatory motion of stochastically support excited mechanical cascaded (chain-like) systems. The TMDI takes advantage of the "mass amplification effect" of the inerter, a two-terminal flywheel device developing resisting forces proportional to the relative acceleration of its terminals, to achieve enhanced performance compared to the classical TMD. Specifically, it is analytically shown that optimally designed TMDI outperforms the classical TMD in minimizing the displacement variance of undamped single-degree-of-freedom (SDOF) white-noise excited primary structures. For this particular case, optimal TMDI parameters are derived in closed-form as functions of the TMD mass and the inerter constant. Furthermore, pertinent numerical data are furnished, derived by means of a numerical optimization procedure, for a 3-DOF classically damped primary structure base excited by stationary colored noise, which exemplify the effectiveness of the TMDI over the classical TMD to suppress the fundamental mode of vibration for MDOF structures. It is concluded that the incorporation of the inerter in the proposed TMDI configuration can either replace part of the TMD vibrating mass to achieve lightweight passive vibration control solutions, or improve the performance of the classical TMD for a given TMD mass.

Keywords: passive vibration control; tuned mass damper; inerter; optimum design

1 INTRODUCTION

The idea of attaching an additional free-to-vibrate mass to dynamically excited structural systems (primary structures) to suppress their oscillatory motion is historically among the first passive vibration control strategies in the area of structural dynamics [1-4]. This idea relies on designing or "tuning" the mechanical devices that link the added mass to the primary structure to achieve a "resonant" out-of-phase motion of the mass. In this context, Frahm [1] introduced the use of a linear spring-mass attachment to suppress the oscillations of harmonically excited primary structural systems in naval and mechanical engineering applications. This early "dynamic vibration absorber" was able to reduce the oscillations of single-degree-of-freedom (SDOF) primary structures within a narrow range centered at a particular (pre-specified) frequency of excitation. Later, Ormondroyd and Den Hartog [2] enhanced the effectiveness of the above absorber to dissipate the kinetic energy of primary structures subject to harmonic excitations by appending a viscous damper (dashpot) in parallel to the linear spring. Further, a semi-empirical "optimum" design procedure has been established by Den Hartog [3] and Brock [4] to "tune" the damping and stiffness properties for an *a priori* specified mass of this spring-mass-damper attachment such that the peak displacement of harmonically excited undamped SDOF primary structures is minimized (see also [5]). This design/tuning procedure relies on the "fixed point" assumption which states that all frequency response curves of the resulting two-DOF dynamical system pass through two specific points; the location of these points being independent of the damping coefficient of the dashpot. The thus tuned spring-mass-damper attachment, commonly termed in the literature as the "tuned mass-damper" (TMD), achieves the suppression of the oscillatory motion of harmonically excited primary structures over a wider range of exciting frequencies compared to a spring-mass attachment. Recently, the fixed point-based tuning procedure was shown to be very close to the "exact" solution for the optimal tuning of the classical TMD [6].

Although alternative arrangements of linear springs and dashpots (viscous dampers) have been considered in the literature to attach a mass to primary structures (see e.g. [7,8] and references therein), the above discussed "classical" TMD configuration (mass attached via a spring and a dashpot in parallel) is the most widely studied in the literature and the most commonly used one for passive vibration control of various mechanical and civil engineering structures and structural components. In particular, motivated mostly by earthquake engineering applications, substantial research work has been devoted to investigate the potential of using the classical TMD to mitigate the motion of stochastically support-excited primary structures. Using standard analytical techniques, optimal TMD parameters (damping and stiffness coefficients of the linking spring-damper elements) can be readily obtained in closed-form as functions of the TMD mass to minimize the response variance of undamped SDOF primary structures subject to white noise support excitation [9,10]. However, for the case of damped SDOF primary structures subjected to stochastic support excitations, the derivation of optimal TMD parameters by analytical approaches becomes a challenging task [11]. To this end, numerical optimization techniques are commonly employed for optimum design of TMDs to minimize the response variance for such primary structures (see e.g. [12-15]). Alternatively, simplified approximate solutions for the problem at hand have been reached by making the assumption of "lightly" damped primary structures (e.g. [16,17]). Along similar lines, several researchers proposed different approximate simplified and numerical methods for the design of TMDs for damped linear multi-degree-of-freedom

(MDOF) primary structures under stochastic base excitation widely used to model seismically excited multi-storey building structures (see e.g. [18-21] and references therein).

In recent years, several different strategies have been employed to enhance the performance of the classical TMD for passive vibration suppression of stochastically support excited structural systems including the use of multiple classical TMDs (see e.g. [22,23] and references therein), the incorporation of non-linear viscous dampers to the classical TMD configuration [24], and the consideration of hysteretic TMDs (see e.g. [25]). These strategies do offer enhanced performance compared to the classical TMD, however, optimum design/tuning becomes a challenging and computationally involved task, especially for damped MDOF primary structures. Furthermore, analytical and numerical results reported in the extensive relevant literature suggest that the effectiveness of the TMD for vibration mitigation of base-excited structures increases by increasing the attached TMD mass. This is particularly the case for high intensity support excitations (e.g. [13,21]).

In this regard, this paper proposes an alternative passive vibration control solution considering the use of a recently developed two-terminal flywheel (TTF) mechanical device, dubbed the “inerter” by Smith [26], in conjunction with the classical TMD configuration. In theory, the “ideal” inerter is a linear device with two terminals free to move independently which develops an internal (resisting) force *proportional to the relative acceleration of its terminals*. Employing rack and pinion gearing arrangements to drive a rotating flywheel, certain inerter/TTF prototypes have been physically built [26-28]. In fact, inerter/TTF devices are currently used for vibration control of suspension systems in high performance vehicles [29-30]. Further, the performance of various passive vibration control configurations for support excited building structures employing inerters placed in between the ground and the superstructure in a “base isolation” type of arrangement has been studied by Wang et al. [31,32]. It has been established that inerter devices are effective in controlling the response of rigid superstructures exposed to vertical band-limited white noise ground motions. Furthermore, passive vibration control systems comprising inerters in conjunction with springs and dampers have been considered by Lazar et al. [33] for vibration isolation of SDOF and of two-DOF primary systems subjected to recorded earthquake excitations applied along the vertical direction.

The present research work is motivated by the fact that an inerter/TTF device with approximately 1 kg of physical mass can have a constant of resisting force within the range of 60–200 kg depending on the size of the flywheel [27]. Thus, the aim of the herein proposed tuned mass-damper-inerter (TMDI) configuration is to exploit the mass amplification effect of the inerter. Attention is focused on introducing the underlying equations of motion for linear SDOF and MDOF primary structures, to demonstrate that the TMDI constitutes a generalization of the classical TMD and to provide analytical and numerical evidence demonstrating its enhanced performance compared to the TMD. The remainder of this paper is organized as follows: Section 2 introduces the TMDI for the case of linear SDOF primary structures exposed to stochastic support excitation. The governing equations of motion are derived for damped primary structures and analytical expressions for optimum TMDI parameters minimizing the displacement variance for the special case of undamped white noise excited SDOF primary structures are obtained. Section 3 proposes a TMDI configuration to suppress oscillations following the fundamental mode of vibration of support-excited damped MDOF chain-like primary structures. A numerical optimization procedure for optimum design of the TMDI system for these primary structures is also discussed. Section 4 provides numerical data to demonstrate the effectiveness and applicability of the TMDI vis-à-vis the classical TMD for classically damped MDOF chain-like primary structures. Section 5, summarizes the main conclusions of this work.

2 PROPOSED TUNED MASS-DAMPER-INERTER (TMDI) CONFIGURATION FOR SINGLE-DEGREE-OF-FREEDOM (SDOF) SUPPORT-EXCITED PRIMARY STRUCTURES

Consider a linear damped single-degree-of-freedom (SDOF) dynamical system (primary structure) modeled by a linear spring of stiffness k_1, a mass m_1, and a viscous damper with damping coefficient c_1, based-excited by an acceleration stochastic process $a_g\left(t\right)$. To suppress the oscillatory motion of this primary structure it is herein proposed to consider the classical tuned mass-damper (TMD), in conjunction with a two terminal flywheel (inerter) device as shown in Fig. 1. The TMD consists of a mass m_{TMD} attached to the primary structure via a linear spring of stiffness k_{TMD} and a viscous damper with damping coefficient c_{TMD}. The inerter device connects the TMD mass to the supporting ground. It is noted in passing that the idea of placing the damper in between the TMD mass and the ground instead of in between the TMD mass and the primary structure has been explored in the literature (e.g. [34]). However, such "non-traditional" TMD topologies are not considered in this work.

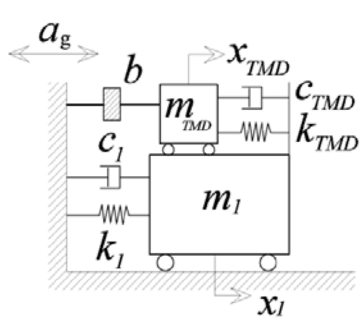

Figure 1. Single-degree-of-freedom (SDOF) primary structure incorporating the proposed tuned mass-damper-inerter (TMDI) configuration.

In Fig. 1, the inerter is depicted by a hatched box which should be interpreted as a mechanical two-terminal device similarly to springs and dampers. To facilitate this interpretation, Fig. 2 depicts an inerter device whose terminals are subject to an equal and opposite externally applied force *F* in equilibrium with the internally developed force.

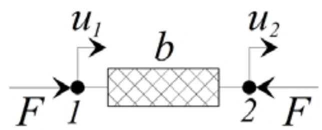

Figure 2.Schematic representation of the two-terminal flywheel device (*b* is the mass-equivalent constant of proportionality)

By definition, the following relationship holds for the ideal linear inerter (e.g. [26,28])

$$F = b(\ddot{u}_1 - \ddot{u}_2), \tag{1}$$

where u_1 and u_2 are the displacement coordinates of the two terminals and a dot over symbol signifies differentiation with respect to time *t*. In the above equation, the constant of proportionality *b* attains mass units and fully characterizes the behavior of the inerter. In the next section, equations of motion are derived for the two degree of freedom system of Fig. 1 by assuming that the physical mass of the inerter and of the TMD damper and spring are negligible compared to the masses m_1 and m_{TMD}.

3.1 Governing equations of motion

The governing equations of motion of the linear dynamical system shown in Fig.1 can be readily written in matrix form as

$$\begin{bmatrix} m_{TMD}+b & 0 \\ 0 & m_1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_{TMD} \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} c_{TMD} & -c_{TMD} \\ -c_{TMD} & c_1+c_{TMD} \end{bmatrix}\begin{Bmatrix} \dot{x}_{TMD} \\ \dot{x}_1 \end{Bmatrix} + \begin{bmatrix} k_{TMD} & -k_{TMD} \\ -k_{TMD} & k_1+k_{TMD} \end{bmatrix}\begin{Bmatrix} x_{TMD} \\ x_1 \end{Bmatrix} = -\begin{Bmatrix} m_{TMD} \\ m_1 \end{Bmatrix} a_g , \tag{2}$$

derived by considering equilibrium of forces and by application of D'Alembert's principle. In the above equations, x_1 and x_{TMD} are the displacement processes relative to the motion of the ground of the primary structure mass and of the TMD mass (see also Fig.1).

Denote by ω_1 and ζ_1 the natural frequency and the ratio of critical damping of the primary structure expressed by

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \quad \text{and} \quad \zeta_1 = \frac{c_1}{2m_1\omega_1}, \tag{3}$$

respectively. Further, let ω_{TMD} and ζ_{TMD} be the natural frequency and the critical damping ratio of the TMD defined as

$$\omega_{TMD} = \sqrt{\frac{k_{TMD}}{m_{TMD}}} \quad \text{and} \quad \zeta_{TMD} = \frac{c_{TMD}}{2m_{TMD}\omega_{TMD}}, \tag{4}$$

respectively. It is common practice in the analysis and design of systems equipped with classical TMDs to consider the dimensionless mass ratio μ and the dimensionless frequency ratio υ expressed by

$$\mu = \frac{m_{TMD}}{m_1} \quad \text{and} \quad \upsilon = \frac{\omega_{TMD}}{\omega_1}, \tag{5}$$

respectively. Furthermore, an additional dimensionless parameter β is herein introduced defined by the ratio of the mass of the primary structure over the inerter constant b. That is,

$$\beta = \frac{b}{m_1} \tag{6}$$

Considering the normalized acceleration input stochastic process $a_g(t)/\omega_1^2$, the complex frequency response function (FRF) in terms of the relative displacement x_1 of the primary structure of Fig.1 can be written as

$$G_1(\omega) = \frac{x_1}{a_g}\omega_1^2 =$$

$$\frac{(1+\mu)\omega_{TMD}^2 - \frac{\beta+\mu}{\mu}\omega^2 + i2\zeta_{TMD}(1+\mu)\omega_{TMD}\omega}{(1-\frac{\omega^2}{\omega_1^2} + i2\zeta_1\frac{\omega}{\omega_1})(\omega_{TMD}^2 - \frac{\beta+\mu}{\mu}\omega^2 + i2\zeta_{TMD}\omega_{TMD}\omega) - \frac{\beta+\mu}{\mu}(\omega_{TMD}^2 + i2\zeta_{TMD}\omega_{TMD}\omega)\omega^2}, \tag{7}$$

in the domain of frequency ω. In the latter equation and hereafter i denotes the imaginary unit ($i=\sqrt{-1}$). The FRF $G_1(\omega)$ can be viewed as the frequency domain counterpart of the time-domain equations of motion Eqs.(2) (see also [5]).

It is important to note that by setting $b=\beta=0$ in Eqs. (2) and (7) the equations of motion and the FRF in terms of the relative displacement x_1, respectively, for a damped SDOF primary system equipped with the classical TMD are retrieved. In this respect, *it is seen that the proposed tuned mass-damper-inerter (TMDI) configuration for passive vibration control can be interpreted as a generalization of the classical TMD*. In the following section, optimum TMDI design parameters (υ and ζ_{TMD}) are derived in closed-form to suppress the vibratory motion of undamped SDOF primary structures under white noise support excitation. The latter is a well-studied in the literature special case for which analytical formulae for the optimal "tuning" of the classical TMD exist (see e.g. [10]).

3.2 Optimum design of TMDI for undamped primary structures under white noise support excitation

Assuming a stationary stochastic support excitation process $a_g\left(t\right)/\omega_1^2$ represented in the frequency domain by a double-sided spectral density function (power spectrum) $S(\omega)$, the variance of the relative displacement process x_1 of the primary structure of Fig. 1 is written as

$$\sigma_1^{\ 2}=\int_{-\infty}^{+\infty}\left|G_1(\omega)\right|^2 S(\omega)d\omega\,. \tag{8}$$

In the latter equation, the "transfer function" $\left|G_1(\omega)\right|^2$ is the squared modulus of the FRF defined in Eq.(7).

Given μ and β mass ratios, it is sought to determine optimum values for the stiffness k_{TMD} and damping c_{TMD} constants of the considered TMDI configuration, or equivalently υ

and ζ_{TMD} dimensionless parameters (see Eqs. (3) and (4)), which minimize the variance σ_1^2 in Eq. (8) for the case of undamped SDOF primary structures under white noise support excitation. To this aim, note that for an undamped primary structure ($\zeta_1=0$) the transfer function in Eq.(8) can be written as

$$\left|G_1(\omega)\right|^2 = \frac{b_2\omega^4 + b_1\omega^2 + b_0}{\left(a_4\omega^4 + a_3\omega^3 + a_2\omega^2 + a_1\omega + a_0\right)\left(a_4\omega^4 - a_3\omega^3 + a_2\omega^2 - a_1\omega + a_0\right)} \quad (9)$$

where

$$\begin{aligned} b_0 &= \omega_1^4\upsilon^4(1+\mu)^2 ; \\ b_1 &= 4\zeta_{TMD}^2(1+\mu)^2\omega_1^2\upsilon^2 - 2\omega_1^2\upsilon^2\frac{(\beta+\mu)(1+\mu)}{\mu} ; \\ b_2 &= \left(\frac{\beta+\mu}{\mu}\right)^2 ; \end{aligned} \quad (10)$$

and

$$\begin{gathered} a_0 = \omega_1^2\upsilon^2 ;\; a_1 = -i2\zeta_{TMD}\omega_1\upsilon ;\; a_2 = -\left(\frac{\beta+\mu}{\mu} + \upsilon^2(1+\beta+\mu)\right) ; \\ a_3 = \frac{i2\zeta_{TMD}(1+\beta+\mu)\upsilon}{\omega_1} \quad ; \quad a_4 = \frac{\beta+\mu}{\mu\omega_1^2} . \end{gathered} \quad (11)$$

Assuming a constant power spectrum over all frequencies $S(\omega)=S_0$ (ideal white noise) and using standard analytical techniques to evaluate the integral in Eq.(8) (see e.g. [35]) the variance σ_1^2 for an undamped primary system equipped with a TMDI is expressed as

$$\sigma_1^2 = \pi S_0\omega_1 \frac{\mu^2 C_1 + \beta^2 C_2 + 2\beta\mu C_3}{2\zeta_{TMD}\upsilon\mu(\beta+\mu)^2} , \quad (12)$$

in which

$$\begin{aligned} C_1 &= 1 + \upsilon^4(1+\mu)^4 + \upsilon^2(1+\mu)^2[\mu - 2 + 4\xi_{TMD}^2(1+\mu)] ; \\ C_2 &= 1 + \upsilon^2\mu(1+\mu)[\mu(1+\upsilon^2(1+\mu)) - 1] ; \\ C_3 &= 1 + \upsilon^2(1+\mu)[\mu(\mu + 2\xi_{TMD}^2(1+\mu) + \upsilon^2(1+\mu)^2 - 1)] . \end{aligned} \quad (13)$$

Assuming constant mass ratios μ and β, the variance $\sigma_1^{\ 2}$ of Eq. (12) is minimized in terms of the TMD frequency ratio υ and damping ratio ζ_{TMD} by enforcing the following two conditions

$$\frac{\partial \sigma_1^2}{\partial \upsilon} = 0 \quad \text{and} \quad \frac{\partial \sigma_1^2}{\partial \zeta_{TMD}} = 0 \ . \tag{14}$$

These conditions yield a system of two equations from which the "optimal" tuning parameters υ and ζ_{TMD} of the proposed TMDI configuration are found in terms of the mass ratios μ and β as

$$\upsilon = \frac{1}{1+\beta+\mu} \frac{\sqrt{(\beta+\mu)[\beta(\mu-1)+(2-\mu)(1+\mu)]}}{\sqrt{2\mu(1+\mu)}}, \tag{15}$$

and

$$\zeta_{TMD} = \frac{(\beta+\mu)\sqrt{\beta(3-\mu)+(4-\mu)(1+\mu)}}{2\sqrt{2\mu(1+\beta+\mu)[\beta(1-\mu)+(2-\mu)(1+\mu)]}} \ . \tag{16}$$

Further, by substitution of the above optimal TMDI tuning parameters into Eq. (12) the following expression for the achieved minimum variance of the relative displacement process x_1 is obtained

$$\sigma_{1,\min}^{\ \ 2} = \pi S_0 \omega_1 (1+\mu) \sqrt{\frac{(1+\mu)[\beta(3-\mu)+(4-\mu)(1+\mu)]}{(\beta+\mu)(1+\beta+\mu)}} \ . \tag{17}$$

It is important to note that by setting *b*=β=*0* in Eqs. (15)-(17) the optimal tuning formulae of the classical TMD which minimize the relative displacement variance of an undamped SDOF primary structure subjected to white noise support excitation reported in the literature [10] can be retrieved. In the following section, the potential of the TMDI to suppress the oscillatory motion of white noise support excited undamped SDOF primary structures is assessed vis-à-vis the classical TMD.

2.3 Assessment of TMDI effectiveness vis-à-vis the classical TMD for undamped primary structures under white noise support excitation

To facilitate comparison between the proposed TMDI configuration of Fig. 1 and the TMD, the previously derived formulae for the optimal tuning of the TMDI are juxtaposed with the known formulae corresponding to the classical TMD in Table 1. In Figs. (3) and (4), Eqs. (15) and (16) are plotted, respectively, for four different values of the mass ratio μ and for β ratios within a suggested interval of practical interest [0,1], with $\beta=0$ being the limiting value for which the TMDI degenerates to the classical TMD. It is observed that the influence of the apparent "mass amplification" effect due to the additional inerter device incorporated in the proposed TMDI is more prominent for lower values of the mass ratio μ. Specifically, for $\mu>0.6$ the "optimum" frequency ratio υ decreases slightly as β increases, while for $\mu<0.4$ the ratio υ increases significantly for values of β up to about 0.3 to 0.4 and then decays for higher values of β. More importantly, the achieved "optimum" damping ratio ζ_{TMD} increases monotonically (and almost linearly) for all considered values of μ as the normalized inerter constant β increases. However, the rate of this increase deteriorates for higher values of μ. These trends suggest that the incorporation of an inerter device to the classical TMD is more beneficial for relatively smaller values of the mass ratio μ (i.e. for relatively lower vibrating TMD masses) as it allows for "driving" viscous dampers with higher kinetic energy absorption capabilities (i.e. damping ratios).

Table 1. Closed-form expressions for optimal tuning of the proposed TMDI configuration for undamped SDOF primary structures subject to white noise base excitation vis-à-vis the classical TMD case.

	Classical TMD (b=0)	*Proposed TMDI (b>0)*
Variance of x_1	$\sigma_1^2 = \pi S_0 \omega_1 \dfrac{C_1}{2\xi_{TMD}\upsilon\mu}$	$\sigma_1^2 = \pi S_0 \omega_1 \dfrac{\mu^2 C_1 + \beta^2 C_2 + 2\beta\mu C_3}{2\zeta_{TMD}\upsilon\mu(\beta+\mu)^2}$
Frequency ratio	$\upsilon = \dfrac{\sqrt{(1-\mu/2)}}{1+\mu}$	$\upsilon = \dfrac{1}{1+\beta+\mu}\dfrac{\sqrt{(\beta+\mu)[\beta(\mu-1)+(2-\mu)(1+\mu)]}}{\sqrt{2\mu(1+\mu)}}$
TMD damping ratio	$\zeta_{TMD} = \dfrac{\sqrt{\mu(1-\mu/4)}}{\sqrt{4(1+\mu)(1-\mu/2)}}$	$\zeta_{TMD} = \dfrac{(\beta+\mu)\sqrt{\beta(3-\mu)+(4-\mu)(1+\mu)}}{2\sqrt{2\mu(1+\beta+\mu)[\beta(1-\mu)+(2-\mu)(1+\mu)]}}$

Minimum achieved variance of x_1	$\sigma_{1,\min}{}^2 = \pi S_0 \omega_1 (1+\mu)\sqrt{\frac{(1+\mu)(4}{\mu}}$	$\sigma_{1,\min}{}^2 = \pi S_0 \omega_1 (1+\mu)\sqrt{\frac{(1+\mu)[\beta(3-\mu)+(4-\mu)(}{(\beta+\mu)(1+\beta+\mu)}}$

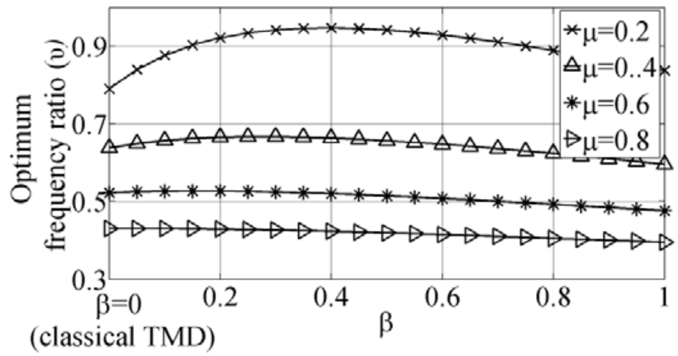

Figure 3. Optimum TMD frequency ratio for various values of β and several mass ratio values

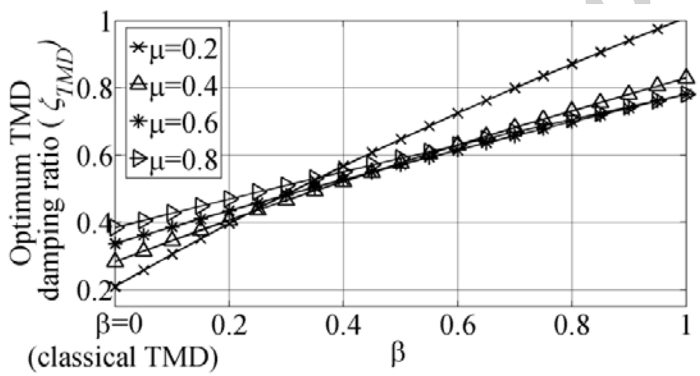

Figure 4. Optimum TMD damping ratio for various values of β and several mass ratio values

The above argument is confirmed by the numerical data of Fig. 5 in which the minimum relative displacement variance of the primary structure achieved by means of TMDI $\sigma_{1,\min}{}^2(b>0)$ is plotted (Eq.(17)), normalized by the minimum variance achieved via the classical TMD $\sigma_{1,\min}{}^2(b=0)$ for the same values of the mass ratios μ and β as previously considered. In all cases, the displacement response variance decreases significantly as the parameter β increases demonstrating that the proposed TMDI configuration is more effective to suppress oscillations compared to the classical TMD. Further, the effectiveness of the TMDI increases considerably as lower values of the mass ratio μ (μ<0.4) are considered, commonly adopted in practical TMD implementations.

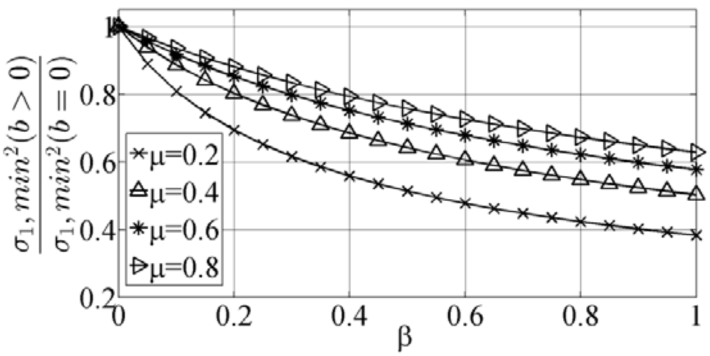

Figure 5. Minimum variance ratio between the proposed model (b>0) and the classical TMD (b=0)

In fact, although in most practical applications of the TMD the considered mass ratio μ rarely exceeds values of 0.2~0.3, it can be shown that the proposed TMDI configuration is more effective than the classical TMD in suppressing the relative displacement variance of the primary structure for all values of the mass ratio μ within the interval (0,4]. Specifically, by relying on Eq. (17), it can be shown that

$$\frac{\sigma_{1,\min,}{}^{2}(b>0)}{\sigma_{1,\min,}{}^{2}(b=0)}<1 \tag{18}$$

for

$$0<\mu \;\leq\; 3 \text{ and } \beta>0 \;, \tag{19}$$

and for

$$3<\mu\leq 4 \quad \text{and} \quad 0<\beta\leq\frac{4+3\mu-\mu^2}{\mu-3}\,. \tag{20}$$

In view of the above analytical and numerical data, it can be concluded that the use of the inerter in the proposed configuration reduces significantly the minimum variance of an undamped SDOF subjected to white noise base acceleration in comparison with classical TMD. In the following section, the optimum tuning parameters υ and ζ_{TMD} obtained for undamped SDOF primary structures (Eqs. (15) and (16)) are used to facilitate the optimum design of a novel TMDI configuration to suppress the oscillatory motion of support excited multi-degree-of-freedom (MDOF) primary structures.

3 PROPOSED TUNED MASS-DAMPER-INERTER (TMDI) CONFIGURATION FOR MULTI-DEGREE-OF-FREEDOM (MDOF) SUPPORT-EXCITED PRIMARY STRUCTURES

Consider a linear proportionally (classically) damped multi-degree-of-freedom (MDOF) system with masses m_j (j=1,2,…n), linked together by linear springs of stiffness coefficients k_j (j=1,2,…n) and viscous dampers with damping coefficients c_j (j=1,2,…n) as shown in Fig. 6, where n is the number of the DOFs of the system, based-excited by an acceleration stochastic process $a_g(t)$. To suppress the motion of this "chain-like" MDOF primary structure according to its, presumably dominant, fundamental (first) mode shape of vibration, it is proposed to consider a tuned mass-damper (TMD) attached to the "lead" m_1 mass in conjunction with an inerter device connecting the TMD mass to the penultimate m_2 of the primary structure (Fig. 6). From a practical viewpoint, note that by eliminating the inerter (b=0), the above tuned mass-damper-inerter (TMDI) configuration coincides with TMD arrangements commonly used to control the fundamental mode of vibration of seismically excited "regular" multi-storey building structures. These arrangements involve the attachment of an additional TMD mass to the top building floor (e.g. [19]), or, equivalently, the "isolation" of the upper stories from the rest of the building such that they vibrate independently (e.g. [20]). In the latter case, the total mass of the upper isolated stories becomes the TMD mass. In this regard, as in the case of the SDOF primary structure presented in previous sections, *the herein proposed TMDI configuration can be interpreted as a generalization of the commonly used TMD arrangements for passive vibration control of support-excited MDOF primary structures.*

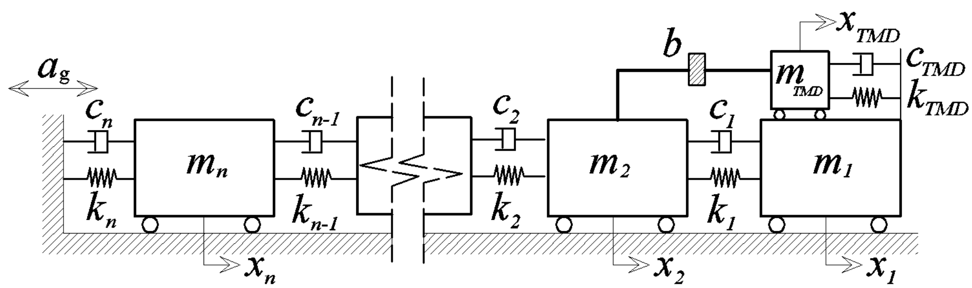

Figure 6. Multi-degree-of-freedom (MDOF) primary structure incorporating the proposed tuned mass-damper-inerter (TMDI) configuration.

4.1 Governing equations of motion

Given that the dynamical structural system of Fig. 6 is linear, the derivation of its governing equations of motion can be significantly facilitated by considering passive mechanical "admittances" Q defined as the ratio of force over velocity in the Laplace domain (e.g. [36]). This is a common practice in topology studies of mechanical system networks. In this respect, the considered MDOF primary structure equipped with the TMDI configuration of Fig. 6 can be interpreted as a system of n+1 masses inter-connected by "networks" represented by admittances Q as shown in Fig. 7.

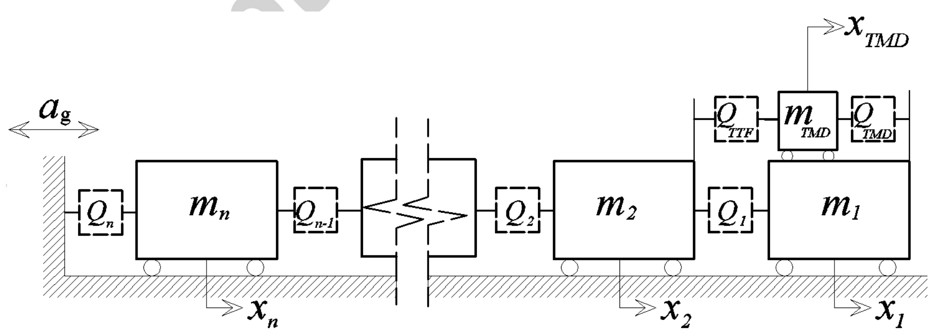

Figure 7. Multi-degree-of-freedom (MDOF) primary structure incorporating the proposed tuned mass-damper-inerter (TMDI) configuration: mechanical admittance representation.

In particular, the mechanical admittances shown in Fig.7 are expressed in terms of the standard Laplace variable s by

$$Q_{TTF}(s) = bs \quad ; \quad Q_{TMD}(s) = \frac{k_{TMD}}{s} + c_{TMD} \quad ; \quad Q_j(s) = \frac{k_j}{s} + c_j \ \left(j = 1 \ldots n\right), \qquad (21)$$

where Q_{TTF} is the admittance corresponding to the inerter (two terminal flywheel), Q_{TMD} is the admittance corresponding to the TMD-spring-and-damper-in-parallel "network" connecting the TMD mass to the lead mass m_1 of the primary structure and Q_j are the admittances of the n spring-plus-dashpot-in-parallel "networks" linking the n masses of the primary structure together and with the ground (see Figs. 6 and 7). By relying on the previous expressions, the $n+1$ equations of motions of the linear MDOF dynamical system of Fig. 6 can be written in the Laplace domain as

$$\mathbf{B}(s)\tilde{\mathbf{X}}(s) = -\mathbf{M}\boldsymbol{\delta}\tilde{A}(s), \qquad (22)$$

where

$$\mathbf{B} = \begin{bmatrix} m_{TMD}s^2 + (\mathrm{Q}_{TMD} + \mathrm{Q}_{TTF})\mathrm{s} & -\mathrm{Q}_{TMD}\, s & -\mathrm{Q}_{TTF}\, s & 0 & \cdots & 0 \\ -\mathrm{Q}_{TMD}\, s & m_1 s^2 + (\mathrm{Q}_1 + \mathrm{Q}_{TMD})\mathrm{s} & -\mathrm{Q}_1\, s & 0 & \cdots & 0 \\ -\mathrm{Q}_{TTF}\, s & -\mathrm{Q}_1\, s & m_2 s^2 + (\mathrm{Q}_1 + \mathrm{Q}_2 + \mathrm{Q}_{TMD})\mathrm{s} & -\mathrm{Q}_2\, s & \cdots & 0 \\ 0 & 0 & -\mathrm{Q}_2\, s & m_3 s^2 + (\mathrm{Q}_2 + \mathrm{Q}_3)\mathrm{s} & \ddots & \vdots \\ 0 & 0 & 0 & -\mathrm{Q}_3\, s & \ddots & -\mathrm{Q}_{n-2}\, s \\ \vdots & \vdots & \vdots & \vdots & \ddots & m_n s^2 + (\mathrm{Q}_{n-1} + \mathrm{Q}_n)\mathrm{s} \\ 0 & 0 & 0 & 0 & \cdots & -\mathrm{Q}_{n-1}\, s \end{bmatrix}$$

,(23)

$\boldsymbol{\delta}$ is the unit column vector, $\tilde{A}(s)$ is the Laplace transform of the support acceleration process $a_g(t)$, $\mathbf{M}$ is the diagonal mass matrix of the system written as

$$\mathbf{M} = \begin{bmatrix} m_{TMD} & 0 & \cdots & 0 \\ 0 & m_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix}, \qquad (24)$$

and $\tilde{\mathbf{X}}(s)$ is the Laplace transform of the vector

$$\mathbf{x} = \left\{ x_{TMD}(t) \quad x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t) \right\}^T \qquad (25)$$

collecting the relative displacements of the *n+1* masses included in the considered system. In the latter equation, the superscript "*T*" denotes matrix transposition.

The frequency response function (FRF) $H_1(\omega)$ relating the (input) support excitation in terms of acceleration to the (output) relative displacement of the lead mass m_1 of the primary structure is reached by evaluating the ratio

$$H_1(s) = \frac{\tilde{x}_1(s)}{\tilde{A}(s)} \tag{26}$$

along the imaginary axis $s=i\omega$. In the latter equation, $\tilde{x}_1(s)$ is the Laplace transform of $x_1(t)$ which is analytically found by solving Eq. (22). That is,

$$\tilde{\mathbf{X}}(s) = -\mathbf{B}^{-1}(s)\mathbf{M}\boldsymbol{\delta}\tilde{A}(s). \tag{27}$$

In the following section, the $H_1(\omega)$ FRF is utilized to obtain "optimal" TMDI parameters for the suppression of the oscillatory motion of the dynamical system of Fig. 6 according to its fundamental mode of vibration.

4.2 Optimum design of TMDI to control the first mode of damped MDOF primary structures under stochastic base excitation

Consider the dimensionless modal mass ratio defined by

$$\mu_M = \frac{m_{TMD}}{M_1}, \tag{28}$$

where M_1 is the generalized mass of the fundamental mode shape of the uncontrolled (primary) chain-like MDOF structure of Fig. 6 given by the expression

$$M_1 = \boldsymbol{\varphi}_1^{\mathbf{T}}\mathbf{M}\boldsymbol{\varphi}_1. \tag{29}$$

In the last equation $\boldsymbol{\varphi}_1$ is the fundamental mode shape vector (eigenvector) normalized by the modal coordinate corresponding to the lead mass m_1 (see also [19]). Further, similarly to the case of SDOF primary structures discussed in section 2.3, a second dimensionless (modal) mass ratio involving the constant of the inerter *b* is defined as

$$\beta_M = \frac{b}{M_1}. \tag{30}$$

Assuming a stationary non-normalized stochastic support excitation process $a_g(t)$ represented in the frequency domain by a one-sided spectral density function (power spectrum) $A(\omega)$, it is sought to determine "optimal" design values for the frequency ratio υ and the damping ratio ζ_{TMD} defined in Eqs. (4) and (5), respectively, to minimize the variance of the process x_1 (relative displacement of the lead mass m_1) given the mass ratios μ_M and β_M (see also [13],[19]). To this aim, the following dimensionless cost function or "performance index" (*PI*) is considered

$$PI = \frac{J^{TMDI}}{J^0}, \tag{31}$$

where J^0 and J^{TMDI} denote the relative displacement variance of the lead mass (m_1) for an uncontrolled primary structure exposed to the support acceleration $a_g(t)$ and for the same primary structure equipped with the proposed TMDI configuration, respectively. Specifically,

$$J^{TMDI} = \int_0^\infty \left|\mathrm{H}_1(\omega)\right|^2 A(\omega) d\omega, \tag{32}$$

where $\left|H_1(\omega)\right|^2$ is the squared modulus of the function defined in Eq. (26) evaluated on the imaginary axis of the Laplace *s*-plane.

In all of the ensuing numerical work, a standard MATLAB® built-in "min-max" constraint optimization algorithm employing a sequential programming method is used to minimize the *PI* of Eq. (31) for the design parameters υ and ζ_{TMD} [15]. The required "seed" values of υ and ζ_{TMD} used to initiate the optimization algorithm are determined by substituting $\beta_M \to \beta$ and $\mu_M \to \mu$ in Eqs. (15) and (16), respectively. These values minimize the considered *PI* for an undamped linear SDOF primary structure under white noise support excitation, as detailed in Section 2. Further, the constraints

$$0.5 < \upsilon < 4.50 \;\; and \;\; 0 < \zeta_{TMD} < 1.00 \tag{33}$$

are enforced to the sought TMDI design parameters relying on physical considerations.

4 NUMERICAL APPLICATION OF THE TMDI FOR DAMPED MDOF SYSTEMS

In this section optimum design parameters are derived following the procedure discussed in Section 3.2 for the TMDI passive vibration control configuration of Fig. 6. A 3-DOF primary structure (n=3) is considered whose inertial and elastic properties are shown in Table 2. The undamped natural frequencies of the considered primary structure obtained from standard modal analysis are ω_1= 6.37rad/s, ω_2= 13.02rad/s, and ω_3= 20.57rad/s. Further, the fundamental mode shape normalized by the modal coordinate of the lead mass m_1 is computed as $\boldsymbol{\varphi}_1 = \{1.000 \quad 0.593 \quad 0.286\}^T$ and the corresponding generalized mass is equal to M_1= 16.9x10^3 kg (Eq. (29)). The damping coefficients of the considered primary structure are assumed to be stiffness proportional ("classically" damped system), determined by the expression

$$c_j = \frac{2\zeta_1}{\omega_1} k_j \quad \left(j = 1,2,3 \right) , \tag{34}$$

in which ζ_1 is the critical damping ratio of the fundamental mode shape taken equal to 0.02.

Table 2. Inertial and elastic properties of the considered 3-DOF primary structure

j	Mass m_j (kg)	Stiffness k_j (N/m)	Damping c_j (Ns/m)
1	10 x 10^3	10 x 10^5	6280

2	15 x 10^3	25 x 10^5	15670
3	20 x 10^3	35 x 10^5	21980

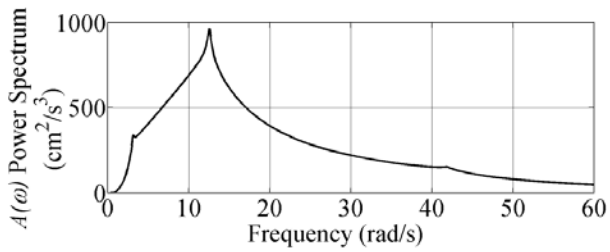

Figure 8. One-sided power spectrum representing the acceleration support excitation $\alpha_g(t)$.

The input action is represented by the stationary "colored noise" power spectrum $A(\omega)$ plotted in Fig. 8. Incidentally, this spectrum is compatible in the "mean sense" with the elastic spectrum of the European seismic code for peak ground acceleration 0.36g (g=981cm/s^2) and ground type "B" [37]. It has been derived by a methodology described in Giaralis and Spanos [38].

1.1 *Optimum design of the classical TMD as a special case of the TMDI configuration*

As discussed in Section 3, by setting $b \to 0$, or equivalently $\beta_M \to 0$ (Eq. (30)), the proposed TMDI configuration depicted in Fig. 6 becomes the classical TMD used to suppress oscillations according to the fundamental mode of vibration for MDOF primary structures. Therefore, optimal design TMD parameters for the frequency ratio υ and the damping ratio ζ_{TMD} can be determined by following the procedure outlined in section 3.2. For example, set $\beta_M = 0$ and let the TMD mass be equal to 450 kg, that is, 1% of the total mass of the 3-DOF primary structure with properties given in Table 2. The modal mass ratio becomes $\mu_M = 0.0267$ (Eq.(28)). Next, the "seed" values υ= 0.967 and $\zeta_{TMD} = 0.081$ are computed from Eqs. (15) and (16) respectively to initialize the adopted optimization algorithm to minimize the cost function of Eq. (31) (see section 3.2). The obtained values for the frequency ratio υ and damping ratio ζ_{TMD} for the particular case considered are shown in Figs. 9 and 10, respectively. Similar computations are performed for different values of the TMD mass within a commonly used in engineering applications range: 1% to 10% of the total mass of the primary structure, corresponding to 450kg to 4500kg of mass. The optimal frequency ratio υ and damping ratio ζ_{TMD} parameters obtained from the adopted optimization procedure are plotted as functions of the TMD mass in Figs. 9 and 10, respectively (b=0 classical TMD curve). Further, the achieved value of the performance index of Eq. (31) for the classical TMD (b =0) are plotted in Figure 11 as a function of the TMD mass.

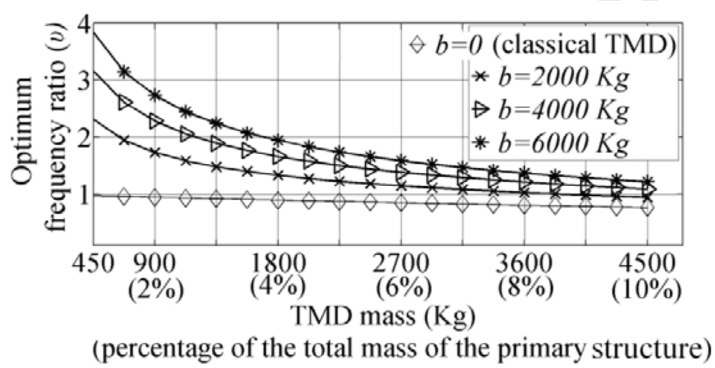

Figure 9. Optimum frequency ratio as a function of the TMD mass for various values of the inerter constant b to control the fundamental mode of vibration of the 3-DOF primary structure of Table 2.

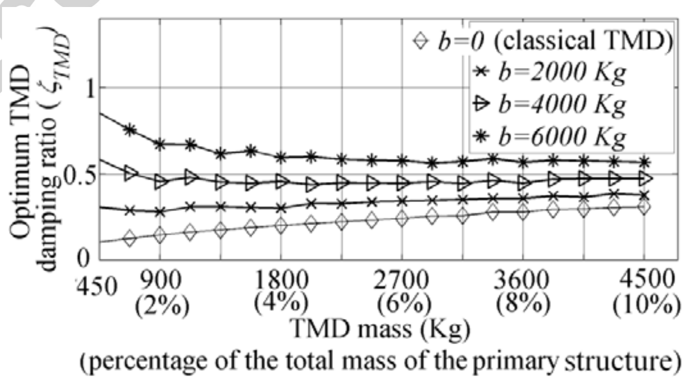

Figure 10. Optimum damping ratio as a function of the TMD mass for various values of the inerter constant b to control the fundamental mode of vibration of the 3-DOF primary structure of Table 2.

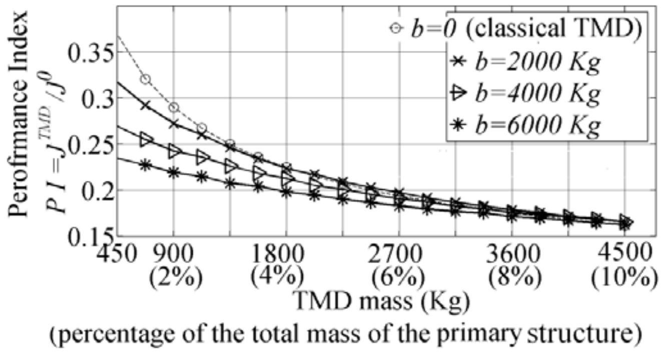

Figure 11. Achieved performance index versus the TMD mass for various values of the inerter constant *b*.

It is noted that the numerical data presented in Figs. 9 to 11 for the classical TMD are in alignment with similar results reported in the literature obtained by alternative numerical optimization techniques (see e.g. [13,19,23] and references therein). Specifically, increased values of the TMD mass require higher damping ratios ζ_{TMD} values and lower TMD frequency ratios to achieve optimal tuning. Consequently, larger TMD mass is more effective in controlling the dynamic response of the primary structure according to its fundamental mode shape, at the cost of an increase total weight of the structural system. However, the rate of decay of the PI (proportional to the variance of the relative displacement of the m_1 mass) decreases rapidly (i.e. PI "saturates") as the TMD mass increases. It reaches a practically constant value for *TMD* mass larger than 5% the total mass of the considered primary structure.

1.2 *Optimum design of the TMDI configuration (b>0)*

Let an inerter device be incorporated to the considered 3-DOF primary structure with the properties listed in Table 2 according to the proposed TMDI configuration of Fig. 6. The previously considered optimization procedure is used to derive optimum TMDI parameters (υ and ζ_{TMD}) minimizing the cost function of Eq.(31) for the input power spectrum of Fig. 8.

The same range of pre-specified TMD mass (1% to 10% of the total mass of the primary structure) is considered as in the previous section, while three different values of the inerter constant b are taken: 2000, 4000, and 6000 kg. Optimal TMDI parameters are plotted in Figs. 9 and 10 in which the case of the classical TMD (b=0) is also included for comparison purposes. Further, in Fig. 11 curves of the performance index of Eq. (31) achieved by the different TMDI systems considered are superposed to numerical results corresponding to the classical TMD (b =0). It can be readily seen from the herein reported numerical data that the value of the Performance Index, or equivalently the variance of the relative displacement of the m_1 mass, is reduced as the value of the inerter b increases. In fact, in all cases considered, the proposed TMDI configuration outperforms the classical TMD in terms of minimizing the adopted cost function. The achieved improved performance of the TMDI over the classical TMD is reported in the rightmost column of Table 3 for several selected cases. The performance improvement is considerably higher for relatively small TMD mass values (less than about 3% of the total mass of the primary structure) while it becomes less significant for TMD mass values greater than 6% of the total mass of the primary structure. Note that similar trends were found in the case of the undamped SDOF primary structure for which optimal TMDI parameters have been derived in closed form (Fig. 5). Therefore, it can be concluded that the effectiveness of an inerter device to suppress the displacement response variance beyond what can be achieved by the classical TMD increases for relatively small TMD masses. It is also noted that the enhanced performance of the TMDI system requires that the TMD mass is attached to the primary structure by "stiffer" connection arrangements and by considering viscous damping devices with higher damping coefficients.

More importantly, the herein furnished data demonstrate that the "mass amplifying" effect of the inerter device can be effectively used to replace part of the oscillating TMD mass and, thus, to reduce the total weight of the structural system for the same level of performance in terms of keeping the oscillatory motion of the primary structure below a certain threshold. For example, as shown in Table 3, in the case of the considered 3-DOF primary structure, an optimally tuned TMDI with an inerter device of "mass" constant b=6000 kg and a TMD mass of 450kg achieves similar level of performance (more than 75% reduction to the displacement variance of the m_1 mass compared to the uncontrolled primary structure) as an optimally tuned classical TMD with four times heavier oscillating mass (1800kg). However, the physical mass of the considered inerter might be up to two orders of magnitude smaller than its b constant. Specifically, ratios of constant b over physical mass for inerters of up to 200/1 or more have been reported in the literature[27]. Adopting this ratio, the considered inerter has a physical mass of 6000/200= 30kg. Therefore, the total weight of the examined TMDI system remains about four times lighter than a classical TMD for similar vibration control performance assuming that the weight of the equipment used to attach the TMD mass to the primary structure and of the viscous damping devices are similar in both cases. The latter consideration may have significant advantages in certain real-life structural passive vibration control design scenarios necessitating the use of large TMD masses to achieve the desired vibration suppression effect, as is the case of building structures excited by severe earthquake induced strong ground motions (see e.g. [13, 20] and references therein).

Table 3. Optimal TMDI parameters, Performance Index (PI) and percentage difference of PI achieved for different values of the TMD mass and the inerter constant b compared to the classical TMD (b=0).

m_{TMD} (kg)	b (kg)	υ	ζ_{TMD}	PI	Percentage difference of PI compared to the classical TMD (%)
450	0	0.97	0.105	0.369	-
(1%)	2000	2.31	0.307	0.317	14.1
	4000	3.16	0.582	0.270	27.0

	6000	3.84	0.852	0.235	36.5
900 (2%)	0	0.94	0.146	0.290	-
	2000	1.73	0.280	0.272	6.1
	4000	2.28	0.454	0.243	16.3
	6000	2.73	0.671	0.220	24.4
1350 (4%)	0	0.92	0.175	0.250	-
	2000	1.48	0.310	0.246	1.4
	4000	1.89	0.452	0.226	9.5
	6000	2.24	0.615	0.208	16.8
1800 (6%)	0	0.89	0.200	0.225	-
	2000	1.32	0.301	0.224	0.6
	4000	1.66	0.455	0.213	5.4
	6000	1.94	0.594	0.198	11.8

5 CONCLUDING REMARKS

A novel passive vibration control configuration, namely the tuned mass-damper-inerter (TMDI), has been proposed to suppress the oscillatory motion of stochastically support excited linear structural (primary) systems combining an inerter device with the classical tuned mass-damper (TMD). The inerter is a two-terminal-flywheel-based mechanical device developing resisting forces proportional to the relative acceleration of its terminals by a constant b (mass units), which can be up to two orders of magnitude higher than the physical mass of the device. The herein proposed TMDI configuration involves taking advantage of the “mass amplification effect” of the inerter by using it as an additional connective element between the TMD oscillating mass and the ground for single-degree-of-freedom (SDOF) primary structures, and between the TMD oscillating mass and the primary structure for chain-like (cascaded) multi-degree-of-freedom (MDOF) primary structures. In fact, it was shown that the TMDI can be viewed as a generalization of the classical TMD for both SDOF and MDOF support excited primary structures. Therefore, all established in the literature procedures for optimum design ("tuning") of the classical TMD are readily applicable to achieve "optimal" performance for the new TMDI configuration.

In this regard, the governing differential equations of motion have been derived in the time and in the frequency domain for TMDI equipped damped linear SDOF primary structures and in the Laplace domain for damped chain-like MDOF primary structures incorporating a TMDI to suppress the fundamental mode of vibration. Optimal TMDI design parameters minimizing the relative displacement variance of undamped SDOF primary structures under white noise support excitation have been derived analytically in closed form as functions of the TMD mass and the inerter constant b. It has been proved that the optimum designed TMDI configuration is more effective than the classical TMD for a fixed value of the TMD mass in suppressing the displacement variance of white noise excited undamped SDOF primary structures. Further, the effectiveness of the TMDI increases for relatively low values of the TMD mass.

Moreover, for damped MDOF primary structures, a standard optimization procedure has been considered to obtain optimum TMDI and classical TMD designs (as a special case of a TMDI with b=0) which minimize the displacement variance of the "lead" mass (most remote mass from the support) of the primary structure. Pertinent numerical data have been reported for the case of a 3-DOF damped primary structure base excited by a stationary colored stochastic process. These data evidence that the incorporation of the inerter in the proposed TMDI configuration can either replace part of the TMD vibrating mass to achieve a significantly lighter passive vibration control solution (TMD mass replacement effect), or improve the TMD performance for a fixed TMD mass (TMD mass amplification effect). The latter effect is more significant for relatively small TMD masses in which case the inclusion of the inerter accommodates viscous dampers with much higher damping coefficients compared to an optimally tuned classical TMD.

Overall, the herein reported analytical and numerical data provide evidence that the proposed TMDI configuration offers a promising solution for passive vibration control of stochastically support-excited systems. This is due to the mass amplification effect stemming from the unique mechanical properties of the inerter device which improves the effectiveness of the classical TMD for vibration suppression in all cases considered. Further on-going research efforts by the authors are directed towards establishing alternative configurations/topologies to combine TMDs with inerter devices to control the dynamic response of various mechanical and civil engineering structures and structural systems for stochastic and deterministic excitations and for various response minimization criteria.

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References

[1] H. Frahm, Device for Damping Vibrations of Bodies, U.S. Patent No. 989958 (1911)

[2] J. Ormondroyd, J.P. Den Hartog, The theory of the dynamic vibration absorber, J. Appl. Mech. 50 (1928) 9–22.

[3] J.P. Den Hartog, Mechanical Vibrations, fourth ed., McGraw-Hill, New York, 1956.

[4] J. E. Brock, A note on the damped vibration absorber, J. Appl. Mech. 13 (1946) A-284.

[5] S. Krenk, Frequency analysis of the tuned mass damper, J. Appl. Mech. ASME (2005) 936–942.

[6] O. Nishihara, T. Asami, Closed-Form Solutions to the Exact Optimizations of Dynamic Vibration Absorbers (Minimizations of the Maximum Amplitude Magnification Factors), J. Vib. Acoust. 124 (2002) 576 – 582.

[7] K.Liu,J.Liu,The damped dynamic vibration absorbers: revisited and new result, J. Sound Vib. 284 (2005) 1181–1189.