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Optimal control of linear systems with large and variable input delays

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ABSTRACT

This paper proposes an optimal control law for linear systems affected by input delays. Specifically we prove that when the delay functions are known it is possible to generate the optimal control for arbitrarily large delay values by using a DDE without distributed terms. The solution can be seen as a chain of predictors whose size depends on the maximum delay.

1. Introduction

The control and state estimation problems in presence of input or measurement delays have received growing attention due to its relevance in many emerging applications such as network control systems where delays must be taken into account in the transmission of input signals [1–5]. In the context of continuous-time systems it is known that the general solution to the control problem can be provided by means of operators on infinite dimensional spaces [6,7]. The optimal control problem has been studied and solved in this context [8–11]. In [12] it is shown that a suitable state feedback control which involves the integral of the past control law solves the infinite horizon optimal control problem for linear time-invariant systems with single input time-delay. In [13] the finite horizon optimal control problem of time-varying linear systems with multiple constant input delays has been solved.

However infinite dimensional approaches are difficult to implement, as they require to compute an integral term on-line. As explained in [7], obtaining this term as the solution to a differential equation must be discarded because it involves unstable pole-zero cancellation when the original system in unstable. The numerical quadrature rules to evaluate the integral term require special attention in the implementation [14] or approximation methods that yields suboptimal solutions [15,16].

Recently, finite dimensional or memoryless methods, meaning that the input is generated by an instantaneous state feedback as in the delay-free case, have been proposed for linear systems [17–23]. Some of these methods consider also time-varying delays. In [23] the LQ problem is solved with a memoryless feedback for known delay functions satisfying a delay bound. In this paper we extend the approach of [23] in two directions. The first extension is to overcome the problem of the delay bound by introducing a chain of predictors. In this way it is possible to generate a finite-dimensional stabilizing input for arbitrarily large delays, a result previously available only for systems not exponentially unstable [18]. The second extension is to extend the approach to the case of multiple delay functions, each acting on a specific input.

A basic assumption of our work is that the delay functions are known. This may be considered as a strong assumption in many practical situations, but we show that it is the price to pay for having the same performance as in the optimal delay free case. On the other hand, this assumption is not specific to our work but to any exact prediction/control approach in presence of delay. Consequently a contribution of this paper might also be considered to be the study of the conditions on the input delay under which the system can be optimally controlled as if the delay was not present. In this sense, we show that the *size* of the delay is not relevant as long as the delay is known and well behaved in a precise sense.

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We introduce the problem and the delay assumption in Section 2. The approach is illustrated in 3. With the aim of making easier to read the paper we first introduce the case of large delays with a single input in Section 3.1 before giving the solution for the more general case of multiple delayed inputs in Section 3.2. Section 4 considers output feedback control and Section 5 validates the method.

Notation. $\mathbb{R}_+$ is set of non-negative reals. $\sigma(A)$ denotes the set of eigenvalues of the square matrix A, and $\mu(A)$ the largest real part of its eigenvalues. $\Re(z)$ is the real part of $z \in \mathbb{C}$. $\mathcal{C}^n_\delta$ denotes the space of continuous functions that map $[-\delta, 0]$ in $\mathbb{R}^n$, with the uniform convergence norm, denoted $\|\cdot\|_\infty$.

2. Problem statement

In this paper we consider the following linear system with multiple input delays

$$\begin{aligned} &\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} B_i u_i(t - \delta_i(t)), \quad t > 0 \\ &x(0) = x_0 \\ &u_i(t - \delta_i(t)) = 0, \quad t < 0, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $i = 1, \ldots, p$. Notice that system (1) has multiple inputs and each input has only one delay, differently from the case of single input with multiple delays considered, among others, by [7,24].

The delay functions $\delta_i : \mathbb{R}_+ \to [0, \bar{\delta}]$ are uniformly bounded by the known constant $\bar{\delta}$. We denote $\psi_i(t)_i = t - \delta_i(t)$ the time point at which the control signal applied at time t has been generated, that is, $u(t - \delta_i(t)) = u(\psi_i(t))$. Obviously, $\psi_i(t) \le t$. We require that the following two assumptions hold.

Assumption 1. Let $B = [B_1, \ldots, B_p]$. Then the pair (A, B) is controllable.

Assumption 2. The functions $\psi_i(t)$ are bijective, *i.e.* for $\forall t^* \ge \psi_i(0)\ \exists! t_i : t^* = \psi_i(t_i)$, $i = 1, \ldots, p$. Moreover, the inverse functions $t_i = \psi_i^{-1}(t^*)$ are known at time t^*.

Assumption 2 is necessary to ensure that when generating the input $u_i(\psi_i(t))$ at time $\psi_i(t)$ there is a known and unique time t_i at which the input will be received. Practical situation in which Assumption 2 holds are constant or continuous, slowly delays that satisfy $|\dot{\delta}(t)| < 1$. However, continuity or differentiability of $\delta_i(t)$ are not implied by Assumption 2, thus $\delta_i(t)$ could be fast-varying or even not continuous, as long as $\psi_i(t)$ are invertible and known functions (see for example $\delta(t)$ in Fig. 2). Assumption 2 is quite standard in this setting [20]. The only alternative to it is to use robust control with unknown input delay, but in this case the control is no longer optimal [1]. We look for the optimal controls $u_i(t)$ with respect to a quadratic functional in the infinite-horizon case, that can be written as

$$J = \int_0^\infty x^T(t)Qx(t) + \sum_{i=1}^{p} R_i u_i^2(\psi_i(t))dt, \tag{2}$$

where Q is an appropriate positive-definite symmetric matrix and R_i are positive scalars.

It is well known that, at least for constant delays $\delta_i(t) = \delta$, the optimal control of (1) can be achieved through the computation of distributed terms ([1], p. 202). Instead, we explore solutions based on optimal instantaneous state feedback of the kind

$$u_i(\psi_i(t)) = -K(\psi_i(t))x(\psi_i(t)), \tag{3}$$

and we show that the optimal control can be generated with such finite-dimensional feedback, even in presence of variable delays.

Remark 1. A different but related problem is when the delay affects the state measurement, but not the input, that is, at time t the input $u_i(t)$ can be immediately applied but must be generated using delayed information about the state, $\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} B_i u_i x(t-\delta_i(t))$. The instantaneous state feedback (3) becomes $u_i(t) = -K(\psi_i(t))x(\psi_i(t))$. Thus, the method described in this paper can be applied also in this case and Assumption 2 can be relaxed to the knowledge of $\delta(t)$ at t.

3. Predictors for input delays

3.1. Systems with single input delay

In order to make the presentation easier we consider in the first place the case of a single delay and scalar input,

$$\dot{x}(t) = Ax(t) + Bu(\psi(t)) \tag{4}$$

with $u(t)$ scalar, $u(\psi(t)) = 0$ for $t < 0$, and $x(0) = x_0$.

Given a square matrix $A \in \mathbb{R}^{n\times n}$ we introduce the following scalar function of vector $K \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}_+$

$$\omega_A(\alpha, K) := \max\left\{\delta \in \mathbb{R}_+ : \int_0^\delta \left|Ke^{(A-BK)s}B\right| e^{\alpha s} ds \le 1\right\}, \tag{5}$$

with the convention that $\omega_A(\alpha, K) = \infty$ if the inequality is always satisfied. If $\omega_A(\alpha, K) < \infty$, due to the structure of the integrand in (5), larger values of α correspond to smaller values of $\omega_A(\alpha, K)$ and vice-versa. It is possible to show [21] that $\omega_A(\alpha, K)$ does not depend on B. but only on α, $\sigma(A)$ and $\sigma(A - BK)$, and it is therefore invariant to a change of coordinates.

The optimal control problem for system (4) was solved in [23] for delay functions uniformly bounded. We report the main result.

Theorem 1 ([23]). *Consider system (4) with the pair (A, B) controllable, $\delta(t) \le \bar{\delta}$ that satisfies Assumption 2 and the cost functional (2). Let $\bar{K}^o = R^{-1}B^T P$ be the optimal gain with no input delay, P steady-state solution of the Riccati equation*

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{6}$$

and $\bar{A} = A - B\bar{K}^o$. If the delay bound satisfies $\bar{\delta} < \omega_A(-\mu(\bar{A}), \bar{K}^o)$, then the optimal control law is

$$u(\psi(t)) = \begin{cases} -\bar{K}^o e^{\bar{A}t} x_0, & t < \bar{\delta}, \\ -\bar{K}^o e^{\bar{A}(t-\psi(t))} x(\psi(t)), & t \ge \bar{\delta}. \end{cases} \tag{7}$$

Moreover the value of J for (4) with (7) is $x_0^T P x_0$.

In (7), by definition, $t - \psi(t) = \delta(t)$. In the time coordinate of the controller, control law (7) can be written, for $t \ge \bar{\delta}$

$$u(t) = -\bar{K}^o e^{\bar{A}(\psi^{-1}(t)-t)} x(t), \tag{8}$$

where $\psi^{-1}(t) - t = \delta(\psi^{-1}(t))$ is the delay with which the plant will receive the input, and $\psi^{-1}(t)$ is known in virtue of Assumption 2. From now on we use the time coordinate of the plant. It may be noticed that the idea behind (7) is to use $e^{\bar{A}\delta(t)}x(t - \delta(t))$ as a predictor of $x(t)$ This would yield, for $t \ge \bar{\delta}$, $u(\psi(t)) = -\bar{K}^o x(t) = u^o(t)$, where $u^o(t)$ is the optimal input for the delay-free case. If this finite-dimensional predictor works well, the optimal evolution is therefore the same as in the delay-free case. Theorem 1 provides a sufficient (sometimes necessary, see [23]) delay bound for the predictor. Our aim is to extend this solution to delays that are larger than $\omega_A(\bar{K}^o)$.

We resort to a *chain* of predictors, each in charge of extending the prediction provided by the exponential of $\bar{A}$ to a fraction

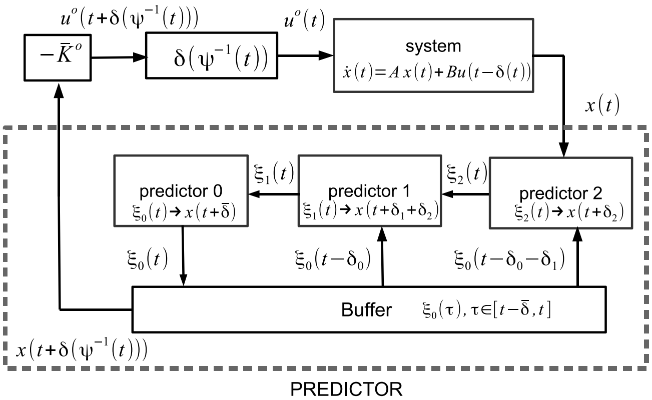

Fig. 1. Scheme of a chain of three predictors.

of the total delay. This technique has already been successfully used to overcome delay bounds for observers with delayed measurements [25–28]. The chain structure suited to our purposes is illustrated in Fig. 1. At time t the present value of $x(t)$ is used by the predictor to generate $\xi_0(t)$, which is an estimate of $x(t+\bar{\delta})$. Each elementary predictor in the chain extends the prediction of a fraction of $\bar{\delta}$. The value $\xi_0(t-\bar{\delta}+\delta(\psi^{-1}(t)))$, that approximates $x(t+\delta(\psi^{-1}(t)))$, is extracted from the buffer and used to generate the input signal. Due to the delay, this input will be applied at time $t+\delta(\psi^{-1}(t))$ and it will correspond to the optimal input at that time.

Before defining the chain of predictors for the variable delay case it is useful to report a result that can be found in [21], Theorem 3.

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $\bar{K}$ such that $\bar{A} = A - B\bar{K}$ is Hurwitz. Consider the delay system*

$$\dot{\xi}(t) = A\xi(t) - B\bar{K}e^{\bar{A}\delta}\xi(t-\delta), \quad t \geq 0, \tag{9}$$

with pre-shape function $\xi(\tau) = \phi_\tau \in \mathcal{C}^n_\delta$. For any $\alpha > 0$ such that $-\alpha > \mu(\bar{A})$, if $\omega_A(\alpha, \bar{K}) > \delta$ then $\|\xi(t)\| \leq e^{-\alpha t}\gamma\|\phi\|_\infty$, $\forall t \geq 0$, $\forall \phi \in \mathcal{C}^n_\delta$.

Definition 1. Given a maximum delay $\bar{\delta}$ and a delay bound δ^*, a *delay partition* $P_{\bar{\delta},\delta^*}$ is a set $\{\delta_j\}, j = 0, \ldots, m$, such that $\delta^* > \delta_0 > \delta_1 > \cdots > \delta_m > 0$ and $\sum_{j=0}^m \delta_j = \bar{\delta}$.

Definition 2. Given system (4) with the cost functional (2), $\bar{K}^o$ optimal gain without input delay, $\delta(t) \leq \bar{\delta}$, $\alpha = -\mu(A - B\bar{K}^o)$, a delay partition $P_{\bar{\delta},\delta^*}$ is *valid* if $\delta^* = \omega_A(\alpha, \bar{K}^o)$.

The idea behind the previous definition is to have a sequence of decreasing delays, each one satisfying the delay bound $\delta_j < \omega_A(\alpha, \bar{K}^o)$ that sum up to the total delay $\bar{\delta}$. Note that the definition of a valid delay partition implies that there exists a sequence $\alpha < \alpha_0 < \alpha_1 < \cdots < \alpha_m$ such that $\delta_j < \omega_A(\alpha_j, \bar{K}^o)$. In practice the choice of a valid delay partition is easily accomplished by choosing $\alpha_0 > \alpha$ and a slowly increasing sequence α_j. The sequence terminates when the sum of the corresponding $\omega_A(\alpha_j, \bar{K}^o)$ exceeds $\bar{\delta}$. It is easy to see that extending the delay partition is in principle possible to compensate any delay $\bar{\delta}$.

Let, as before, $\bar{K}^o = R^{-1}B^TP$ with P solution of (6), $\bar{A} = A - B\bar{K}^o$ and $K(\delta) = \bar{K}^o e^{\bar{A}\delta}$. Given a valid delay partition $P_{\bar{\delta},\omega_A(\alpha,\bar{K}^o)} = \{\delta_j\}$, we denote $d_j = \sum_{k=0}^{j-1}\delta_k$ for $j = 1, \ldots, m$, with $d_0 = 0$. Notice that $\bar{\delta} > d_j \geq 0$.

The chain is made up by the $m+1$ systems $\xi_j(t), j = 0, \ldots, m$, where $\xi_j(t)$ aims at predicting $x(t+\bar{\delta}-d_j)$, thus $\xi_0(t)$ predicts $x(t+\bar{\delta})$.

Initialization:

$$\xi_j\left(-\bar{\delta}+d_j\right) = x_0, \quad j = 0, \ldots, m. \tag{10}$$

Pre-shape:

$$\dot{\xi}_j(t) = \bar{A}\xi_j(t), \quad -\bar{\delta}+d_j \leq t < d_j. \tag{11}$$

$t > d_j$:

$$\begin{aligned} \dot{\xi}_j(t) &= A\xi_j(t) - B\bar{K}^o\xi_0\left(t-d_j\right) \\ &\quad + BK(\delta_j)\left(\xi_{j+1}(t) - \xi_j(t-\delta_j)\right), \quad j < m \\ \dot{\xi}_m(t) &= A\xi_m(t) - B\bar{K}^o\xi_0\left(t-d_m\right) \\ &\quad + BK(\delta_m)\left(x(t) - \xi_m(t-\delta_m)\right). \end{aligned} \tag{12}$$

For illustrative purposes, let us detail the case of two predictors, $m = 1$, with a delay partition $\delta_0 + \delta_1 = \bar{\delta}$, with $\delta_0 > \delta_1$. Since both delays must be less than the delay bound δ^* for a single predictor, but they can be arbitrarily close to it, we have that the overall delay $\bar{\delta}$ can be arbitrarily close to $2\delta^*$, thus actually doubling the delay bound. $\xi_1(t)$ predicts $x(t+\delta_1)$, $\xi_0(t)$ predicts $x(t+\delta_0+\delta_1) = x(t+\bar{\delta})$. The initialization is $\xi_0(-\bar{\delta}) = x_0$, $\xi_1(-\delta_1) = x_0$. Notice that, to perform the initialization, the controller needs a *future* state, *i.e.* $x(0)$ at time $-\bar{\delta}$. Depending on the situation, this can be accomplished by using a prediction based on the open-loop dynamics of the plant or an approximate initialization, as discussed in the sequel. The pre-shape evolution of the predictors is, for $t \in (-\bar{\delta}, 0)$ and $t \in (-\delta_1, \delta_0)$ respectively,

$$\dot{\xi}_0(t) = \bar{A}\xi_0(t), \qquad \dot{\xi}_1(t) = \bar{A}\xi_1(t). \tag{13}$$

Finally,

$$\begin{aligned} \dot{\xi}_0(t) &= \bar{A}\xi_0(t) + BK(\delta_0)\left(\xi_1(t) - \xi_0(t-\delta_0)\right), \quad t \geq 0 \\ \dot{\xi}_1(t) &= A\xi_1(t) - B\bar{K}^o\xi_0\left(t-\delta_0\right) \\ &\quad + BK(\delta_1)\left(x(t) - \xi_1(t-\delta_1)\right), \quad t \geq \delta_0. \end{aligned} \tag{14}$$

If we suppose, for simplicity a constant delay ($\delta(t) = \bar{\delta}$), the control law at the plant side, $t \geq 0$, is $u(t-\bar{\delta}) = -\bar{K}^o\xi_0(t-\bar{\delta}) = -\bar{K}^o x(t)$ where of course the last equality holds if the prediction is correct. The conditions under which this is true are provided by the following theorem.

Theorem 2. *Consider system (4) with the pair (A, B) controllable, $\delta(t) \leq \bar{\delta}$ that satisfies Assumption 2 and the cost functional (2). Let $\bar{K}^o = R^{-1}B^TP$ be the optimal gain with no input delay, P solution of the Riccati equation (6), $\bar{A} = A - B\bar{K}^o$ and $\alpha = -\mu(\bar{A})$. Given a valid delay partition $P_{\bar{\delta},\omega_A(\alpha,\bar{K}^o)} = \{\delta_j\}$ the optimal control law is*

$$u(\psi(t)) = -\bar{K}^o\xi_0(t-\bar{\delta}), \quad t \geq 0, \tag{15}$$

where, by definition, $\xi_0(t-\bar{\delta}) = \xi_0(\psi(t)-\bar{\delta}+\delta(t))$, $\xi_0(t)$ defined by (10)–(12). The value of J for (4) with (15) is $x_0^TPx_0$.

Proof. The closed-loop dynamics of (4) is, for $t \geq 0$,

$$\dot{x}(t) = Ax(t) - B\bar{K}^o\xi_0(t-\bar{\delta}). \tag{16}$$

The optimal control, without delay, is $u^o(t) = -\bar{K}^o x^o(t)$, and the optimal trajectory is $\dot{x}^o(t) = \bar{A}x^o(t)$, that is, $x^o(t) = e^{\bar{A}t}x_0$. Consequently the proof is complete if we show that $x(t) = e^{\bar{A}t}x_0$. The crucial step is to prove $\xi_0(t-\bar{\delta}) = x(t) = e^{\bar{A}t}x_0$, because the thesis immediately follows from (16). From (10) to (11) it follows that $\xi_0(t-\bar{\delta}) = x(t) = e^{\bar{A}t}x_0$ holds for $t \in [0, \bar{\delta}]$. Similarly, it is easy to see that for $j = 1, \ldots, m$,

$$\xi_j\left(t-\bar{\delta}+d_j\right) = x^o(t) = e^{\bar{A}t}x_0, \quad t \in [0, \bar{\delta}]. \tag{17}$$

Introducing the prediction errors $\epsilon_j(t) = x(t) - \xi_j(t - \bar{\delta} + d_j)$, it follows that $\epsilon_j(t) = 0$ for $t \in [0, \bar{\delta}]$. Consider now $t \geq \bar{\delta}$. For $j = m$ the prediction error evolution is

$$\begin{aligned}\dot{\epsilon}_m(t) &= Ax(t) - B\bar{K}^o\xi_0(t-\bar{\delta}) - A\xi_m(t-\bar{\delta}+d_m) + B\bar{K}^o\xi_0(t-\bar{\delta}) \\ &\quad - BK(\delta_m)(x(t-\bar{\delta}+d_m) - \xi_m(t-\bar{\delta}+d_m-\delta_m)) \\ &= A\epsilon_m(t) - BK(\delta_m)(x(t-\delta_m) - \xi_m(t-2\delta_m)) \\ &= A\epsilon_m(t) - BK(\delta_m)\epsilon_m(t-\delta_m), \end{aligned} \tag{18}$$

because $t - \bar{\delta} + d_m = t - \delta_m$. Since $\delta_m < \omega_A(\alpha_m, \bar{K}^o)$, the delay equation (18) is exponentially stable in virtue of Lemma 1, with exponential decay rate α_m. Moreover, the initial state of (18) is the 0 function. We conclude that $\epsilon_m(t) = 0$ for $t \geq 0$. For $j < m$ and $t \geq \bar{\delta}$ the remaining prediction errors are

$$\begin{aligned}\dot{\epsilon}_j(t) &= Ax(t) - B\bar{K}^o\xi_0(t-\bar{\delta}) - A\xi_j(t-\bar{\delta}+d_j) + B\bar{K}^o\xi_0(t-\bar{\delta}) \\ &\quad - BK(\delta_j)(\xi_{j+1}(t-\bar{\delta}+d_j) - \xi_j(t-\bar{\delta}+d_j-\delta_j)) \\ &= A\epsilon_j(t) - BK(\delta_j)(\xi_{j+1}(t-\bar{\delta}+d_{j+1}-\delta_j) - x(t-\delta_j) \\ &\quad + x(t-\delta_j) - \xi_j(t-\bar{\delta}+d_j-\delta_j)) \\ &= A\epsilon_j(t) - BK(\delta_j)\epsilon_j(t-\delta_j) + BK(\delta_j)\epsilon_{j+1}(t-\delta_j), \end{aligned} \tag{19}$$

where we have used the property $d_{j+1} = d_j + \delta_j$. A direct check confirms that the delayed terms in the last expression of (19) are well posed. System (19) is equivalent to system (9) with an external disturbance $BK(\delta_j)\epsilon_{j+1}(t-\delta_j)$. This disturbance is null for $j+1 = m$, and it is consequently null for $j + 1 = 1, \dots, m$ since the delay equation (19) is exponentially stable in virtue of Lemma 1, with exponential decay rate $\alpha_j > \alpha$. Since this holds also for $j = 0$, we have $\epsilon_0(t) \equiv 0$, that is, $x(t) = \xi_0(t - \bar{\delta})$. ■

Remark 2. At first sight it may seem strange that the feedback law (15) does not contain any time-varying delay and that the resulting system-predictor has equations with constant delay, a fact that considerably simplifies their analysis. However this paradox is apparent, because the time-varying delay is hidden in the difference between t and $\psi(t)$. To see this write (15) as $u(\psi(t)) = -\bar{K}^o\xi_0(\psi(t) + \delta(t) - \bar{\delta})$. Recalling that $\xi_0(\psi(t) - \bar{\delta})$ is the prediction of $x(\psi(t))$ we have, when the prediction is exact, $u(\psi(t)) = -\bar{K}^o x(\psi(t) + \delta(t)) = -\bar{K}^o x(t)$, where the presence of the variable delay is made explicit in the first equality. Notice that the input is computed at $\psi(t)$ by using ξ_0 at time $\psi(t) + \delta(t) - \bar{\delta} < \psi(t)$. Then, in the case of variable delay, past values of $\xi_0(t)$ extracted by the buffer are used. When $\delta(t) = \bar{\delta}$ only the most recent value of ξ_0 is used, but a buffer for ξ_0 is needed by the remaining predictors.

Remark 3. The predictor chain (10)–(12) yields the optimal trajectory $x^o(t)$ for *any* delay function that satisfies Assumption 2. This implies that the system receives the same input independently of the delay. In other words, the input generated at the predictor is delay dependent, but the delayed input received at the system is not (see Fig. 3).

Remark 4. We have proved that the prediction errors are initially null and, for $t \geq \bar{\delta}$, their dynamics is exponentially stable with decay rate faster than the controlled system. If the second point were not guaranteed the systems dynamics would eventually diverge from $x^o(t)$.

The initialization (10) deserves some caution, since here we require to know $x(0)$ at $t = -\bar{\delta}$. This is not surprising, because we want to generate the optimal trajectory through a delay system, then we need a suitable infinite-dimensional initial condition containing x_0. In practical cases it could be impossible to estimate in advance $x(0)$ from the open-loop dynamics. To overcome this difficulty, we consider the generic initialization

$$\xi_j\left(-\bar{\delta} + d_j\right) = \hat{x}_0, \quad j = 0, \dots, m, \tag{20}$$

where $\hat{x}_0$ is not necessarily equal to x_0.

Corollary 2. *In the hypotheses of Theorem 2, the control law (15) with $\xi_0(t)$ generated by (11), (12) with initialization (20) is such that $\|x^o(t) - x(t)\| \leq e^{-\alpha t}\gamma^o\|x_0 - \hat{x}_0\|$.*

Proof. From (19) it follows that the rate of exponential convergence to 0 of $\epsilon_j(t)$ is $\min(\alpha_j, \alpha_j + 1) = \alpha_j$. Therefore the rate of convergence to zero of $\epsilon_0(t)$ is $\alpha_0 > \alpha$. Let $z(t) = x^o(t) - x(t)$. Since $\dot{x}^o(t) = Ax^o(t) - B\bar{K}^o x^o(t)$, it follows that

$$\dot{z}(t) = Az(t) - B\bar{K}^o(x^o(t) - \xi_0(t-\delta)) = \bar{A}z(t) - B\bar{K}^o\epsilon_0(t), \tag{21}$$

from which it follows that the exponential rate of convergence to zero of $z(t)$ is α. ■

The above result can of course be used to solve the more general (i.e. non optimal) control problem of linear systems with arbitrary but known input delay by means of finite-dimensional feedback.

Corollary 3. *Consider system (4) with the pair (A, B) controllable, $\delta(t) \leq \bar{\delta}$ that satisfies Assumption 2. Let $\bar{K}$ be such that $\bar{A} = A - B\bar{K}$ is Hurwitz and $\alpha = -\mu(\bar{A})$. Given a valid delay partition $P_{\bar{\delta},\omega_A(\alpha,\bar{K})} = \{\delta_j\}$ the control law*

$$u(\psi(t)) = -\bar{K}\xi_0(t - \bar{\delta}), \quad \psi(t) \geq 0, \tag{22}$$

where $t = \psi^{-1}(\psi(t)) = \psi(t)+\delta(t)$, $\xi_0(t)$ defined by (11), (12) with initialization (20) is such that $\|x(t)\| \leq e^{-\alpha t}\gamma\|x_0 - \hat{x}_0\|$.

It must be finally mentioned that all the derivations of this section considered the case of scalar $u(t)$. This limitation is only due to the application to multiple input delays in next section. All the results presented here hold for vector $u(t)$ by changing the definition of $\omega_A(\alpha, K)$ to

$$\omega_A(\alpha, K) := \max\left\{\delta \in \mathbb{R}_+ : \int_0^\delta \|Ke^{(A-BK)s}B\|e^{\alpha s}ds \leq 1\right\}. \tag{23}$$

3.2. Systems with multiple input delays

We now go back to consider system (1). It turns out that the solution described in the previous section extends nicely to the case of the multiple delays in the input. The following result is immediate to prove.

Corollary 4. *Consider system (1) satisfying Assumption 1, $\delta_i(t) \leq \bar{\delta}$ that satisfy Assumption 2 and the cost functional (2). Let $\bar{K}^o = R^{-1}B^TP$ be the optimal gain with no input delay, P solution of the Riccati equation (6), $\bar{A} = A - B\bar{K}^o$ and $\alpha = -\mu(\bar{A})$. Given a valid delay partition $P_{\bar{\delta},\omega_A(\alpha,\bar{K}^o)} = \{\delta_j\}$ the optimal control law is*

$$u_i(\psi_i(t)) = -\bar{K}_i^o\xi_0(t - \bar{\delta}), \quad \psi_i(t) \geq 0, \tag{24}$$

where $\bar{K}_i$, $i = 1, \dots, p$ is the ith row of $\bar{K}^o$, by definition $t = \psi_i(t) + \delta_i(t)$, and $\xi_0(t)$ is defined by (10)–(12). Moreover, the value of J for (4) with (24) is $x_0^TPx_0$.

Proof. From (24) it follows that the dynamics of $x(t)$ is given by (16). The proof is therefore the same as in Theorem 2. ■

As in the case of single input, if the initialization (20) is used the resulting control is asymptotically optimal. If a generic gain $\bar{K}$ that makes $\bar{A} = A - B\bar{K}$ Hurwitz is used instead of the optimal gain $\bar{K}^o$, the resulting non optimal trajectory is still exponentially stable with decay rate $\alpha = -\mu(\bar{A})$.

4. Optimal control from output

In this section we briefly consider the case of state not fully accessible, and with a possible delay,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{p} B_i u_i(t - \delta_i(t)), \quad t > 0 \\ y(t) &= Cx(t - \Delta), \quad t > \Delta, \\ x(0) &= x_0. \end{aligned} \tag{25}$$

Case $\Delta = 0$. In this case the chain of predictors is extended with a Luenberger observer in charge of providing an estimate $\hat{x}(t)$ of $x(t)$. The asymptotic optimality is preserved if the estimation error decays at a rate larger than the optimal solution. The last stage of the predictor chain (12) is modified as follows for $t > 0$,

$$\begin{aligned} \dot{\xi}_m(t) &= A\xi_m(t) - B\bar{K}^o \xi_0 (t - d_m) \\ &\quad + BK(\delta_m) \left(\hat{x}(t) - \xi_m(t - \delta_m)\right), \\ \dot{\hat{x}}(t) &= A\hat{x}(t) - B\bar{K}^o \xi_0 \left(t - \bar{\delta}\right) \\ &\quad + K_L \left(y(t) - C\hat{x}(t)\right), \quad \hat{x}(0) = \hat{x}_0. \end{aligned} \tag{26}$$

Corollary 5. *In the assumptions of Corollary 4, if the pair (A, C) is observable, given a delay partition $P_{\bar{\delta},\delta^*} = \{\delta_j\}$ let α_m be such that $\omega_A(\alpha_m, \bar{K}^o) = \delta_m$. If $\mu(A - K_L C) < -\alpha_m$, then the trajectory generated by the control law (24), where the predictor chain (20), (11), (12) is modified according to (26), is asymptotically optimal.*

Case $\Delta > 0$. The solution for the case of delayed measurements is to use a Luenberger observer as before, with the task of estimating $x(t - \Delta)$. The chain of predictors is designed to span the *extended* delay $\bar{\delta} + \Delta$ (see also [24]).

Corollary 6. *In the assumptions of Corollary 4, if the pair (A, C) is observable, given a delay partition $P_{\bar{\delta}+\Delta,\delta^*} = \{\delta_j\}$ let α_m be such that $\omega_A(\alpha_m, \bar{K}^o) = \delta_m$. If $\mu(A - K_L C) < -\alpha_m$, then the trajectory generated by the control law (24), where in the predictor chain (20), (11) $\bar{\delta}$ must be replaced by $\bar{\delta} + \Delta$, and the last element of the chain is*

$$\begin{aligned} \dot{\xi}_m(t) &= A\xi_m(t) - B\bar{K}^o \xi_0 (t - d_m) \\ &\quad + BK(\delta_m) \left(\hat{x}(t) - \xi_m(t - \delta_m)\right), \\ \dot{\hat{x}}(t) &= A\hat{x}(t) - B\bar{K}^o \xi_0 \left(t - \bar{\delta} - \Delta\right) \\ &\quad + K_L \left(y(t) - C\hat{x}(t)\right), \quad \hat{x}(0) = \hat{x}_0, \end{aligned} \tag{27}$$

is asymptotically optimal.

Notice that this approach requires the measurement delay to be constant. For the case of time-varying output delay it is necessary to use observers with delay, see for example [25].

5. Examples

5.1. Single delay

For comparison purposes we consider first the delayed double oscillator example presented also in [20,21,23]. The state space matrices are

$$A = \begin{bmatrix} p & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & -\omega & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{28}$$

with $p = 0.1$ and $\omega = 1$. Note that $\sigma(A) = \{0.1, \pm j\}$, hence the open loop system is exponentially unstable. We consider the cost functional (2) with $Q = I_5$, $R = 1$ and P solution of the algebraic Riccati equation. The corresponding optimal gain without delay is $\bar{K}^o = [1.4478, -0.1391, 3.1395, 3.4685, 2.8173]$. The spectrum of the closed loop system without delay is $\sigma(A - B\bar{K}^o) = \{-0.4921, -0.6882 \pm 0.8460i, -0.4244 \pm 1.2466i\}$, hence $\alpha = -\mu(\bar{A}) = 0.4244$. The delay bound $\omega_A(-\mu(\bar{A}), \bar{K}^o)$ (5) is $\delta^* = 0.4950$. Since the input is scalar and $\bar{K}^o e^{\bar{A}t} B$ is positive for $t \in [0, \delta^*]$ this bound is strict, thus the control law (7) is not optimal for $\delta(t) > \delta^*$. We use the delay function shown in Fig. 2, defined as

$$\delta(t) = \begin{cases} 2/3t, & t < \bar{\delta} \\ 3(t - k\bar{\delta}), & t \in [k\bar{\delta}, (k + 1/3)\bar{\delta}) \\ \dfrac{3}{2}\left(t - \left(k + \dfrac{1}{3}\right)\bar{\delta}\right), & t \in [(k + 1/3)\bar{\delta}, (k + 1)\bar{\delta}] \end{cases} \tag{29}$$

with $k = 1, 2, \ldots$ and $\bar{\delta} = 1$. $\delta(t)$ is not continuous, and for $t \geq \bar{\delta}$ the delay is fast-varying, because $\dot{\delta}(t) > 1$ where it exists. However, it is easy to see that $t - \delta(t)$ is invertible and that $d(t) < \bar{\delta}$. Since $\bar{\delta} > 2\delta^*$, we can resort to a chain of 3 predictors. In the simulations we chose the delay partition $\delta_0 = 0.45$, $\delta_1 = 0.33$, $\delta_2 = 0.22$. With the initial value $x_0 = [-1, 2, 2, 1, 2]^T$ the cost for the optimal control without delay is $J^o = x_0^T P x_0 = 162.7004$. A delay-free simulation for $t \leq 50$ and integration step $dt = 10^{-3}$ yields $J = 163.0908$, the difference being due to the numerical approximation. The chain of predictors and the control law are, omitting initializations,

$$\begin{aligned} u(\psi(t)) &= -\bar{K}^o \xi_0(\psi(t) + \delta(t) - \bar{\delta}), \quad t \geq 0 \\ \dot{\xi}_0(t) &= \bar{A}\xi_0(t) + BK(\delta_0)\left(\xi_1(t) - \xi_0(t - \delta_0)\right), \quad t \geq 0 \\ \dot{\xi}_1(t) &= A\xi_1(t) - B\bar{K}^o \xi_0 (t - \delta_0) \\ &\quad + BK(\delta_1)\left(\xi_2(t) - \xi_1(t - \delta_1)\right), \quad t \geq \delta_0 \\ \dot{\xi}_2(t) &= A\xi_2(t) - B\bar{K}^o \xi_0 (t - (\delta_0 + \delta_1)) \\ &\quad + BK(\delta_2)\left(x(t) - \xi_2(t - \delta_2)\right), \quad t \geq \delta_0 + \delta_1. \end{aligned} \tag{30}$$

Fig. 3 compares the input $u(\psi(t))$ generated by the predictor-based control law (left) with the input signal received at the system (right), that coincides with the optimal input. The value of the cost functional along the trajectory generated by the predictor is $J = 163.0937$. The choice of a more precise numerical integration would yield $J = J^o$, but we deliberately chose a coarser integration scheme to verify that the control scheme is robust with respect to numerical approximations. To further illustrate this point we compare in Fig. 2 (right) the norm of the optimal trajectory $x^o(t)$ without delay with the displacement norm $x^o(t) - x(t)$ for two cases, namely precise initialization (10) and approximate initialization (20). In the latter case we chose $\hat{x}_0 = 0.9x_0$ and we obtained a value of the cost functional $J = 170.3092$. The important point is that in both cases the norms converge exponentially to 0 with the same rate α as $\|x^o(t)\|$, in accordance with Corollary 2.

5.2. Multiple delays

Consider a system with the same matrix A as in the previous section and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t - \delta_1(t)) \\ u_2(t - \delta_2(t)) \end{bmatrix} \tag{31}$$

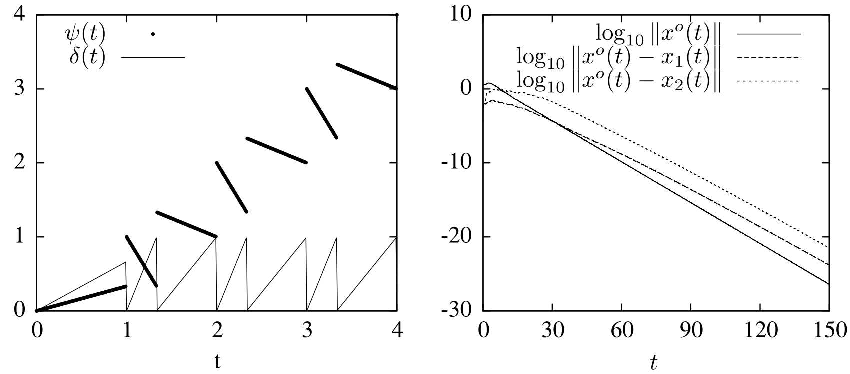

Fig. 2. Delay function $\delta(t)$ and the corresponding $\psi(t) = t - \delta(t)$ (left). Norm of the optimal state trajectory $x^o(t)$ without delay, and norm of the difference $x^o(t) - x_i(t)$, where $x_1(t), x_2(t)$ are the trajectories generated by the predictor with precise and approximate initial conditions (right).

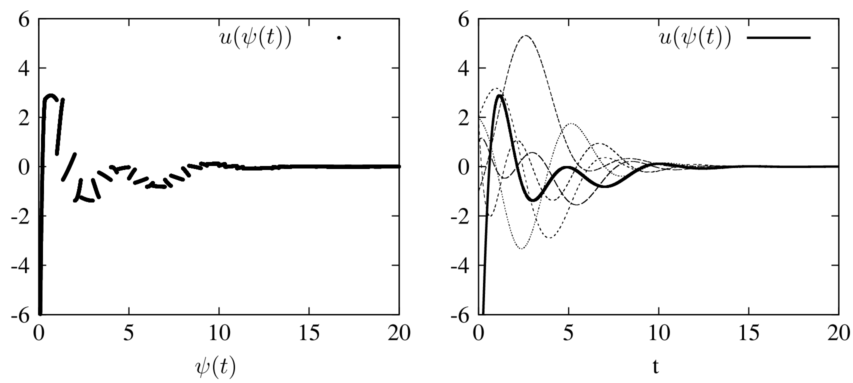

Fig. 3. Input generated by the chain of 3 predictors (left) and input signal received at the system after $\delta(t)$ (right).

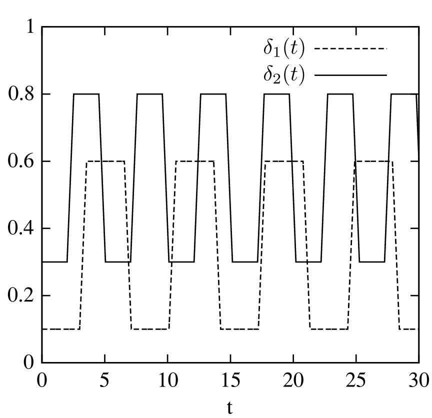

Fig. 4. Delay functions for a system with two inputs.

where the periodic delay functions $0.1 \leq \delta_1(t) \leq 0.6$, $0.3 \leq \delta_2(t) \leq 0.8$ are shown in Fig. 4. Both functions are continuous with derivatives < 1 (where they exist), so that $t - \delta_i(t)$, $i = 2$ are bijective and invertible. With $x_0 = [1,\ 1,\ 1,\ 1,\ 1]^T$, $Q = I_5$, $R = I_2$ the optimal cost has cost $J^o = x_0^T P x_o = 28.788$. A simulation with $dt = 5 \cdot 10^{-4}$ yields the approximate value $J = 28.805$.

6. Conclusions

In this paper we have shown that is possible to build a finite dimensional state predictor for arbitrarily large input delays, under suitable conditions on the delay functions. The idea of a chain of predictors, initially introduced for the state observation problem [26], has been extended to the control problem. The optimality of the resulting control law confirms that this kind of finite-dimensional predictor is in some sense the "right-one" in the deterministic case. Future work will consider the stochastic version of the problem, for which [23] provides only initial results.

References

[1] E. Fridman, Introduction to Time-Delay Systems, Birkhäuser, 2014.
[2] J.P. Hespanha, P. Naghshtabrizi, Y. Xu, A survey of recent results in networked control systems, IEEE Proc. 95 (1) (2007) 138.
[3] Y.-P. Chang, J.S.-H. Tsai, et al., Optimal digital redesign of hybrid cascaded input-delay systems under state and control constraints, IEEE Trans. Circuits Syst. I 49 (9) (2002) 1382–1392.
[4] N. Chopra, P. Berestesky, M.W. Spong, Bilateral teleoperation over unreliable communication networks, IEEE Trans. Control Syst. Technol. 16 (2) (2008) 304–313.
[5] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems, Springer, 2010.
[6] A.W. Olbrot, Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays, IEEE Trans. Automat. Control 23 (5) (1978) 887–890.
[7] A. Manitius, A.W. Olbrot, Finite spectrum assignment problem for systems with delays, IEEE Trans. Automat. Control 24 (4) (1979) 541–552.
[8] M.C. Delfour, The linear-quadratic optimal control problem with delays in state and control variables: a state space approach, SIAM J. Control Optim. 24 (5) (1986) 835–883.
[9] A. Ichikawa, Quadratic control of evolution equations with delays in control, SIAM J. Control Optim. 20 (5) (1982) 645–668.
[10] M.H. Milman, Approximating the linear quadratic optimal control law for hereditary systems with delays in the control, SIAM J. Control Optim. 26 (2) (1988) 291–320.
[11] L. Pandolfi, The standard regulator problem for systems with input delays. An approach through singular control theory, Appl. Math. Optim. 31 (2) (1995) 119–136.
[12] G. Tadmor, The standard H_∞ problem in systems with a single input delay, IEEE Trans. Automat. Control 45 (3) (2000) 382–397.
[13] H. Zhang, G. Duan, L. Xie, Linear quadratic regulation for linear time-varying systems with multiple input delays, Automatica 42 (9) (2006) 1465–1476.
[14] S. Mondié, W. Michiels, Finite spectrum assignment of unstable time-delay systems with a safe implementation, IEEE Trans. Automat. Control 48 (12) (2003) 2207–2212.

[15] A. Germani, C. Manes, P. Pepe, Implementation of an LQG control scheme for linear systems with delayed feedback action, in: Proceedings of the 3rd European Control Conference, Roma, Italy, vol. 4, 1995, pp. 2886–2891.
[16] A. Germani, C. Manes, P. Pepe, A twofold spline approximation for finite horizon LQG control of hereditary systems, SIAM J. Control Optim. 39 (4) (2000) 1233–1295.
[17] F. Carravetta, P. Palumbo, P. Pepe, Memoryless solution to the optimal control problem for linear systems with delayed input, Kybernetika 49 (4) (2013) 568–589.
[18] Z. Lin, H. Fang, On asymptotic stabilizability of linear systems with delayed input, IEEE Trans. Automat. Control 52 (6) (2007) 998–1013.
[19] B. Zhou, Z. Lin, G.-R. Duan, Truncated predictor feedback for linear systems with long time-varying input delays, Automatica 48 (10) (2012) 2387–2399.
[20] S.Y. Yoon, Z. Lin, Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay, Systems Control Lett. 62 (10) (2013) 837–844.
[21] F. Cacace, A. Germani, C. Manes, Exponential stabilization of linear systems with time-varying delayed state feedback via partial spectrum assignment, Systems Control Lett. 69 (2014) 47–52.
[22] B. Zhou, Pseudo-predictor feedback stabilization of linear systems with time-varying input delays, Automatica 50 (11) (2014) 2861–2871.
[23] F. Cacace, F. Conte, A. Germani, Memoryless approach to the LQ and LQG problems with variable input delay, IEEE Trans. Automat. Control (2015) http://dx.doi.org/10.1109/TAC.2015.2427611.
[24] B. Zhou, Truncated Predictor Feedback for Time-Delay Systems, Springer, 2014.
[25] F. Cacace, A. Germani, C. Manes, Predictor-based control of linear systems with large and variable measurement delays, Internat. J. Control 87 (4) (2014) 704–714.
[26] A. Germani, C. Manes, P. Pepe, A new approach to state observation of nonlinear systems with delayed output, IEEE Trans. Automat. Control 47 (1) (2002) 96–101.
[27] T. Ahmed-Ali, E. Cherrier, F. Lamnabhi-Lagarrigue, Cascade high gain predictors for a class of nonlinear systems, IEEE Trans. Automat. Control 57 (1) (2012) 221–226.
[28] F. Cacace, A. Germani, C. Manes, A chain observer for nonlinear systems with multiple time-varying measurement delays, SIAM J. Control Optim. 52 (3) (2014) 1862–1885.