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Another Gorenstein analogue of flat modules

Lixin Mao

Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing, China

Abstract

A right R-module M is called glat if any homomorphism from any finitely presented right R-module to M factors through a finitely presented Gorenstein projective right R-module. The concept of glat modules may be viewed as another Gorenstein analogue of flat modules. We first prove that the class of glat right R-modules is closed under direct sums, direct limits, pure quotients and pure submodules for arbitrary ring R. Then we obtain that a right R-module M is glat if and only if M is a direct limit of finitely presented Gorenstein projective right R-modules. In addition, we explore the relationships between glat modules and Gorenstein flat (Gorenstein projective) modules. Finally we investigate the existence of preenvelopes and precovers by glat and finitely presented Gorenstein projective modules.

KEYWORDS: Coherent ring; glat module; Gorenstein flat module; Gorenstein projective module; precover; preenvelope

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Lixin Mao maolx2@hotmail.com Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China.

1. INTRODUCTION

The origin of Gorenstein homological algebra may date back to 1960s (2) when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring. In (17), Enochs and Jenda extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective (whether finitely generated or not) and Gorenstein injective modules over arbitrary rings. Not much later, the Gorenstein flat module and the Gorenstein flat dimension were introduced by Enochs, Jenda and Torrecillas (19).

As a generalization of flat modules, Gorenstein flat modules indeed inhabit some properties parallel to flat modules especially when the ring is Gorenstein (see (9, 18, 20)). Though it was proved that the Gorenstein flat dimension is a refinement of the classical flat dimension by Bennis (see (5)), the Gorenstein flat notion of modules has not been proven yet that it is the suitable flat counterpart in Gorenstein homological algebra over arbitrary rings. Several authors have attempted to contribute in this sense. For example, we do not know yet whether the class of Gorenstein flat modules is projectively resolving. This fact led to introducing the notion of GF-closed rings (see (4)). Using the GF-closed notion of rings, Yang and Liu proved that all left R-modules over a left GF-closed ring R have Gorenstein flat covers (see (35)). However, in (24), Holm and Jorgensen showed that there are Gorenstein flat modules which are not direct limits of finitely generated Gorenstein projective modules. This shows that there is not a Gorenstein analogue of the classical Govorov–Lazard theorem.

In the present paper, we will give a remedy of some of the above situations by introducing a new flat counterpart in Gorenstein homological algebra, which is called a *glat* module.

It is well known that a right R-module M is flat if and only if any homomorphism from any finitely presented right R-module to M factors through a finitely presented projective right R-module (26). It is natural to ask whether this result has a Gorenstein analogue, i.e., whether is a right R-module M Gorenstein flat if and only if any homomorphism from any finitely presented right R-module to M factors through a finitely presented Gorenstein projective right R-module? In (19), Enochs, Jenda and Torrecillas proved that this is true for a Gorenstein ring. Later, Ding and Chen proved that this is also true for the case of n-FC rings (15). However, we find the answer is no for arbitrary ring in this paper (see Remark 3.9). So it is valuable to investigate the behavior of those modules M satisfying that any homomorphism from any finitely presented right R-module to M factors through a finitely presented Gorenstein projective right R-module. We will call this kind of modules M glat in the present paper. The concept of glat modules is clearly a generalization of flat modules, which may be viewed as another Gorenstein analogue of flat modules.

The main results are in Section 3. First we prove that the class of glat right R-modules is closed under direct sums, direct summands, direct limits, pure quotients and pure submodules for arbitrary ring R (Theorem 3.5). As applications, we give some new characterizations of FC rings and right IF rings in terms of glat modules (Corollaries 3.6 and 3.7). Then we prove that a right R-module M is glat if and only if M is a direct limit of finitely presented Gorenstein projective right R-modules if and only if there is a pure exact sequence $0 \to A \to B \to M \to 0$ with B a glat right R-module (Theorem 3.8). Next we explore the relationships between glat modules and Gorenstein flat modules (Gorenstein projective modules). It is shown that any glat right R-module over a left coherent ring is Gorenstein flat if one of the following conditions holds: (1) R is right coherent; (2) $r.Cot(R) < \infty$; (3) $l.IF(R) < \infty$ (Theorem 3.10). We also prove that, for a left coherent ring R with $FP\text{-}id({}_RR) < \infty$, R is a right perfect ring if and only if every glat right R-module is Gorenstein projective (Theorem 3.11). Finally, we investigate the existence of preenvelopes and precovers by glat and finitely presented Gorenstein projective modules. For example, we prove that every right R-module has a glat cover for any ring R (Proposition 3.15), and every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope if and only if the class of glat right R-modules is closed under direct products (Theorem 3.12). For a two-sided coherent ring R, we prove that every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope (resp. envelope) if every finitely presented left R-module has a finitely presented Gorenstein projective precover (resp. cover) and the converse holds in case R_R is FP-injective (Theorem 3.17).

2. PRELIMINARIES

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R (resp. ${}_RM$) denotes a right (resp. left) R-module. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ and the dual module $\mathrm{Hom}_R(M,R)$ is denoted by M^*. $fd(M)$ and $pd(M)$ stand for the flat and projective dimensions of M respectively. Let M and N be R-modules. $\mathrm{Hom}(M,N)$ and $M \otimes N$ will mean $\mathrm{Hom}_R(M,N)$ and $M \otimes_R N$ respectively, and similarly for derived functors $\mathrm{Ext}^n(M,N))$ and $\mathrm{Tor}_n(M,N)$.

Next we recall some definitions needed in the later section.

An exact sequence of projective right R-modules

$$\mathcal{C} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

is called a *complete projective resolution* (23) if $\mathrm{Hom}(\mathcal{C}, P)$ leaves the sequence $\mathcal{C}$ exact whenever P is a projective right R-module.

A right R-module M is called *Gorenstein projective* (17, 23) if there is a complete projective resolution $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ with $M = \ker(P^0 \to P^1)$.

A right R-module N is called *Gorenstein flat* (19) if there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat right R-modules with $N = \ker(F^0 \to F^1)$ such that $- \otimes E$ leaves the sequence exact whenever E is an injective left R-module.

A right R-module M is called *reflexive* (1) if the canonical map $\delta_M : M \to M^{**}$ is an isomorphism.

A right R-module M is called *cotorsion* (18) if $\mathrm{Ext}^1(F, M) = 0$ for any flat right R-module F.

The *cotorsion dimension* $cd(M)$ of a right R-module M (27) is the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for any flat right R-module F. If there is no such n, set $cd(M) = \infty$. The *right cotorsion dimension* of a ring R (27), denoted by $r.cot.D(R)$, is defined to be $\sup\{cd(M) : M$ is a right R-module$\}$.

A right R-module M is said to be *FP-injective* (or *absolutely pure*) (29, 31) if $\mathrm{Ext}^1(N, M) = 0$ for any finitely presented right R-module N.

The *FP-injective dimension* of a right R-module M (31), denoted by *FP-id*(M), is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for any finitely presented right R-module F. If no such n exists, set *FP-id*$(M) = \infty$.

R is called a *left coherent ring* (26) if every finitely generated left ideal of R is finitely presented.

R is called an *n-FC ring* (15) if it is a two-sided coherent ring with *FP-id*$(R_R) \leq n$ and *FP-id*$({}_RR) \leq n$ for some nonnegative integer n.

R is called a *left FC ring* (12) if it is a left coherent ring and ${}_RR$ is *FP*-injective.

R is called an *FC ring* in case it is a left and right *FC* ring.

R is said to be a *right IF ring* (10) if every injective right R-module is flat.

The dimension $l.IFD(R)$ of a ring R (14) is defined as $l.IFD(R) = \sup\{fd(E) : E$ is an injective left R-module$\}$.

Let $\mathcal{C}$ be a class of right R-modules. Following (16), a homomorphism $\phi : M \to C$ is called a $\mathcal{C}$-*preenvelope* of a right R-module M if $C \in \mathcal{C}$ and the homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is an epimorphism for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-*envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover.

For unexplained concepts and notations, we refer the reader to (1, 9, 18, 20, 21, 26, 30, 32, 33).

3. MAIN RESULTS

Definition 3.1. A right R-module M is called *glat* if any homomorphism from any finitely presented right R-module to M factors through a finitely presented Gorenstein projective right R-module. Namely, for any finitely presented right R-module A and any homomorphism $\theta : A \to M$, there are a finitely presented Gorenstein projective right R-module B, $\gamma : A \to B$ and $\psi : B \to M$ such that $\theta = \psi\gamma$.

Example 3.2. (1) Any flat module is clearly glat.

(2) Any finitely presented Gorenstein projective module is obviously glat.

Proposition 3.3. *Let R be any ring. Then*

(1)An injective right R-module E is glat if and only if E is flat.

(2)A finitely presented right R-module M is glat if and only if M is Gorenstein projective.

Proof.

(1) If an injective right R-module E is glat, then for any finitely presented right R-module A and any homomorphism $\theta : A \to E$, there are a finitely presented Gorenstein projective right R-module Q, $\gamma : A \to Q$ and $\psi : Q \to E$ such that $\theta = \psi\gamma$. Since there is a monomorphism $\lambda : Q \to P$ with P finitely generated projective by the proof of (9, Theorem 4.2.6), there exists $\alpha : P \to E$ such that $\alpha\lambda = \psi$. Hence $\theta = \alpha(\lambda\gamma)$ and so E is flat.

The converse is trivial.

(2) If a finitely presented right R-module M is glat, then the identity map $M \to M$ factors through a finitely presented Gorenstein projective right R-module G. So M is isomorphic to a direct summand of G and hence is Gorenstein projective by (23, Theorem 2.5).

The converse is clear. □

Remark 3.4. In general, a glat right R-module which is not injective need not be flat. For example, $\mathbb{Z}_4$ is a quasi-Frobenius ring. Thus any $\mathbb{Z}_4$-module is Gorenstein projective by (7, Theorem 2.2) and so is glat. But $2\mathbb{Z}_4$ is not a flat $\mathbb{Z}_4$-module.

It is clear that every glat right R-module is flat if and only if every finitely presented Gorenstein projective right R-module is projective. Therefore, for a ring R with finite right global dimension, every glat right R-module is flat by (18, Proposition 10.2.3).

(2) If a right R-module M is glat but not finitely presented, then M need not be Gorenstein projective in general. For example, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module and so is glat. But $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module and so is not Gorenstein projective by (18, Proposition 10.2.3).

The class of glat modules admits nice closure properties as follows.

Theorem 3.5. *For any ring R, the class of glat right R-modules is closed under direct sums, direct summands, direct limits, pure quotients and pure submodules.*

Proof. Let $(M_i)_{i\in I}$ be any family of glat right R-modules and N any finitely presented right R-module. For any homomorphism $f : N \to \oplus_{i\in I} M_i$, since N is finitely presented, there exists a finite index set $J \subseteq I$ such that $\mathrm{im}(f) \subseteq \oplus_{j\in J} M_j$.

Define $\xi : N \to \oplus_{j\in J} M_j$ by $\xi(x) = f(x)$ for any $x \in N$, and $\iota : \oplus_{j\in J} M_j \to \oplus_{i\in I} M_i$ to be the inclusion. Then $f = \iota\xi$.

Let $\pi_j : \oplus_{j\in J} M_j \to M_j$ be the jth canonical projection. Since M_j is glat, there are finitely presented Gorenstein projective right R-module U_j and $g_j : N \to U_j$ and $h_j : U_j \to M_j$ such that $\pi_j\xi = h_j g_j$. So we get $\phi : N \to \oplus_{j\in J} U_j$ such that $g_j = \rho_j\phi$, where $\rho_j : \oplus_{j\in J} U_j \to U_j$ is the jth canonical projection.

Let $\oplus h_j : \oplus_{j\in J} U_j \to \oplus_{j\in J} M_j$ be the induced homomorphism. Then

$\pi_j(\oplus h_j)\phi = h_j\rho_j\phi = h_j g_j = \pi_j\xi.$

Hence $(\oplus h_j)\phi = \xi$. So we get the following commutative diagram:

$$\begin{tikzcd}
 & \oplus_{i\in I} M_i & \\
N \arrow[ur, "f"] \arrow[r, "\xi"] \arrow[dr, dotted, "\phi"] \arrow[ddr, "g_j"'] & \oplus_{j\in J} M_j \arrow[u, "\iota"'] \arrow[r, "\pi_j"] & M_j \\
 & \oplus_{j\in J} U_j \arrow[u, dotted, "\oplus h_j"'] \arrow[d, "\rho_j"] & \\
 & U_j \arrow[uur, "h_j"'] &
\end{tikzcd}$$

Then $f = \iota\xi = \iota(\oplus h_j)\phi$. Since $\oplus_{j\in J} U_j$ is finitely presented Gorenstein projective by (23, Theorem 2.5), $\oplus_{i\in I} M_i$ is glat. So the class of glat right R-modules is closed under direct sums.

Now let $0 \to A \xrightarrow{\epsilon} B \xrightarrow{\pi} C \to 0$ be a pure exact sequence with B a glat right R-module. For any finitely presented right R-module T and any homomorphism $\alpha : T \to A$, since B is glat, there are a finitely presented Gorenstein projective right R-module Q, $\gamma : T \to Q$ and $\psi : Q \to B$ such that $\epsilon\alpha = \psi\gamma$.

Let $T \xrightarrow{\gamma} Q \xrightarrow{\varphi} H \to 0$ be an exact sequence with $H = Q/\mathrm{im}(\gamma)$. Then there exists $\beta : H \to C$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & T \arrow[r, "\gamma"] \arrow[d, "\alpha"] & Q \arrow[r, "\varphi"] \arrow[d, "\psi"] & H \arrow[r] \arrow[d, dotted, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, "\epsilon"] & B \arrow[r, "\pi"] & C \arrow[r] & 0.
\end{tikzcd}$$

Since H is finitely presented and the sequence $0 \to A \xrightarrow{\epsilon} B \xrightarrow{\pi} C \to 0$ is pure, there is $\eta : H \to B$ such that $\beta = \pi\eta$. Hence we have

$$\pi(\psi - \eta\varphi) = \pi\psi - \beta\varphi = 0.$$

Therefore $\operatorname{im}(\psi - \eta\varphi) \subseteq \ker(\pi) = A$. So we get a homomorphism $\theta : Q \to A$ such that $\epsilon\theta = \psi - \eta\varphi$. Thus

$$\epsilon\alpha = \psi\gamma = (\epsilon\theta + \eta\varphi)\gamma = \epsilon\theta\gamma.$$

Since ϵ is monic, $\alpha = \theta\gamma$. It follows that A is glat. Hence the class of glat right R-modules is closed under pure submodules. In particular, it is closed under direct summands.

Because T is finitely presented and the sequence $0 \to A \xrightarrow{\epsilon} B \xrightarrow{\pi} C \to 0$ is pure, for any homomorphism $\mu : T \to C$, there exists $\nu : T \to B$ such that $\pi\nu = \mu$. Since B is glat, there are a finitely presented Gorenstein projective right R-module G, $\omega : T \to G$ and $\sigma : G \to B$ such that $\sigma\omega = \nu$. So $\mu = (\pi\sigma)\omega$. Thus C is glat. Hence the class of glat right R-modules is closed under pure quotients.

Finally for any direct system $(M_i)_{i\in I}$ of glat right R-modules, there exists a pure epimorphism $\oplus_{i\in I} M_i \to \lim_{\to} M_i$. By the proof above, $\oplus_{i\in I} M_i$ is glat, and so $\lim_{\to} M_i$ is glat. Thus the class of glat right R-modules is closed under direct limits. □

We point out that the class of glat right R-modules is not closed under direct products in general. For example, if a ring R is not left coherent, then the class of glat right R-modules is not closed under direct products (see Theorem 3.12 below).

As applications of the theorem above, we give some new characterizations of *FC* rings and right *IF* rings in terms of glat modules.

Corollary 3.6. *The following are equivalent for a ring R:*

(1) *R is an FC ring.*

(2) *Every R-module (left and right) is glat.*

(3) *Every finitely presented R-module (left and right) is glat.*

(4) *Every cotorsion R-module (left and right) is glat.*

Proof.

(1) $\Rightarrow$ (2) Every R-module (left and right) is Gorenstein flat by (15, Theorem 6) and so is glat by (15, Theorem 5).

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are trivial.

(3) $\Rightarrow$ (1) By Proposition 3.3(2), every finitely presented R-module (left and right) is Gorenstein projective. So R is an FC ring by (15, Theorem 6).

(4) $\Rightarrow$ (2) Let M be any right R-module. Then by (21, Theorem 4.1.1) and Wakamatsu's Lemma (33, Section 2.1), there exists an exact sequence $0 \to M \to C \to L \to 0$, where C is cotorsion and L is flat. Since C is glat, M is glat by Theorem 3.5. □

Corollary 3.7. *The following are equivalent for a ring R:*

(1) *R is a right IF ring.*

1. Every injective right R-module is glat.

(2) *Every FP-injective right R-module is glat.*

(3) *Every finitely presented right R-module embeds in a glat right R-module.*

(4) *Every right R-module embeds in a glat right R-module.*

Proof.

(1) $\Leftrightarrow$ (2) follows from Proposition 3.3(1). (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let A be an FP-injective right R-module. Then there exists a pure monomorphism $A \to E$ with E injective. Since E is glat, A is glat by Theorem 3.5.

(4) $\Rightarrow$ (2) For any finitely presented right R-module M, there is a monomorphism $\gamma : M \to F$ with F glat by (4). So there are a finitely presented Gorenstein projective right R-module P, $\alpha : M \to P$ and $\psi : P \to F$ such that $\psi\alpha = \gamma$. Note that α is monic. Hence for any injective right R-module E and any homomorphism $\varphi : M \to E$, there exists $\beta : P \to E$ such that $\varphi = \beta\alpha$. So E is glat.

(2) $\Rightarrow$ (5) is clear since every right R-module embeds in an injective right R-module.

(5) $\Rightarrow$ (4) is trivial. $\square$

The following theorem gives a characterization of glat modules.

Theorem 3.8. *The following are equivalent for a right R-module M:*

(1) *M is glat.*

(2) *M is a direct limit of finitely presented Gorenstein projective right R-modules.*

(3) *There is a pure exact sequence* $0 \to A \to B \to M \to 0$ *with B a glat right R-module.*

Proof.

(1) $\Rightarrow$ (2) The proof is modeled on that of (3) $\Rightarrow$ (4) in (19, Theorem 2.1).

We construct a set of pairs (C,f) with C a finitely presented Gorenstein projective right R-module and $f : C \to M$ a map which includes every such $C \to M$ up to isomorphism.

Let $D = \oplus C$ over all the set of pairs (C,f) and $g : D \to M$ be the homomorphism induced by f. Set $\overline{D} = \oplus_{i\in\mathbb{N}} D_i$ with $D_i = D$ and let $\theta : \overline{D} \to M$ be the homomorphism induced by g.

We define a directed set I as follows. An element $\alpha \in I$ if $\alpha = (S_\alpha, U_\alpha)$, where U_α is the sum of a finite number of the summands C (for various (C,f) and various i) and S_α is a finitely generated submodule of U_α with $S_\alpha \subseteq \ker\theta$. We order the pairs by $\alpha \leq \beta$ if $S_\alpha \subseteq S_\beta$ and $U_\alpha \subseteq U_\beta$. For any $\alpha \leq \beta$, there exists a natural homomorphism $\varphi^\alpha_\beta : U_\alpha/S_\alpha \to U_\beta/S_\beta$. Then we obtain a directed system $\{U_\alpha/S_\alpha, \varphi^\alpha_\beta\}$. Since $S_\alpha \subseteq \ker\theta$, there exists $\xi_\alpha : U_\alpha/S_\alpha \to M$ such that $\xi_\beta\varphi^\alpha_\beta = \xi_\alpha$. So there is a homomorphism $\lim_{\to} U_\alpha/S_\alpha \to M$. It is easy to see that $\lim_{\to} U_\alpha/S_\alpha \cong M$.

Next we will show that the set $J = \{\gamma \in I : U_\gamma/S_\gamma$ is finitely presented Gorenstein projective$\}$ is cofinal in I. For any $\alpha \in I$, since U_α/S_α is finitely presented, by (1), the homomorphism $U_\alpha/S_\alpha \to M$ has a factorization $U_\alpha/S_\alpha \xrightarrow{\sigma} \overline{C} \xrightarrow{\tau} M$ with $\overline{C}$ finitely presented Gorenstein projective. Note that each of the summands C in U_α is also a direct summand of some D_i. So let $n_0 \neq i$ for such i and U_γ be the sum of U_α and $\overline{C}$ as a direct summand of D_{n_0}. Write $U_\gamma = U_\alpha \oplus \overline{C}$. Let η be the composition $U_\alpha \xrightarrow{\pi} U_\alpha/S_\alpha \xrightarrow{\sigma} \overline{C}$ and $S_\gamma = \{(u, -\eta(u)) : u \in U_\alpha\}$. Then $U_\gamma/S_\gamma \cong \overline{C}$

and $S_\alpha \subseteq S_\gamma \subseteq \ker\theta$. Thus $(S_\alpha, U_\alpha) \leq (S_\gamma, U_\gamma)$. Since U_γ/S_γ is finitely presented Gorenstein projective, the set J is cofinal in I. Therefore M is a direct limit of finitely presented Gorenstein projective right R-modules by (18, Remark 1.5.4(3)).

(2) $\Rightarrow$ (3) Let $M = \lim_{\rightarrow} M_i$ with each M_i finitely presented Gorenstein projective. Then there exists a pure exact sequence $0 \to K \to \oplus M_i \to \lim_{\rightarrow} M_i \to 0$ with $\oplus M_i$ glat by Theorem 3.5.

(3) $\Rightarrow$ (1) is clear by Theorem 3.5. □

Remark 3.9. Although the class of glat right R-modules over an n-FC ring coincides with the class of Gorenstein flat right R-modules by (15, Theorem 5), these two kinds of modules may be different for arbitrary ring. For example, let K be a field and $R = K[x,y,z]/ < x^2, yz, y^2 - xz, z^2 - yx >$ be the 6-dimensional K-algebra. Then there exists a Gorenstein flat R-module which is not a direct limit of finitely presented Gorenstein projective R-modules by (3, Example 4.3) and so is not glat by Theorem 3.8.

Now we furthermore establish the relationships between glat modules and Gorenstein flat modules.

Theorem 3.10. *Let R be a left coherent ring. Then any glat right R-module is Gorenstein flat if one of the following conditions holds:*

(1) R is right coherent;

(2) $r.Cot(R) < \infty$;

1. $l.IF(R) < \infty$.

In particular, a finitely presented right R-module is glat if and only if it is Gorenstein flat for a two-sided coherent ring.

Proof. If M is a glat right R-module, then $M = \lim_{\rightarrow} M_i$ with every M_i finitely presented Gorenstein projective by Theorem 3.8. We next show that every M_i is Gorenstein flat. For every M_i, there is a complete projective resolution

$$\mathcal{C} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M_i = \ker(P^0 \to P^1)$.

Case 1. R is right coherent.

By (18, Proposition 10.3.2), for a two-sided coherent ring R, every finitely presented Gorenstein projective right R-module M_i is Gorenstein flat.

Case 2. $r.Cot(R) < \infty$.

For any injective left R-module E, E^+ is flat by (8, Theorem 1). Thus there exists $m \in \mathbb{N}$ such that $pd(E^+) = m < \infty$ by (27, Corollary 19.2.7). So there is an exact sequence

$$0 \to Q_m \to Q_{m-1} \to \cdots \to Q_1 \to Q_0 \to E^+ \to 0$$

with each Q_i projective. Then we obtain the exact sequence of complexes

$$0 \to \mathrm{Hom}(\mathcal{C}, Q_m) \to \mathrm{Hom}(\mathcal{C}, Q_{m-1}) \to \cdots \to \mathrm{Hom}(\mathcal{C}, Q_0) \to \mathrm{Hom}(\mathcal{C}, E^+) \to 0.$$

Since each $\mathrm{Hom}(\mathcal{C}, Q_i)$ is exact, $\mathrm{Hom}(\mathcal{C}, E^+)$ is exact by (30, Theorem 6.3). Note that $(\mathcal{C} \otimes E)^+ \cong \mathrm{Hom}(\mathcal{C}, E^+)$. Thus the complex $(\mathcal{C} \otimes E)^+$ is exact and so is $\mathcal{C} \otimes E$. Hence every M_i is Gorenstein flat.

Case 3. $l.IF(R) < \infty$.

Let E be any injective left R-module, then there exists $n \in \mathbb{N}$ such that $fd(E) = n < \infty$. So there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to E \to 0$$

with each F_i flat. Thus we obtain the exact sequence of complexes

$$0 \to \mathcal{C} \otimes F_n \to \mathcal{C} \otimes F_{n-1} \to \cdots \to \mathcal{C} \otimes F_0 \to \mathcal{C} \otimes E \to 0.$$

Since each $\mathcal{C} \otimes F_i$ is exact, $\mathcal{C} \otimes E$ is exact by (30, Theorem 6.3). Thus every M_i is Gorenstein flat. It follows that M is a direct limit of Gorenstein flat right R-modules. Since R is left coherent, the class of Gorenstein flat right R-modules is closed under direct limits by (20, Corollary 2.1.9). Thus M is Gorenstein flat.

Finally, we assume that A is a finitely presented Gorenstein flat right R-module over a two-sided coherent ring R. Then there is an exact sequence

$$\cdots \to P_1 \to P_0 \to A \to 0$$

with each P_i finitely generated projective since R is right coherent. For any injective left R-module E, since $\mathrm{Tor}_n(A, E) = 0$ for any $n \geq 1$, we get the exact sequence

$$\cdots \to P_1 \otimes E \to P_0 \otimes E \to A \otimes E \to 0.$$

On the other hand, since A is Gorenstein flat, there is a monomorphism $\lambda : A \to F^0$ with F^0 flat such that $\lambda \otimes 1 : A \otimes E \to F^0 \otimes E$ is monic for any injective left R-module E. Since A is a finitely presented, $\lambda : A \to F^0$ factors through a finitely generated projective right R-module P^0, i.e., there are $\gamma : A \to P^0$ and $\psi : P^0 \to F^0$ such that $\lambda = \psi\gamma$. Note that γ is monic. So we get the exact sequence $0 \to A \xrightarrow{\gamma} P^0 \to C^1 \to 0$ with C^1 finitely presented.

Consider the following commutative diagram:

$$\begin{tikzcd} A \otimes E \arrow[r, "\lambda\otimes 1"] \arrow[d, "\gamma\otimes 1"'] & F^0 \otimes E \\ P^0 \otimes E \arrow[ur, "\psi\otimes 1"'] & \end{tikzcd}$$

Since $\lambda \otimes 1$ is monic, $\gamma \otimes 1$ is monic. Hence $\mathrm{Tor}_1(C^1, E) = 0$. So C^1 is finitely presented Gorenstein flat by (23, Proposition 3.8) since R is left coherent. Repeating the step above to C^1 and so on, we get the exact sequence

$$0 \to A \to P^0 \to P^1 \to \cdots$$

with each P^i finitely generated projective such that

$$0 \to A \otimes E \to P^0 \otimes E \to P^1 \otimes E \to \cdots$$

is exact. So we get the exact sequence

$\mathcal{C} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$

with $A = \ker(P^0 \to P^1)$ such that $\mathcal{C} \otimes E$ is exact.

For any projective right R-module P, P^+ is injective. So $\mathcal{C} \otimes P^+$ is exact. By (30, Lemma 3.60), $\mathrm{Hom}(\mathcal{C}, P)^+ \cong \mathcal{C} \otimes P^+$. Thus $\mathrm{Hom}(\mathcal{C}, P)^+$ is exact and so is $\mathrm{Hom}(\mathcal{C}, P)$. Therefore A is Gorenstein projective and hence is glat. □

Let R be an *n-FC* ring. Then any Gorenstein projective right R-module is glat by (15, Theorem 5). The following theorem exhibits when every glat right R-module is Gorenstein projective.

Theorem 3.11. *The following are equivalent for a left coherent ring R with FP-id*$({}_RR) < \infty$*:*

(1) *R is a right perfect ring.*

(2) *Every glat right R-module is Gorenstein projective.*

(3) Every glat right R-module is cotorsion.

Proof.

(1) $\Rightarrow$ (2) Let M be a glat right R-module. Then $M = \lim_{\to} M_i$ with each M_i finitely presented Gorenstein projective by Theorem 3.8. For any projective right R-module P, there is an exact sequence $0 \to P \to N \to L \to 0$, where N is the pure-injective envelope of P. By (33, Lemma 3.1.6), L is flat and so is projective by (1). Thus P is isomorphic to a direct summand of N and hence is pure-injective.

By (21, Lemma 3.3.4), for any $n \geq 1$, we have

$$\mathrm{Ext}^n(M,P) = \mathrm{Ext}^n(\varinjlim M_i, P) \cong \varprojlim \mathrm{Ext}^n(M_i,P) = 0.$$

Thus there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i projective, which remains exact whenever $\mathrm{Hom}(-,P)$ is applied to it.

On the other hand, since R is left coherent right perfect, every right R-module has a projective preenvelope by (13, Proposition 3.14). Thus there is a complex

$$0 \to M \to P^0 \to P^1 \to \cdots$$

with each P^i projective such that $\mathrm{Hom}(-,P)$ leaves the sequence exact whenever P is a projective right R-module by (18, Proposition 8.1.3). So we get the complex

$$\mathcal{C} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right R-modules with $M \cong \mathrm{coker}(P_1 \to P_0)$.

We next prove that $\mathcal{C} \otimes G$ is exact for any left R-module G with $FP\text{-}id(G) = n < \infty$. We proceed by induction on n.

Let $n = 0$, then G is FP-injective. So G^+ is projective by (8, Theorem 3). Thus the complex $\mathrm{Hom}(\mathcal{C}, G^+)$ is exact. Since $(\mathcal{C} \otimes G)^+ \cong \mathrm{Hom}(\mathcal{C}, G^+)$, the complex $\mathcal{C} \otimes G$ is exact.

Let $n \geq 1$. There is an exact sequence $0 \to G \to E \to L \to 0$ with E injective and FP-$id(L) = n-1$, which induces the exact sequence of complexes

$0 \to \mathcal{C} \otimes G \to \mathcal{C} \otimes E \to \mathcal{C} \otimes L \to 0.$

By induction, $\mathcal{C} \otimes L$ is exact. Thus $\mathcal{C} \otimes G$ is exact by (30, Theorem 6.3). In particular, $\mathcal{C} \otimes R$ is exact since FP-$id(_R R) < \infty$. Therefore $\mathcal{C}$ is a complete projective resolution. So M is Gorenstein projective.

(2) $\Rightarrow$ (1) Let P be any projective right R-module. For any flat right R-module F, F is Gorenstein projective by (2). So $\mathrm{Ext}^1(F, P) = 0$. Thus P is cotorsion. Hence R is a right perfect ring by (22, Corollary 10).

(1) $\Rightarrow$ (3) is trivial since every right R-module is cotorsion.

(3) $\Rightarrow$ (1) Let F be any flat right R-module. Then F is cotorsion by (3). Hence R is a right perfect ring by (33, Proposition 3.3.1). □

Next we turn to the existence of preenvelopes and precovers by glat and finitely presented Gorenstein projective modules.

Theorem 3.12. *The following are equivalent for a ring R:*

(1) *Every right R-module has a glat preenvelope.*

(2) *Every finitely presented right R-module has a glat preenvelope.*

(3) *Every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope.*

(4) *The class of glat right R-modules is closed under direct products.*

In this case, R is a left coherent ring.

Proof.

(1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let $\alpha : M \to F$ be a glat preenvelope of a finitely presented right R-module M. Then there exist a finitely presented Gorenstein projective right R-module P, homomorphisms $\beta : M \to P$ and $\gamma : P \to F$ such that $\alpha = \gamma\beta$. It is easy to verify that $\beta : M \to P$ is a finitely presented Gorenstein projective preenvelope.

(3) $\Rightarrow$ (4) Let $(F_j)_{j\in J}$ be a family of glat right R-modules, G any finitely presented right R-module and $\alpha : G \to \prod_{j\in J} F_j$ any homomorphism. Then there exist finitely presented Gorenstein projective right R-modules Q_j, homomorphisms $\phi_j : G \to Q_j$ and $\varphi_j : Q_j \to F_j$ such that $\pi_j\alpha = \varphi_j\phi_j$, where $\pi_j : \prod_{j\in J} F_j \to F_j$ is the canonical projection. By (3), G has a finitely presented Gorenstein projective preenvelope $\beta : G \to H$. So there are $\psi_j : H \to Q_j$ such that $\phi_j = \psi_j\beta$. Hence there is $\eta : H \to \prod_{j\in J} F_j$ such that $\pi_j\eta = \varphi_j\psi_j$. So

$$\pi_j\alpha = \varphi_j\phi_j = \varphi_j\psi_j\beta = \pi_j\eta\beta,$$

and whence $\alpha = \eta\beta$. Thus $\prod_{j\in J} F_j$ is glat.

(4) $\Rightarrow$ (1) follows from Theorem 3.5 and (11, Theorem 4.1).

Finally suppose that R satisfies one of the equivalent conditions. Let M be any finitely presented right R-module. Then M has a finitely presented Gorenstein projective preenvelope $M \to N$. So we the exact sequence

$$\mathrm{Hom}(N,R) \to \mathrm{Hom}(M,R) \to 0.$$

By the proof of (9, Theorem 4.2.6), there exists an exact sequence

$$0 \to N \to Q \to M' \to 0$$

with Q finitely generated projective and M' Gorenstein projective. So we get the induced exact sequence

$$\mathrm{Hom}(Q,R) \to \mathrm{Hom}(N,R) \to \mathrm{Ext}^1(M',R) = 0.$$

Since $\mathrm{Hom}(Q,R)$ is a finitely generated left R-module, $\mathrm{Hom}(N,R)$ is also finitely generated and so is $\mathrm{Hom}(M,R)$. By (10, Proposition 1), R is a left coherent ring. $\square$

Corollary 3.13. *The following are equivalent for a ring R such that the class of glat right R-modules is closed under direct products:*

(1) *R is a left FC ring.*

(2) *R is a right IF ring.*

(3) *Every right R-module has a monic glat preenvelope.*

(4) *Every finitely presented right R-module has a monic finitely presented Gorenstein projective preenvelope.*

Proof. By Theorem 3.12, R is a left coherent ring.

(1) $\Leftrightarrow$ (2) follows from the fact that a ring R is left FC if and only if R is left coherent right IF (see (25, Theorem 3.10)).

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are easy by Theorem 3.12 and Corollary 3.7. □

Recall from (6) that a commutative ring R is called *Gorenstein semihereditary* if R is a coherent ring and every submodule of a (Gorenstein) flat R-module is Gorenstein flat.

Clearly, a commutative ring R is semihereditary if and only if R is a Gorenstein semihereditary ring and every glat R-module is flat.

Corollary 3.14. *The following are equivalent for a commutative ring R such that the class of glat R-modules is closed under direct products:*

(1) *R is a Gorenstein semihereditary ring.*

(2) *Every R-module has an epic glat preenvelope.*

(3) *Every finitely presented R-module has an epic finitely presented Gorenstein projective preenvelope.*

Proof.

(1) $\Rightarrow$ (3) By Theorem 3.12, every finitely presented R-module M has a finitely presented Gorenstein projective preenvelope $f : M \to N$. Since im$f$ is finitely presented Gorenstein flat by (1), im$f$ is finitely presented Gorenstein projective by Theorem 3.10. So $M \to \mathrm{im}(f)$ is an epic finitely presented Gorenstein projective preenvelope of M.

(3) $\Rightarrow$ (2) Let M be any R-module. Then $M = \lim_{\rightarrow} M_i$ with every M_i finitely presented. By (3), every M_i has an epic finitely presented Gorenstein projective preenvelope $f_i : M_i \to N_i$, which is also an epic glat preenvelope. Then we get the epimorphism $M = \lim_{\rightarrow} M_i \to \lim_{\rightarrow} N_i$ by (18, Theorem 1.5.6).

We next show that $\lim_{\rightarrow} M_i \to \lim_{\rightarrow} N_i$ is a glat preenvelope. That is to say, we must show that the morphism $\mathrm{Hom}(\lim_{\rightarrow} N_i, F) \to \mathrm{Hom}(\lim_{\rightarrow} M_i, F)$ is an epimorphism for any glat R-module F. In fact, the morphism $\mathrm{Hom}(N_i, F) \to \mathrm{Hom}(M_i, F)$ is monic since $f_i : M_i \to N_i$ is epic. Also the morphism $\mathrm{Hom}(N_i, F) \to \mathrm{Hom}(M_i, F)$ is epic since $f_i : M_i \to N_i$ is a glat preenvelope. So we get the isomorphism $\mathrm{Hom}(N_i, F) \cong \mathrm{Hom}(M_i, F)$. Hence we obtain the isomorphism $\lim_{\leftarrow} \mathrm{Hom}(N_i, F) \cong \lim_{\leftarrow} \mathrm{Hom}(M_i, F)$.

From (18, Theorem 1.5.14), we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\lim\limits_{\rightarrow} N_i, F) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\lim\limits_{\rightarrow} M_i, F) \arrow[d, "\cong"'] \\
\lim\limits_{\leftarrow} \mathrm{Hom}(N_i, F) \arrow[r, "\cong"] & \lim\limits_{\leftarrow} \mathrm{Hom}(M_i, F).
\end{tikzcd}$$

Thus the morphism $\mathrm{Hom}(\lim_{\to} N_i, F) \to \mathrm{Hom}(\lim_{\to} M_i, F)$ is an epimorphism. Note that $\lim_{\to} N_i$ is glat by Theorem 3.8. So $M = \lim_{\to} M_i \to \lim_{\to} N_i$ is a glat preenvelope which is an epimorphism.

(2) $\Rightarrow$ (1) By Theorem 3.12, R is a coherent ring.

Let A be a submodule of a flat R-module B. Then A has an epic glat preenvelope $\varphi : A \to C$ by (2). It is easy to see that φ is also monic. Thus $A \cong C$ is glat, and so is Gorenstein flat by Theorem 3.10. $\square$

It is known that every right R-module has a flat cover (18) and has a Gorenstein flat precover for any ring R (34). Here we have

Proposition 3.15. *Every right R-module has a glat cover for any ring R.*

Proof. It holds by Theorem 3.5 and (11, Theorem 2.6). $\square$

Let R be an n-FC ring. Then every finitely presented left R-module has a finitely presented Gorenstein projective precover by (15, Theorem 14). On the other hand, every right R-module has a Gorenstein flat preenvelope by (28, Theorem 5.3) and so has a glat preenvelope by (15, Theorem 5). Thus every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope by Theorem 3.12.

At the end of the paper, we prove that, for a two-sided coherent ring R, there exists certain duality between finitely presented Gorenstein projective precovers (resp. covers) of finitely presented left

R-modules and finitely presented Gorenstein projective preenvelopes (resp. envelopes) of finitely presented right R-modules.

The following lemma is needed.

Lemma 3.16. *Let R be a right coherent ring. If M is a finitely presented Gorenstein projective right R-module, then M is reflexive and M^* is a finitely presented Gorenstein projective left R-module.*

Proof. By (9, Theorem 4.2.6), there is an exact sequence

$$0 \to M \to P^0 \to P^1 \to \cdots$$

with each P^i finitely presented projective such that $\mathrm{Hom}(-, P)$ leaves the sequence exact whenever P is a projective right R-module.

Since R is right coherent and M is a finitely presented, there is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i finitely presented projective. Note that $\mathrm{Ext}^n(M, P) = 0$ for any projective right R-module P and $n \geq 1$. So we get the exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of finitely presented projective right R-modules such that $\mathrm{Hom}(-, P)$ leaves the sequence exact whenever P is a projective right R-module. Hence we get the induced exact sequence

$$\cdots \to (P^1)^* \to (P^0)^* \to (P_0)^* \to (P_1)^* \to \cdots$$

with $M^* \cong \ker((P_0)^* \to (P_1)^*)$, and so we get the complex

$$\cdots \to (P_1)^{**} \to (P_0)^{**} \to {(P^0)}^{**} \to (P^1)^{**} \to \cdots,$$

which is naturally isomorphic to the exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$. Thus the complex $\cdots \to (P_1)^{**} \to (P_0)^{**} \to {(P^0)}^{**} \to (P^1)^{**} \to \cdots$ is exact. So $M \cong M^{**}$, i.e., M is reflexive.

Since all $(P^i)^*$ and $(P_i)^*$ are finitely presented projective, it is easy to see that the exact sequence $\cdots \to (P^1)^* \to (P^0)^* \to (P_0)^* \to (P_1)^* \to \cdots$ is $\mathrm{Hom}(-, P)$ exact for any projective right R-module P. So M^* is a finitely presented Gorenstein projective left R-module. □

Theorem 3.17. *Let R be a two-sided coherent ring. If every finitely presented left R-module has a finitely presented Gorenstein projective precover (resp. cover), then every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope (resp. envelope). The converse holds if R_R is FP-injective.*

Proof. We first assume that every finitely presented left R-module has a finitely presented Gorenstein projective precover.

Let M be any finitely presented right R-module. Then M^* is a finitely presented left R-module by (10, Proposition 1) since R is left coherent. Suppose that $\varphi : N \to M^*$ is a finitely presented Gorenstein projective precover. Let $\psi : M \to F$ with F finitely presented Gorenstein projective be any homomorphism. Note that F^* is finitely presented Gorenstein projective by Lemma 3.16 since R is right coherent. So there exists $\theta : F^* \to N$ such that $\varphi\theta = \psi^*$. Then we have the following commutative diagram:

$$\begin{tikzcd}
F \arrow[r, "\delta_F"] & F^{**} & \\
M \arrow[u, "\psi"] \arrow[r, "\delta_M"'] & M^{**} \arrow[u, "\psi^{**}"] \arrow[r, "\varphi^*"'] & N^* \arrow[ul, "\theta^*"']
\end{tikzcd}$$

By Lemma 3.16, F is reflexive. So we have

$$(\delta_F^{-1}\theta^*)(\varphi^*\delta_M) = \delta_F^{-1}(\varphi\theta)^*\delta_M = \delta_F^{-1}\psi^{**}\delta_M = \delta_F^{-1}\delta_F\psi = \psi.$$

Note that N^* is finitely presented Gorenstein projective by Lemma 3.16. Therefore the composition $\varphi^*\delta_M : M \to M^{**} \to N^*$ is a finitely presented Gorenstein projective preenvelope of M.

Now we furthermore assume that every finitely presented left R-module has a finitely presented Gorenstein projective cover.

For any finitely presented right R-module M, the finitely presented left R-module M^* has a finitely presented Gorenstein projective cover $\varphi : N \to M^*$. We only need to show that any $\xi : N^* \to N^*$ such that $\xi(\varphi^*\delta_M) = \varphi^*\delta_M$ is an isomorphism by the proof above. Note that $\delta_{M^*}\varphi = \varphi^{**}\delta_N$. Then $(\delta_M)^*\delta_{M^*}\varphi = (\delta_M)^*\varphi^{**}\delta_N$, which means that $\varphi = (\delta_M)^*\varphi^{**}\delta_N$ by (1, Proposition 20.14). Since N is reflexive by Lemma 3.16, $\varphi\delta_N^{-1} = (\delta_M)^*\varphi^{**}$. Note that $(\delta_M)^*\varphi^{**}\xi^* = (\delta_M)^*\varphi^{**}$, so we have

$$\varphi\delta_N^{-1}\xi^*\delta_N = (\delta_M)^*\varphi^{**}\xi^*\delta_N = (\delta_M)^*\varphi^{**}\delta_N = \varphi.$$

Thus $\delta_N^{-1}\xi^*\delta_N$ is an isomorphism since φ is a cover. Hence ξ^* is an isomorphism and so ξ^{**} is an isomorphism. By Lemma 3.16, N^* is also reflexive. So $\xi = \delta_{N^*}^{-1}\xi^{**}\delta_{N^*}$ is an isomorphism. It follows that $\varphi^*\delta_M$ is a finitely presented Gorenstein projective envelope of M.

Conversely, we assume that every finitely presented right R-module has a finitely presented Gorenstein projective preenvelope and R_R is FP-injective.

Let A be any finitely presented left R-module, then A^* is a finitely presented right R-module by (10, Proposition 1) and A is reflexive by (31, Theorem 4.8) or (25, Corollary 2.4). Let $\gamma : A^* \to B$ be a finitely presented Gorenstein projective preenvelope and $\alpha : C \to A$ with C finitely presented Gorenstein projective be any homomorphism. Since C^* is finitely presented Gorenstein projective by Lemma 3.16, there exists $\tau : B \to C^*$ such that $\tau\gamma = \alpha^*$. Then we have the following commutative diagram:

$$\begin{tikzcd} & C^{**} \arrow[dl, "\tau^*"'] \arrow[d, "\alpha^{**}"] & C \arrow[l, "\delta_C"'] \arrow[d, "\alpha"] \\ B^* \arrow[r, "\gamma^*"'] & A^{**} & A \arrow[l, "\delta_A"] \end{tikzcd}$$

So

$$(\delta_A^{-1}\gamma^*)(\tau^*\delta_C) = \delta_A^{-1}(\tau\gamma)^*\delta_C = \delta_A^{-1}\alpha^{**}\delta_C = \delta_A^{-1}\delta_A\alpha = \alpha.$$

Note that B^* is finitely presented Gorenstein projective by Lemma 3.16. Therefore the composition $\delta_A^{-1}\gamma^* : B^* \to A^{**} \to A$ is a finitely presented Gorenstein projective precover of A.

Next we furthermore assume that every finitely presented right R-module has a finitely presented Gorenstein projective envelope and R_R is FP-injective.

For any finitely presented left R-module A, the finitely presented right R-module A^* has a finitely presented Gorenstein projective envelope $\gamma : A^* \to B$. Then $\delta_A^{-1}\gamma^*$ is a finitely presented Gorenstein projective precover of A by the proof above.

Let $\eta : B^* \to B^*$ satisfy that $\delta_A^{-1}\gamma^*\eta = \delta_A^{-1}\gamma^*$. Then we have $\eta^*\gamma^{**}(\delta_A^{-1})^* = \gamma^{**}(\delta_A^{-1})^*$. Note that $\delta_B\gamma = \gamma^{**}\delta_{A^*}$ and B is reflexive. Thus $\gamma = \delta_B^{-1}\gamma^{**}\delta_{A^*}$. Also

$$\delta_{A^*} = (\delta_A\delta_A^{-1})^*\delta_{A^*} = (\delta_A^{-1})^*(\delta_A)^*\delta_{A^*} = (\delta_A^{-1})^*.$$

Then

$$\gamma = \delta_B^{-1}\gamma^{**}(\delta_A^{-1})^* = \delta_B^{-1}\eta^*\gamma^{**}(\delta_A^{-1})^* = \delta_B^{-1}\eta^*\delta_B\delta_B^{-1}\gamma^{**}(\delta_A^{-1})^* = \delta_B^{-1}\eta^*\delta_B\gamma.$$

Thus $\delta_B^{-1}\eta^*\delta_B$ is an isomorphism since γ is an envelope. Hence η^* is an isomorphism and so η^{**} is an isomorphism. By Lemma 3.16, B^* is also reflexive. So $\eta = \delta_{B^*}^{-1}\eta^{**}\delta_{B^*}$ is an isomorphism. Therefore $\delta_A^{-1}\gamma^*$ is a finitely presented Gorenstein projective cover of A.

This completes the proof. □

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