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A group decision making method with interval valued fuzzy preference relations based on the geometric consistency

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ABSTRACT

This paper investigates a group decision making (GDM) method with interval valued fuzzy preference relations (IVFPRs). According to the geometric consistency of IVFPR, the max-consistency index and min-consistency index of an IVFPR are developed respectively. Combining the max-consistency index with min-consistency index, the geometric consistent index of an IVFPR is defined to measure the consistency level of the IVFPR by considering decision maker's (DM's) risk attitude. For improving the unacceptable geometric consistency of an IVFPR, a goal programming model is constructed to derive an acceptable geometric consistent IVFPR. By regarding the geometric consistent conditions of an IVFPR as fuzzy constraints, a fuzzy logarithmic program is established to generate the interval priority weights. In GDM problems, the individual interval priority weights are obtained by solving the corresponding fuzzy logarithmic programs. The similarities between DMs are calculated based on their individual interval priority weights. Subsequently the confidence degrees of DMs are defined to determine DMs' weights. To obtain the collective interval priority weights, a parametric linear program is constructed and transformed into a linear program to resolve. The order of alternatives is generated by the collective interval priority weights. Some examples are analyzed to verify the effectiveness of the proposed method.

1. Introduction

Decision making problems are common to all aspects of modern life, such as human resource performance evaluation, facility location, and investment project selection. In general, the best alternative is selected from a set of finite alternatives by decision making methods. Since the evaluation factors are numerous in complex decision problems, it is difficult for decision makers (DMs) to provide the ratings of alternatives on each attribute. Thus, originated from analytic hierarchy process (AHP) method, the preference relation began to appear. Now, the preference relation is not restricted to the framework of AHP. It becomes an important form of DM's opinion for expressing preference on alternatives. As the modern society developed, the complication and uncertainty of the real-world problems are highlighted during the decision process [1,2]. Classical numerical decision methods could not be suitable for solving such problems. Thus, the interval valued fuzzy preference relation (IVFPR) is developed, where the judgments on the IVFPR are represented by intervals. Since intervals can flexibly capture and describe the ambiguity and uncertainty of human's appraisal, IVFPRs are more flexible to represent the uncertain preferences over alternatives. At present, research on IVFPRs mainly concentrates on the consistency analysis and the determination of priority weights.

(1) Consistency analysis of IVFPR

The consistency of IVFPR requires that DMs' judgments yield no contradiction. Due to the complexity of the problems, it may be impossible for DMs to provide completely consistent judgments on alternatives in some specific situations. However, it can not guarantee the reliability of the decision results using the priority weights derived by inconsistent matrices. Hence, the consistency of IVFPR is an important topic. Lots of works have been done on the topic. By extending the additive and multiplicative consistent fuzzy preference relations, Xu and Chen [3] firstly proposed the additive and multiplicative consistent IVFPRs, respectively. According to the interval multiplicative transitivity of an IVFPR, Genç et al. [4] tested the consistency of an IVFPR by judging whether there is a constructed IVFPR obtained from the original IVFPR. Chen and Zhou [5] defined the expected value fuzzy preference relations corresponding to an IVFPR and presented the consistency indicator of

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an IVFPR. Liu et al. [6] proposed the consistency definition of an IVFPR by judging whether two fuzzy preference relations obtained by the IVFPR are consistent or not. Based on the additive transitivity and multiplicative transitivity of an IVFPR, Wang and Li [7] defined the additive and multiplicative IVFPRs and used some functions about priority weights to construct the additive and multiplicative IVFPRs, respectively. Xu et al. [8] developed the additive consistency of an IVFPR by introducing the interval [0.5,0.5] and analyzed the relationship between the multiplicative consistency of an interval multiplicative reciprocal preference relation and the additive consistency of its converted IVFPR. Wu and Chiclana [9] presented the consistent IVFPR according to the multiplicative transitive IVFPR. Then the multiplicative consistency index of an IVFPR is proposed to measure the level of consistency of the information provided by DMs. Liu et al. [10] defined a multiplicative consistent IVFPR by judging whether the two additive reciprocal matrices generated from the IVFPR are multiplicatively consistent. Using the interval ratio of DM's preference intensity, Wang and Chen [11] proposed the geometric consistency of an IVFPR based on its geometric transitivity. The proposed geometric consistency definition is robust to permutations of DM's judgments for determining whether an IVFPR is consistent or not. Wang and Li [12] investigated the multiplicative transitivity defined in [7] and introduced some properties for multiplicative consistent IVFPR.

(2) Determination of the priority weights

Since the priority weights are used to generate the ranking order of alternatives, it is important to determine the priority weights from an IVFPR in the decision making. By introducing the deviation variables, Xu and Chen [3] established some linear programming models to obtain the priority weights from consistent or inconsistent IVFPRs. Genç et al. [4] proposed a straightforward approach to checking whether an IVFPR is consistent or not and obtained the priority weights from a consistent IVFPR by a simple formula. Xu [13] established two goal programming models to derive the priority weights from additive and multiplicative consistent IVFPRs, respectively. Chen and Zhou [5] presented a consistency induced generalized continuous ordered weighted averaging operator to obtain the priority weights of alternatives in group decision making (GDM) with IVFPRs. Wang and Li [7] proposed some goal programming-based models for deriving interval weights from IVFPRs. Based on the defined additive consistent IVFPRs, Wang et al. [14] established linear programming models to generate interval priority weights that are more accurate than those obtained by method [3]. According to a formula for interval priority weights, Liu et al. [6] derived the interval priority weights from two fuzzy preference relations obtained by an IVFPR. An algorithm was proposed to obtaining the priority weights. Xu and Liu [15] established an expected value matrix from an IVFPR and used method [13] to derive the numerical priority weights. Xu et al. [16] studied a distance-based aggregation approach to assessing the relatively important weights for GDM with IVFPRs. Based on the additive consistency of fuzzy preference relation, Zhang et al. [17] demonstrated how to obtain the priority weights from an IVFPR by extracting consistent additive-based pairwise comparison matrices. Bentkowska et al. [18] addressed some transitivity properties of IVFPRs, such as weak transitivity and 0.5-transitivity. Then a method was proposed to take the solution alternative using the nondominance algorithm. To find a group interval valued priority vector, Wang and Li [12] established a logarithmic goal programming model that minimizes the deviations between an IVFPR and the corresponding matrix obtained by priority weights. Meng et al. [19] presented several consistency-based linear programming models to derive the interval priority weights from IVFPRs, which can cope with the consistent and inconsistent cases. Based on the geometric consistency of an IVFPR defined in [7], Wang [20] developed a two-stage linear goal programming approach to eliciting interval weights from IVFPRs. In view of the geometric transitivity in modeling the consistency of fuzzy preference relations, Zhang [21] proposed a goal programming model to obtain the priority weights from an IVFPR.

Previous studies have significantly advanced the research on decision making with IVFPRs. Nevertheless, there are still some limitations as follows:

(1) In the above literature [3,4,6-8,10-12], the definitions of a consistent IVFPR are generally proposed by extending the consistency of a fuzzy preference relation. These definitions can be applied to judge whether an IVFPR is consistent or not. However, none of them measures the consistency level of an inconsistent IVFPR. In some actual decision making problems, an inconsistent IVFPR may be accepted if it has a high level of consistency (See Examples 2 and 3 in Section 6). Such an IVFPR can be regarded as an acceptable consistent IVFPR. As a result, it is necessary to define the consistency index for discussing the acceptable consistent IVFPR from an inconsistent one.

(2) Some methods [5,6,15,16,18] directly use the formulas between the priority weights and elements in IVFPR to derive the priority weights. Such methods would be invalid for inconsistent IVFPRs since they are only suitable for consistent IVFPRs. Other methods [3,4,7,12-14,19-21] apply some linear or goal programming models to obtain the priority weights. These methods merely minimize the deviation between an original IVFPR and the converted consistent one obtained by priority weighs as much as possible. However, for extremely inconsistent IVFPR, the priority weights derived by such methods are unreasonable and cannot be accepted in decision making.

To overcome the above limitations, this paper focuses on a new GDM method with IVFPRs based on geometric consistency. The motivations of this paper mainly come from the following facts:

(1) With the increasing complexity of the practical problems, the amount of knowledge and information is greatly increased. A single DM has no ability to handle such complex problems. Thus, the GDM participated by several DMs is becoming more and more prevalent [22–24]. Due to different understanding and cognition of various DMs, the decision opinions provided by DMs are diversified and even conflicting. However, some existing methods are not effective enough to deal with high complexity and uncertainties in GDM with IVFPRs.

(2) Prior scholars ignored the consistency level of an IVFPR. In reality, some DMs can accept the acceptable consistent IVFPR since they have diverse risk attitudes. Incorporating DMs' risk attitude into the consistency level of IVFPR is reasonable and suitable for actual decision problems.

(3) The final solution should be agreed by all the DMs in GDM. Hence, the ranking order of alternatives should be generated by the collective priority weights. However, the previous research failed to consider the determination of the collective priority weights. It is necessary to introduce DMs' preference principles to obtain the collective priority weights from the individual priority weights.

In this paper, the geometric consistent index of an IVFPR is first defined considering DM's risk preference. Then a goal programming model is built to adjust an unacceptable geometric consistent IVFPR to derive an acceptable geometric consistent IVFPR. By constructing the membership functions of fuzzy geometric consistent conditions, a fuzzy logarithmic program is established to derive the interval priority weights from an IVFPR. In GDM problems, the similarities between DMs are defined according to the individual interval priority weight vectors. DMs' weights are determined by the confidence degrees of each DM. Subsequently, a parametric linear programming model is constructed to derive the collective interval priority weights which are used to rank alternatives. Finally, some examples are provided to verify the effectiveness and practicability of the proposed method. The main innovations of this work are highlighted at four aspects:

(1) Wang and Chen [11] defined the geometric consistency of IVFPR to check whether an IVFPR is geometric consistent or not. The consistency level of an inconsistent IVFPR cannot be determined by [11]. To measure the consistency level of an IVFPR, this paper defines the geometric consistent index based on the geometric consistent definition in [11]. A max-consistency index and a min-consistency index of the IVFPR are proposed based on the majority principle and minority principle, respectively. Combining the max-consistency index with min-consistency index, the geometric consistent index of an IVFPR is defined by considering DM's risk attitude adequately.
(2) To repair and improve the geometric consistency of an unacceptable geometric consistent IVFPR, a goal programming model is constructed to obtain acceptable geometric consistent one.
(3) The geometric consistent conditions about priority weights are treated as fuzzy constraints. By constructing the membership functions, a fuzzy logarithmic programming model is established to derive the interval priority weights from an IVFPR, which takes DM's satisfaction into account.
(4) In GDM problems, the similarities between DMs are calculated by the individual interval priority weights. Then the confidence degrees of DMs are defined to determine DMs' weights. Considering DMs' preference principles, a parametric linear programming model is established to derive the collective interval priority weights of alternatives.

The remainder of this paper unfolds as follows. In Section 2, some preliminaries are reviewed, including some basic definitions associated with intervals and IVFPRs. Section 3 defines the geometric consistent index of an IVFPR. Then a goal programming model is proposed to derive the acceptable geometric consistent IVFPR from an unacceptable geometric consistent one. In Section 4, a fuzzy logarithmic program is proposed to obtain the interval priority weights of an IVFPR. Section 5 constructs a parametric linear programming model to determine the collective priority weights by integrating all the individual interval priority weights. Then a new method is proposed to solve GDM with IVFPRs. In Section 6, some examples are analyzed to demonstrate the proposed method. Finally, Section 7 is concluding remarks.

2. Preliminaries

To render this paper self-contained, some preliminaries are presented in this section.

Definition 1. [25]. An IVFPR $\tilde{\boldsymbol{R}}$ on the alternative set $X=\{x_1,x_2,\dots,x_n\}$ is denoted by an interval valued fuzzy judgment matrix $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}\subset X\times X$, where interval $\tilde{r}_{ij}=[\underline{r}_{ij},\bar{r}_{ij}]$ indicates that the preference degree or intensity of alternative x_i over x_j is between $\underline{r}_{ij}$ and $\bar{r}_{ij}$. Furthermore, $\underline{r}_{ij}$ and $\bar{r}_{ij}$ fulfill the following conditions:

$$0<\underline{r}_{ij}\le\bar{r}_{ij}<1,\quad \underline{r}_{ij}+\bar{r}_{ji}=1,\quad \underline{r}_{ii}=\bar{r}_{ii}=0.5\quad \text{for all}\quad i,j=1,2,\dots,n.$$

Definition 2. [11]. An IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=[\underline{r}_{ij},\bar{r}_{ij}]$ is geometric consistent if it satisfies the geometric transitivity as:

$$\sqrt{\frac{\underline{r}_{ik}}{\bar{r}_{ki}}\frac{\bar{r}_{ik}}{\underline{r}_{ki}}}=\sqrt{\frac{\underline{r}_{ij}}{\bar{r}_{ji}}\frac{\bar{r}_{ij}}{\underline{r}_{ji}}}\sqrt{\frac{\underline{r}_{jk}}{\bar{r}_{kj}}\frac{\bar{r}_{jk}}{\underline{r}_{kj}}}\quad \text{for all}\quad i,j,k=1,2,\dots,n. \tag{1}$$

Let $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=[\underline{r}_{ij},\bar{r}_{ij}]$ be an IVFPR. According to Definition 2, $\tilde{\boldsymbol{R}}$ is geometric consistent iff one of the following conditions is satisfied.

$$\text{i)}\quad \underline{r}_{ij}\bar{r}_{ij}\underline{r}_{jk}\bar{r}_{jk}\bar{r}_{ki}\underline{r}_{ki}=\underline{r}_{ik}\bar{r}_{ik}\underline{r}_{kj}\bar{r}_{kj}\underline{r}_{ji}\bar{r}_{ji}\quad \text{for all}\quad i<j<k,\ i,j,k=1,2,\dots,n. \tag{2}$$

$$\begin{aligned}\text{ii)}\quad &\ln\underline{r}_{ij}+\ln\bar{r}_{ij}+\ln\underline{r}_{jk}+\ln\bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\\ &=\ln\underline{r}_{ik}+\ln\bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})\\ &\quad+\ln(1-\underline{r}_{ij})\quad (i<j<k,\ i,j,k=1,2,\dots,n)\end{aligned} \tag{3}$$

It can be concluded that Eqs. (1)–(3) are equivalent. That is to say, any one of Eqs. (1)–(3) can be applied to check the geometric consistency of an IVFPR. Since Eq. (3) is additive and only contains the upper triangular elements in an IVFPR, using Eq. (3) can simplify the calculation in the process of decision making.

A weight vector $\tilde{\boldsymbol{w}}=(\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_n)^{\mathrm{T}}$ is called a normalized interval weight vector if $\tilde{w}_i=[\underline{w}_i,\bar{w}_i]$, $0\le\underline{w}_i\le\bar{w}_i\le1$, $\underline{w}_i+\sum_{j=1,j\ne i}^{n}\bar{w}_j\ge1$ and $\bar{w}_i+\sum_{j=1,j\ne i}^{n}\underline{w}_j\le1$ for all $i=1,2,\dots,n$[7]. Denote a $2n$-dimensional simplex by

$$W=\left\{\tilde{\boldsymbol{w}}=(\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_n)^{\mathrm{T}}|\tilde{w}_i=[\underline{w}_i,\bar{w}_i]\in[0,1],\underline{w}_i+\sum_{j=1,j\ne i}^{n}\bar{w}_j\ge1,\bar{w}_i+\sum_{j=1,j\ne i}^{n}\underline{w}_j\le1,i=1,2,\dots,n\right\}.$$

Definition 3. [11]. An IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ is geometric consistent if there exists a normalized interval priority weight vector $\tilde{\boldsymbol{w}}=(\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_n)^{\mathrm{T}}$ such that

$$\tilde{r}_{ij}=\left[\underline{r}_{ij},\bar{r}_{ij}\right]=\begin{cases}[0.5,0.5], & \text{if } i=j\\ \left[\frac{\underline{w}_i}{\underline{w}_i+\alpha_{ij}\bar{w}_j},\frac{\alpha_{ij}\bar{w}_i}{\alpha_{ij}\bar{w}_i+\underline{w}_j}\right], & \text{if } i\ne j\end{cases} \tag{4}$$

where $\sqrt{\frac{\underline{w}_i\underline{w}_j}{\bar{w}_i\bar{w}_j}}\le\alpha_{ij}\le1$ and $\alpha_{ij}=\alpha_{ji}$ $(i,j=1,2,\dots,n;i\ne j)$.

Eq. (4) can be written as follows:

$$\begin{cases}\alpha_{ij}(1-\bar{r}_{ij})\bar{w}_i-\bar{r}_{ij}\underline{w}_j=0\ (i,j=1,2,\dots,n,i\ne j)\\ \alpha_{ij}\underline{r}_{ij}\bar{w}_j-(1-\underline{r}_{ij})\underline{w}_i=0\ (i,j=1,2,\dots,n,i\ne j)\end{cases} \tag{5}$$

According to Definition 1, Eq. (5) can be simplified as

$$\begin{cases}\ln\bar{w}_i-\ln\underline{w}_j+\ln\alpha_{ij}+\ln(1-\bar{r}_{ij})-\ln\bar{r}_{ij}=0\\ \quad(i,j=1,2,\dots,n,i<j)\\ \ln\bar{w}_j-\ln\underline{w}_i+\ln\alpha_{ij}+\ln\underline{r}_{ij}-\ln(1-\underline{r}_{ij})=0\\ \quad(i,j=1,2,\dots,n,i<j)\end{cases} \tag{6}$$

To rank intervals, Xu and Da [26] proposed the concept of the likelihood of intervals. For intervals $\tilde{r}_i=[\underline{r}_i,\bar{r}_i]$ $(i=1,2,\dots,n)$, the likelihood of intervals $\tilde{r}_i>\tilde{r}_h$ (i.e., interval $\tilde{r}_i$ is larger than interval $\tilde{r}_h$) is defined as

$$l(\tilde{r}_i>\tilde{r}_h)=\max\left\{1-\max\left\{\frac{\bar{r}_h-\underline{r}_i}{\bar{r}_h-\underline{r}_h+\bar{r}_i-\underline{r}_i},0\right\},0\right\}\quad \text{for all}\quad i,h=1,2,\dots,n. \tag{7}$$

Then the priority weight v_i of interval $\tilde{r}_i$ is calculated as follows [27]:

$$v_i=\frac{1}{n(n-1)}\left(\sum_{h=1}^{n}l(\tilde{r}_i>\tilde{r}_h)+\frac{n}{2}-1\right)\quad \text{for all}\quad i=1,2,\dots,n. \tag{8}$$

The bigger the value of priority weight v_i, the larger the interval $\tilde{r}_i$. Thus, the ranking order of intervals can be obtained according to the descending order of the priority weights v_i $(i=1,2,\dots,n)$.

3. Geometric consistency of interval-valued fuzzy preference relation

In this section, the geometric consistent index of an IVFPR is defined considering DM's risk attitude. Then a goal programming model is constructed to obtain the acceptable geometric consistent IVFPR from an unacceptable geometric consistent one.

3.1. Geometric consistent index of interval valued fuzzy preference relation

In some specified circumstances, it is difficult for DMs to provide completely geometric consistent IVFPRs. According to Definition 2, Wang and Chen [11] only proposed the concept of geometric consistent IVFPR, which can be applied to check whether an IVFPR is geometric consistent or not. The consistency level of an inconsistent IVFPR cannot be determined by Wang and Chen [11]. To measure the consistency level of an IVFPR, the consistency index of IVFPR is defined in the following.

Based on Eq. (3), the consistency level of IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ can be measured by the deviation between $\ln \underline{r}_{ij}+\ln \bar{r}_{ij}+\ln \underline{r}_{jk}+\ln \bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})$ and $\ln \underline{r}_{ik}+\ln \bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})$ for all $i<j<k,\ i,j,k=1,2,\ldots,n.$

Based on p-metric, the total deviation of IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ is determined by Minkowski distance as

$$D(\tilde{\boldsymbol{R}})=\Big\{\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}\Big|\big[\ln \underline{r}_{ij}+\ln \bar{r}_{ij}+\ln \underline{r}_{jk}+\ln \bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\big]-\big[\ln \underline{r}_{ik}+\ln \bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})\big]\Big|^{p}\Big\}^{1/p}$$

where parameter p ($p\geq 0$) reflects the importance assigned to the largest deviation. As p increases, more importance is assigned to the largest deviations. If $p=1$, the Minkowski distance for IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ is reduced to Hamming distance; If $p\to\infty$, the Minkowski distance for IVFPR $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{n\times n}$ is reduced to Chebyshev distance. Thus, a geometric consistent index for an IVFPR can be defined by these two special cases ($p=1$ and $p\to\infty$).

Definition 4. A max-consistency index measures the level of geometric consistency of an IVFPR $\tilde{\boldsymbol{R}}$ by Hamming distance, which is defined as

$$\mathrm{MCI}(\tilde{\boldsymbol{R}})=\frac{1}{C_n^3}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}\Big|\big[\ln \underline{r}_{ij}+\ln \bar{r}_{ij}+\ln \underline{r}_{jk}+\ln \bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\big]-\big[\ln \underline{r}_{ik}+\ln \bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})\big]\Big| \quad (9)$$

Definition 5. A min-consistency index measures the level of geometric consistency of an IVFPR $\tilde{\boldsymbol{R}}$ by Chebyshev distance, which is defined as

$$\mathrm{SCI}\big(\tilde{\boldsymbol{R}}\big)=\max_{1\leq i<j<k\leq n}\Big\{\Big|\big[\ln \underline{r}_{ij}+\ln \bar{r}_{ij}+\ln \underline{r}_{jk}+\ln \bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\big]-\big[\ln \underline{r}_{ik}+\ln \bar{r}_{ik}+\ln\big(1-\bar{r}_{jk}\big)+\ln\big(1-\underline{r}_{jk}\big)+\ln\big(1-\bar{r}_{ij}\big)+\ln\big(1-\underline{r}_{ij}\big)\big]\Big|\Big\} \quad (10)$$

In Eq. (9) the Hamming distance between both sides of Eq. (3) is employed to measure the consistency level of an IVFPR, which is based on the majority principle. It shows that such a DM is optimistic. Hence, Eq. (9) is called the max-consistency index. This case would lead to a more robust estimation. Different from Eq. (9), the Chebyshev distance between both sides of Eq. (3) is used to measure the consistency level of IVFPR, which is based on the minority principle. It implies that such a DM is pessimistic. Hence, Eq. (10) is called the min-consistency index. This case would result in a more sensitive estimation of extreme deviation.

Definition 6. Combining the max-consistency index with min-consistency index, the geometric consistent index of an IVFPR $\tilde{\boldsymbol{R}}$ is defined as

$$\mathrm{CI}(\tilde{\boldsymbol{R}})=\delta\mathrm{MCI}(\tilde{\boldsymbol{R}})+(1-\delta)\mathrm{SCI}(\tilde{\boldsymbol{R}}), \quad (11)$$

where parameter $\delta\in[0,1]$ reflects DM's risk attitude. If $\delta=1$, only the max-consistency index is considered, which shows that the DM is optimistic; if $\delta=0$, only the min-consistency index is considered, which indicates the DM is pessimistic; if $\delta=0.5$, the DM is neutral which means that the DM is indifferent to the risk. The larger the value of δ, the more optimistic the DM. The smaller of the value of δ, the more pessimistic the DM. By introducing parameter δ as a tradeoff between the max-consistency index and the min-consistency index, the geometric consistent index of Definition 6 is more flexible and appropriate for various DMs with different risk attitudes.

The geometric consistent index reflects the reliability of the information provided by DM. Clearly, the smaller the value of $\mathrm{CI}(\tilde{\boldsymbol{R}})$, the more consistent and reliable the information in IVFPR $\tilde{\boldsymbol{R}}$. Moreover, it is easy to verify that the geometric consistent index $\mathrm{CI}(\tilde{\boldsymbol{R}})$ satisfies $0\leq\mathrm{CI}(\tilde{\boldsymbol{R}})\leq 1$. If $\mathrm{CI}(\tilde{\boldsymbol{R}})=0$, then IVFPR $\tilde{\boldsymbol{R}}$ is completely geometric consistent.

Remark 1. Wu and Chiclana [9] defined a consistency index of an IVFPR by calculating the distance between the IVFPR and its corresponding multiplicative consistency based estimated IVFPR. In general, different methods would derive diverse multiplicative consistency based estimated IVFPRs, which would lead to various consistency indices for an IVFPR. Chen and Zhou [5] introduced the expected value fuzzy preference relations corresponding to an IVFPR. Then the consistency indicator of an IVFPR is defined by calculating the deviation between its expected value fuzzy preference relation and the corresponding consistent fuzzy preference relation. The consistency indicator relies on the original IVFPR and its corresponding consistent fuzzy preference relation. However, the consistency level of an IVFPR should not be changed with its corresponding consistent fuzzy preference relation. Moreover, Chen and Zhou [5] transformed an IVFPR into the corresponding fuzzy preference relation, which may result in information loss to some degree. In this paper, the geometric consistency index of an IVFPR defined in Definition 6 is stable and reliable since it only depends on the information of the original IVFPR. In addition, incorporating DM's risk attitude into the consistent index is more accordance with the real-world decision making.

Introducing a predefined consistency threshold $\overline{\mathrm{CI}}$, an acceptable geometric consistent IVFPR is defined as follows:

Definition 7. Let $\overline{\mathrm{CI}}$ be a predefined consistency threshold satisfying $\overline{\mathrm{CI}}\in[0,1]$. If $\mathrm{CI}(\tilde{\boldsymbol{R}})\leq\overline{\mathrm{CI}}$, then IVFPR $\tilde{\boldsymbol{R}}$ is acceptable geometric consistent. Otherwise, $\tilde{\boldsymbol{R}}$ is unacceptable geometric consistent. Especially, if $\mathrm{CI}(\tilde{\boldsymbol{R}})=0$, IVFPR $\tilde{\boldsymbol{R}}$ is completely geometric consistent.

In general, the consistency threshold $\overline{\mathrm{CI}}$ is predefined in the interval [0,1] artificially. Although the determination of consistency threshold is important to the decision making, there is no a unified approach for DMs to determining the proper consistency threshold. The consistency threshold can be determined by all DMs or a super DM according to the characteristic and need of real-world decision making problems. If the decision is significant, a more restrictive value should be adapted to the consistency threshold; if the decision is urgent and needs to select alternative(s) quickly, a less restrictive value should be allocated.

3.2. Derive the acceptable geometric consistent interval valued fuzzy preference relation

For an unacceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}}'$ with $\tilde{r}'_{ij} = [\underline{r}'_{ij}, \bar{r}'_{ij}]$, an important task is to find an acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}} = (\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij} = [\underline{r}_{ij}, \bar{r}_{ij}]$ which is close to $\tilde{\boldsymbol{R}}'$ as much as possible. To fulfill this task, we can minimize the deviation between the original IVFPR $\tilde{\boldsymbol{R}}'$ and the acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}}$. Therefore, a mathematical programming model is constructed as follows:

$$\min \sum_{i=1}^{n}\sum_{j=i+1}^{n}(|\underline{r}'_{ij}-\underline{r}_{ij}|+|\bar{r}'_{ij}-\bar{r}_{ij}|)$$
$$\begin{cases} \frac{\delta}{C_n^3}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}\left|\left[\ln\underline{r}_{ij}+\ln\bar{r}_{ij}+\ln\underline{r}_{jk}+\ln\bar{r}_{jk}\right.\right. \\ \quad \left.+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\right]-\left[\ln\underline{r}_{ik}+\ln\bar{r}_{ik}+\ln(1-\bar{r}_{jk})\right. \\ \quad \left.\left.+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})\right]\right| \\ \quad +(1-\delta)\max\limits_{1\le i<j<k\le n}\left\{\left|\left[\ln\underline{r}_{ij}+\ln\bar{r}_{ij}+\ln\underline{r}_{jk}+\ln\bar{r}_{jk}\right.\right.\right. \\ \quad \left.+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\right]-\left[\ln\underline{r}_{ik}+\ln\bar{r}_{ik}+\ln(1-\bar{r}_{jk})\right. \\ \quad \left.\left.\left.+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})\right]\right|\right\}\le\overline{\text{CI}} \\ 0<\underline{r}_{ij}\le\bar{r}_{ij}<1\ (i,j=1,2,\ldots,n,i<j) \end{cases} \tag{12}$$

In Eq. (12), the first condition ensures that the obtained matrix $\tilde{\boldsymbol{R}}$ is acceptable geometric consistent, and the second guarantees the upper triangular elements in $\tilde{\boldsymbol{R}}$ are intervals.

To solve Eq. (12), some parameters are introduced as

$$g_{ij}^- = \left(\underline{r}'_{ij}-\underline{r}_{ij}\right)\vee 0,\ h_{ij}^- = \left(\underline{r}_{ij}-\underline{r}'_{ij}\right)\vee 0,$$
$$g_{ij}^+ = \left(\bar{r}'_{ij}-\bar{r}_{ij}\right)\vee 0,\ h_{ij}^+ = \left(\bar{r}_{ij}-\bar{r}'_{ij}\right)\vee 0,$$
$$\varepsilon_{ijk}^- = f_{ijk}\vee 0,\ \varepsilon_{ijk}^+ = \left(-f_{ikj}\right)\vee 0,$$
$$\lambda = \max_{1\le i<j<k\le n}\left\{\varepsilon_{ijk}^-+\varepsilon_{ijk}^+\right\},$$

where $f_{ijk} = [\ln\underline{r}_{ij}+\ln\bar{r}_{ij}+\ln\underline{r}_{jk}+\ln\bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})]$ $-[\ln\underline{r}_{ik}+\ln\bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})+\ln(1-\underline{r}_{ij})]$ $(i,j,k=1,2,\ldots,n)$.

Then Eq. (12) can be converted into a goal programming model as follows:

$$\min \sum_{i=1}^{n}\sum_{j=i+1}^{n}(g_{ij}^-+h_{ij}^-+g_{ij}^++h_{ij}^+)$$
$$\begin{cases} \frac{\delta}{C_n^3}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}(\varepsilon_{ijk}^-+\varepsilon_{ijk}^+)+(1-\delta)\lambda\le\overline{\text{CI}} \\ \lambda\ge\varepsilon_{ijk}^-+\varepsilon_{ijk}^+\ (i,j,k=1,2,\ldots,n,\ i<j<k) \\ g_{ij}^--h_{ij}^-=\underline{r}'_{ij}-\underline{r}_{ij},\ g_{ij}^+-h_{ij}^+=\bar{r}'_{ij}-\bar{r}_{ij}\ (i,j=1,2,\ldots,n,\ i<j) \\ \left[\ln\underline{r}_{ij}+\ln\bar{r}_{ij}+\ln\underline{r}_{jk}+\ln\bar{r}_{jk}+\ln(1-\underline{r}_{ik})+\ln(1-\bar{r}_{ik})\right] \\ \quad -\left[\ln\underline{r}_{ik}+\ln\bar{r}_{ik}+\ln(1-\bar{r}_{jk})+\ln(1-\underline{r}_{jk})+\ln(1-\bar{r}_{ij})\right. \\ \quad \left.+\ln(1-\underline{r}_{ij})\right]=\varepsilon_{ijk}^--\varepsilon_{ijk}^+\ (i,j,k=1,2,\ldots,n,\ i<j<k) \\ 0\le\underline{r}_{ij}\le\bar{r}_{ij}\le 1\ (i,j=1,2,\ldots,n,i<j) \end{cases} \tag{13}$$

Solving Eq. (13) yields the optimal solutions $\underline{r}_{ij}$ and $\bar{r}_{ij}$ for all $i,j=1,2,\ldots,n$ and $i<j$. According to Definition 1, the acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}} = (\tilde{r}_{ij})_{n\times n}$ derived from the original IVFPR $\tilde{\boldsymbol{R}}'$ can be generated by

$$\tilde{r}_{ij} = \begin{cases} \left[\underline{r}_{ij},\bar{r}_{ij}\right], & \text{if } i<j \\ [0.5,0.5], & \text{if } i=j \\ \left[1-\bar{r}_{ij},1-\underline{r}_{ij}\right], & \text{if } i>j \end{cases} \tag{14}$$

4. Derive the interval priority weights from an IVFPR

In this section, a fuzzy logarithmic program is proposed to derive the interval priority weights of IVFPR. Then a new method is developed for individual decision making with IVFPR.

4.1. A fuzzy logarithmic program to derive the interval priority weights of IVFPR

Motivated by the idea of fuzzy programming method (FPM) [28], in the following we develop a new fuzzy logarithmic program to derive the interval priority weights from an IVFPR.

As per Definition 3, if an IVFPR $\tilde{\boldsymbol{R}} = (\tilde{r}_{ij})_{n\times n}$ is not geometric consistent, there is not a normalized interval priority weight vector $\tilde{\boldsymbol{w}}$ that satisfies Eq. (6) simultaneously. Therefore, a good enough solution is to find an interval priority weight vector that satisfies Eq. (6) as well as possible. Due to the fuzziness and uncertainty of human thinking and recognition, Eq. (6) may be hold in the sense of approximation. It means that a good enough solution has to satisfy all judgments approximately, i.e.,

$$\begin{cases} \ln\overline{w}_i-\ln\underline{w}_j+\ln\alpha_{ij}+\ln\left(1-\bar{r}_{ij}\right)-\ln\bar{r}_{ij}\cong 0 \\ \quad (i,j=1,2,\ldots,n,i<j) \\ \ln\overline{w}_j-\ln\underline{w}_i+\ln\alpha_{ij}+\ln\underline{r}_{ij}-\ln\left(1-\underline{r}_{ij}\right)\cong 0 \\ \quad (i,j=1,2,\ldots,n,i<j) \end{cases} \tag{15}$$

where symbol $\cong$ denotes the statement "fuzzy equal to".

Let

$$d_{ij}^+(\tilde{\boldsymbol{w}}) = \ln\bar{w}_i-\ln\underline{w}_j+\ln\alpha_{ij}+\ln(1-\bar{r}_{ij})-\ln\bar{r}_{ij},\quad (i,j=1,2,\ldots,n,i<j) \tag{16}$$

and

$$d_{ij}^-(\tilde{\boldsymbol{w}}) = \ln\bar{w}_j-\ln\underline{w}_i+\ln\alpha_{ij}+\ln\underline{r}_{ij}-\ln(1-\underline{r}_{ij}),\quad (i,j=1,2,\ldots,n,i<j). \tag{17}$$

To further simplify the notations, the parameters $d_{ij}^+(\tilde{\boldsymbol{w}})$ and $d_{ij}^-(\tilde{\boldsymbol{w}})$ are unified into $d_{ij}^c(\tilde{\boldsymbol{w}})$ $(c=+,-)$. Thus Eq. (15) can be rewritten as the fuzzy constraints as

$$d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0\quad (i,j=1,2,\ldots,n;\ i<j;\ c=+,-), \tag{18}$$

The fuzzy constraint $d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0$ can be represented by a fuzzy set on the universe W as follows:

$$U_{ij}^c = \left\{\left(\tilde{\boldsymbol{w}},\mu_{ij}^c(\tilde{\boldsymbol{w}})\right)\middle|\tilde{\boldsymbol{w}}\in W\right\},$$

where the membership function $\mu_{ij}^c(\tilde{\boldsymbol{w}})\in[0,1]$ can be appropriately constructed. Especially, if $\mu_{ij}^c(\tilde{\boldsymbol{w}})=1$, then fuzzy constraint $d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0$ reduces to the traditional crisp constraint $d_{ij}^c(\tilde{\boldsymbol{w}})=0$.

For fuzzy constraint $d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0$, its membership function $\mu_{ij}^c(\tilde{\boldsymbol{w}})$ is constructed as

$$\mu_{ij}^c(\tilde{\boldsymbol{w}}) = \begin{cases} 1, & \text{if } d_{ij}^c(\tilde{\boldsymbol{w}})=0 \\ 1-\frac{d_{ij}^c(\tilde{\boldsymbol{w}})}{\theta_{ij}^c}, & \text{if } 0<d_{ij}^c(\tilde{\boldsymbol{w}})\le\theta_{ij}^c \\ 1+\frac{d_{ij}^c(\tilde{\boldsymbol{w}})}{\theta_{ij}^c}, & \text{if } -\theta_{ij}^c\le d_{ij}^c(\tilde{\boldsymbol{w}})<0 \\ 0, & \text{else} \end{cases} \tag{19}$$

where the tolerance parameter $\theta_{ij}^c>0$ $(i,j=1,2,\ldots,n;\ j>i;\ c=+,-)$.

The membership function $\mu_{ij}^c(\tilde{\boldsymbol{w}})$ represents the satisfaction degree of fuzzy constraint $d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0$ and is depicted in Fig. 1.

It is clear that $\mu_{ij}^c(\tilde{\boldsymbol{w}})\in[0,1]$. For the fuzzy constraint $d_{ij}^c(\tilde{\boldsymbol{w}})\cong 0$, if $d_{ij}^c(\tilde{\boldsymbol{w}})=0$, then $\mu_{ij}^c(\tilde{\boldsymbol{w}})=1$ which means "complete satisfaction"; if $d_{ij}^c(\tilde{\boldsymbol{w}})\ge\theta_{ij}^c$ or $d_{ij}^c(\tilde{\boldsymbol{w}})\le-\theta_{ij}^c$, then $\mu_{ij}^c(\tilde{\boldsymbol{w}})=0$ which

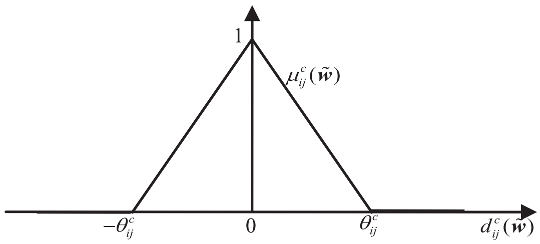

Fig. 1. Membership function of $d_{ij}^{c}(\tilde{\boldsymbol{w}}) \cong 0$.

means "complete dissatisfaction"; if $-\theta_{ij}^{l} < d_{ij}^{c}(\tilde{\boldsymbol{w}}) < 0$ or $0 < d_{ij}^{c}(\tilde{\boldsymbol{w}}) < \theta_{ij}^{c}$, then $\mu_{ij}^{c}(\tilde{\boldsymbol{w}}) \in (0,1)$ which means "approximate satisfaction".

According to Bellman and Zadeh's extension principle [29], a fuzzy decision S can be viewed as a fuzzy set $S = \{(\tilde{\boldsymbol{w}}, \mu_S(\tilde{\boldsymbol{w}})) | \tilde{\boldsymbol{w}} \in W\}$, where

$$\mu_S(\tilde{\boldsymbol{w}}) = \min\left\{\mu_{ij}^{c}(\tilde{\boldsymbol{w}}) \middle| i,j=1,2,\ldots,n;\ j>i;\ c=+,-\right\}.$$

Thus, Eq. (18) can be transformed into the following system of crisp inequalities:

$$\begin{cases} \mu_{ij}^{c}(\tilde{\boldsymbol{w}}) \ge \beta \ (i,j=1,2,\ldots,n;\ j>i;\ c=+,-) \\ \sqrt{\frac{\underline{w}_i \underline{w}_j}{\bar{w}_i \bar{w}_j}} \le \alpha_{ij} \le 1 \ (i,j=1,2,\ldots,n;\ j>i) \\ 0 \le \beta \le 1 \end{cases}$$

where β denotes the minimal satisfaction degree of fuzzy constraints.

To obtain the optimal interval priority weight vector $\tilde{\boldsymbol{w}}$, a single objective fuzzy logarithmic programming model is constructed by maximizing the minimal satisfaction degree β as follows:

$$\max\ \beta$$
$$s.t.\begin{cases} \mu_{ij}^{c}(\tilde{\boldsymbol{w}}) \ge \beta \ (i,j=1,2,\ldots,n;\ j>i;\ c=+,-) \\ \ln \underline{w}_i + \ln \underline{w}_j - \ln \bar{w}_i - \ln \bar{w}_j \\ \quad \le 2\ln\alpha_{ij} \le 0 \ (i,j=1,2,\ldots,n;\ j>i) \\ 0 \le \beta \le 1,\ \tilde{\boldsymbol{w}} \in W \end{cases} \tag{20}$$

In Eq. (20), the second line constraints are the logarithmic forms to the constraints $\sqrt{\frac{\underline{w}_i \underline{w}_j}{\bar{w}_i \bar{w}_j}} \le \alpha_{ij} \le 1 \ (i,j=1,2,\ldots,n;\ i \ne j)$ in Definition 3.

Set $\alpha'_{ij} = \ln\alpha_{ij}$. Plugging Eqs. (16), (17) and (19) into Eq. (20), a fuzzy logarithmic program is derived as

$$\max\ \beta$$
$$s.t.\begin{cases} (\beta-1)\theta_{ij}^{+} \le \ln\bar{w}_i - \ln\underline{w}_j + \ln(1-\bar{r}_{ij}) - \ln\bar{r}_{ij} + \alpha'_{ij} \\ \quad \le (1-\beta)\theta_{ij}^{+} \ \ (i,j=1,2,\ldots,n;\ j>i) \\ (\beta-1)\theta_{ij}^{-} \le \ln\bar{w}_j - \ln\underline{w}_i + \ln\underline{r}_{ij} - \ln(1-\underline{r}_{ij}) + \alpha'_{ij} \\ \quad \le (1-\beta)\theta_{ij}^{-} \ (i,j=1,2,\ldots,n;\ j>i) \\ \ln\underline{w}_i + \ln\underline{w}_j - \ln\bar{w}_i - \ln\bar{w}_j \le 2\alpha'_{ij} \le 0 \ (i,j=1,2,\ldots,n;\ j>i) \\ \underline{w}_i + \sum_{j=1,j\ne i}^{n}\bar{w}_j \ge 1,\ \bar{w}_i + \sum_{j=1,j\ne i}^{n}\underline{w}_j \le 1 \ (i=1,2,\ldots,n) \\ 0 \le \beta \le 1 \end{cases} \tag{21}$$

By solving the above fuzzy logarithmic program (Eq. (21)), the optimal solution $(\tilde{\boldsymbol{w}}^{*}, \beta^{*}, {\alpha'}_{ij}^{*})$ can be obtained, where $\tilde{\boldsymbol{w}}^{*}$ is the interval priority weight vector and β^{*} is the maximum acceptance degree of fuzzy constraints. If $\beta^{*}=1$, then IVFPR $\tilde{\boldsymbol{R}} = (\tilde{r}_{ij})_{n\times n}$ is completely geometric consistent. In general, the tolerance parameters θ_{ij}^{c} $(i,j=1,2,\ldots,n;\ j>i;\ c=+,-)$ should be chosen enough large that makes the optimal solution non-empty.

4.2. A new method for individual decision making with IVFPR

For a set of alternatives $X=\{x_1,x_2,\ldots,x_n\}$, a DM provides an IVFPR $\tilde{\boldsymbol{R}}' = (\tilde{r}'_{ij})_{n\times n}$ where $\tilde{r}'_{ij} = [\underline{r}'_{ij}, \bar{r}'_{ij}]$ is the preference degree of alternative x_i over x_j. Based on the above analysis, a new method for individual decision making with IVFPR is put forward as follows:

Step 1. Set the values of parameter δ and the consistency threshold $\overline{\mathrm{CI}}$ a priori.

Step 2. By Eq. (11), calculate the geometric consistent index $\mathrm{CI}(\tilde{\boldsymbol{R}}')$ of IVFPR $\tilde{\boldsymbol{R}}'$. If $\mathrm{CI}(\tilde{\boldsymbol{R}}') \le \overline{\mathrm{CI}}$, then IVFPR $\tilde{\boldsymbol{R}}'$ is acceptable geometric consistent, thus let $\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{R}}'$ and go to Step 4. Otherwise, IVFPR $\tilde{\boldsymbol{R}}'$ is unacceptable geometric consistent and go to the next step.

Step 3. Solving Eq. (13), the optimal solutions $\underline{r}_{ij}$ and $\bar{r}_{ij}$ for all $i,j=1,2,\ldots,n$ and $i<j$ are derived. Then the acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}}$ is generated by Eq. (14).

Step 4. According to Eq. (21), determine the interval priority weights $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ $(i=1,2,\ldots,n)$ from the acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}}$.

Step 5. Based on Eq. (7), the likelihood matrix $L=(l_{ih})_{n\times n}$ is established where

$$l_{ih} = l(\tilde{w}_i > \tilde{w}_h) = \max\left\{1-\max\left\{\frac{\bar{w}_h - \underline{w}_i}{\bar{w}_h - \underline{w}_h + \bar{w}_i - \underline{w}_i}, 0\right\}, 0\right\}$$
$$(i,h=1,2,\ldots,n). \tag{22}$$

Then the priority weights v_i $(i=1,2,\ldots,n)$ are obtained by Eq. (8). Thus, the ranking order of alternatives can be generated by descending the priority weights v_i $(i=1,2,\ldots,n)$.

5. Method for GDM with IVFPRs

In the section, the GDM problems with IVFPRs are described. DMs' weights are determined by the similarity between DMs. Then a parametric linear programming model is constructed to obtain the collective interval priority weights. A new method is proposed to solve GDM with IVFPRs.

5.1. Problem description for GDM with IVFPRs

The following assumptions or notations are used to represent the GDM problem with IVFPRs:

(1) The alternatives are known. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a discrete set of alternatives.
(2) DMs are known. Let $E=\{e_1,e_2,\ldots,e_m\}$ be a set of DMs.
(3) The information about DMs' weights is unknown. Let $\omega = (\omega_1,\omega_2,\ldots,\omega_m)^{\mathrm{T}}$ be DMs' weight vector where $\omega_t \ge 0$ and $\sum_{t=1}^{m}\omega_t = 1$.
(4) Assume that $\tilde{\boldsymbol{R}}'_t = (\tilde{r}'_{ijt})_{n\times n}$ $(t=1,2,\ldots,m)$ is the individual IVFPR provided by DM e_t, where $\tilde{r}_{ijt} = [\underline{r}_{ijt}, \bar{r}_{ijt}]$ is the preference degree of alternative x_i over x_j given by DM e_t.

For individual IVFPR $\tilde{\boldsymbol{R}}'_t = (\tilde{r}'_{ijt})_{n\times n}$ $(t=1,2,\ldots,m)$, the individual interval priority weight vector $\tilde{\boldsymbol{w}}_t = (\tilde{w}_{1t}, \tilde{w}_{2t}, \ldots, \tilde{w}_{nt})^{\mathrm{T}}$ $(t=1,2,\ldots,m)$ of DM e_t can be derived by the individual decision making method in Section 4.2. Denote the collective interval priority weight vector by $\tilde{\boldsymbol{w}} = (\tilde{w}_1,\tilde{w}_2,\ldots,\tilde{w}_n)^{\mathrm{T}}$ with $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ $(i=1,2,\ldots,n)$. In GDM, the final solution should be agreed by all the DMs. Thus, the individual interval priority weight vectors should be integrated into the collective interval priority weight vector which is used to derive the collective ranking of alternatives.

5.2. Determining the collective interval priority weights in GDM

The similarity between DMs e_t and e_l $(t, l = 1, 2, \ldots, m)$ is measured by the corresponding individual interval priority weight vectors. It is defined as

$$S_{tl} = 1 - \frac{1}{2n}\sum_{i=1}^{n}(|\underline{w}_{it} - \underline{w}_{il}| + |\bar{w}_{it} - \bar{w}_{il}|) \quad (t, l = 1, 2, \ldots, m) \tag{23}$$

where $\tilde{w}_{it} = [\underline{w}_{it}, \bar{w}_{it}]$ and $\tilde{w}_{il} = [\underline{w}_{il}, \bar{w}_{il}]$ $(i = 1, 2, \ldots, n,\ t, l = 1, 2, \ldots, m)$ are the interval priority weight of alternative x_i for DM e_t and e_l, respectively.

The larger the value of S_{tl}, the higher the similarity between DMs e_t and e_l. It is easy to verify that the similarity S_{tl} satisfies some properties: i) $0 \le S_{tl} \le 1$; ii) $S_{tt} = 1$; iii) $S_{tl} = S_{lt}$. If two DMs have the same opinions, then the similarity degree between them is equal to 1.

Thus, the confidence degree CS_t of DM e_t is calculated as

$$CS_t = \sum_{l=1, l\ne t}^{m} S_{tl} \quad (t = 1, 2, \ldots, m). \tag{24}$$

The confidence degree reflects the support degree of a DM from all the others. The higher the confidence degree of a DM, the larger the support degree of a DM from all the others. Therefore, a DM with higher confidence degree should be assigned a larger weight; a DM with lower confidence degree should be allocated a smaller weight. In other words, the higher the confidence degree of a DM, the larger the DM's weight. Thus normalizing the confidence degrees CS_t $(t = 1, 2, \ldots, m)$, DMs' weights are obtained as

$$\omega_t = CS_t / \sum_{t=1}^{m} CS_t \quad (t = 1, 2, \ldots, m). \tag{25}$$

To derive the collective interval priority weights $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ $(i = 1, 2, \ldots, n)$, a reasonable method is to minimize all the weighted deviations between $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ and $\tilde{w}_{it} = [\underline{w}_{it}, \bar{w}_{it}]$ $(i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m)$. The weighted deviations between $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ and $\tilde{w}_{it} = [\underline{w}_{it}, \bar{w}_{it}]$ can be introduced as $\omega_t|\underline{w}_i - \underline{w}_{it}|$ and $\omega_t|\bar{w}_i - \bar{w}_{it}|$, respectively.

Inspired by the mathematical programming model with p-metric [30,31], a minimization deviation model is constructed as follows:

$$\begin{aligned} &\min\ T = \sum_{i=1}^{n}\sum_{t=1}^{m}\left[(\omega_t|\underline{w}_i - \underline{w}_{it}|)^p + (\omega_t|\bar{w}_i - \bar{w}_{it}|)^p\right]^{1/p} \\ &s.t. \tilde{\boldsymbol{w}} \in W \end{aligned} \tag{26}$$

where ω_t is the weight of DM e_t calculated by Eq. (25) and parameter p reflects the importance assigned to the largest deviation. As p increases, more importance is assigned to the largest deviations [31].

Particularly, if $p = 1$, Eq. (26) is converted into

$$\begin{aligned} &\min\ T = \sum_{i=1}^{n}\sum_{t=1}^{m}\omega_t(|\underline{w}_i - \underline{w}_{it}| + |\bar{w}_i - \bar{w}_{it}|) \\ &s.t. \tilde{\boldsymbol{w}} \in W \end{aligned} \tag{27}$$

If $p \to +\infty$, Eq. (26) is rewritten as

$$\begin{aligned} &\min\ \max_{i,t}\{\omega_t|\underline{w}_i - \underline{w}_{it}|, \omega_t|\bar{w}_i - \bar{w}_{it}|\} \\ &s.t. \tilde{\boldsymbol{w}} \in W \end{aligned} \tag{28}$$

The objective function in Eq. (27) is to minimize the sum of all weighted deviations $\omega_t|\underline{w}_i - \underline{w}_{it}|$ and $\omega_t|\bar{w}_i - \bar{w}_{it}|$ $(i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m)$, which is based on the majority principle similar to the model in Wang and Li [7]. This case would lead to a more robust estimation. Different from Eq. (27), the objective function in Eq. (28) is to minimize the maximum weighted deviation between $\omega_t|\underline{w}_i - \underline{w}_{it}|$ and $\omega_t|\bar{w}_i - \bar{w}_{it}|$ $(i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m)$, which is based on the minority principle. This case would result in a more sensitive estimation of extreme deviation [30].

Combining Eqs. (27) and (28), a parametric mathematical program is constructed by introducing parameter ψ as follows:

$$\begin{aligned} &\min\ T = \psi\sum_{i=1}^{n}\sum_{t=1}^{m}\omega_t(|\underline{w}_i - \underline{w}_{it}| + |\bar{w}_i - \bar{w}_{it}|) \\ &\qquad + (1-\psi)\max_{i,t}\{\omega_t|\underline{w}_i - \underline{w}_{it}|, \omega_t|\bar{w}_i - \bar{w}_{it}|\} \\ &s.t. \tilde{\boldsymbol{w}} \in W \end{aligned} \tag{29}$$

where ψ is a control parameter that is a trade-off between majority and minority principles satisfying $0 \le \psi \le 1$. If $\psi = 1$, Eq. (29) can be reduced to the model (27), which only considers the majority principle; if $\psi = 0.5$, Eq. (29) considers both the majority principle and the minority principle; if $\psi = 0$, Eq. (29) can be reduced to the model (28), which only considers the minority principle. Therefore, Eq. (29) can be regarded as an integrated model which includes the above two particular cases (Eqs. (27) and (28)) by parameter ψ.

To solve Eq. (29), let

$$\begin{aligned} u &= \max_{i,t}\{\omega_t|\underline{w}_i - \underline{w}_{it}|, \omega_t|\bar{w}_i - \bar{w}_{it}|\}, \\ \underline{d}_{it}^{+} &= \tfrac{1}{2}\omega_t(|\underline{w}_i - \underline{w}_{it}| + (\underline{w}_i - \underline{w}_{it})), \\ \underline{d}_{it}^{-} &= \tfrac{1}{2}\omega_t(|\underline{w}_i - \underline{w}_{it}| - (\underline{w}_i - \underline{w}_{it})), \\ \bar{d}_{it}^{+} &= \tfrac{1}{2}\omega_t(|\bar{w}_i - \bar{w}_{it}| + (|\bar{w}_i - \bar{w}_{it}|)), \\ \bar{d}_{it}^{-} &= \tfrac{1}{2}\omega_t(|\bar{w}_i - \bar{w}_{it}| - (\bar{w}_i - \bar{w}_{it})). \end{aligned}$$

Thus, Eq. (29) is transformed into a parametric linear programming model as follows:

$$\begin{aligned} &\min\ T = \psi\sum_{i=1}^{n}\sum_{t=1}^{m}(\underline{d}_{it}^{+} + \underline{d}_{it}^{-} + \bar{d}_{it}^{+} + \bar{d}_{it}^{-}) + (1-\psi)u \\ &s.t. \begin{cases} \omega_t(\underline{w}_i - \underline{w}_{it}) - \underline{d}_{it}^{+} + \underline{d}_{it}^{-} = 0\ (i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m) \\ \omega_t(\bar{w}_i - \bar{w}_{it}) - \bar{d}_{it}^{+} + \bar{d}_{it}^{-} = 0\ (i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m) \\ \underline{d}_{it}^{+} + \underline{d}_{it}^{-} \le u,\ \bar{d}_{it}^{+} + \bar{d}_{it}^{-} \le u\ (i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, m) \\ \underline{w}_i + \sum_{j=1, j\ne i}^{n}\bar{w}_j \ge 1,\ \bar{w}_i + \sum_{j=1, j\ne i}^{n}\underline{w}_j \le 1 \quad (i = 1, 2, \ldots, n) \\ 0 \le \underline{w}_i \le \bar{w}_i \le 1\ \ (i = 1, 2, \ldots, n) \end{cases} \end{aligned} \tag{30}$$

Solving Eq. (30), the collective interval priority weights $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ $(i = 1, 2, \ldots, n)$ can be obtained. Then the collective ranking order of alternatives is generated by the collective interval priority weights $\tilde{w}_i$ $(i = 1, 2, \ldots, n)$.

5.3. Method for GDM with IVFPRs

Summarizing the aforesaid analyzes, a new method for GDM with IVIFPRs is developed below.

Step 1. Set the values of parameter δ and the predefined consistency threshold $\overline{\mathrm{CI}}$. Suppose that $\boldsymbol{T}$ is an empty set.

Step 2. By Eq. (11), calculate the geometric consistent index $\mathrm{CI}(\tilde{\boldsymbol{R}}'_t)$ for individual IVFPR $\tilde{\boldsymbol{R}}'_t$ $(t = 1, 2, \ldots, m)$. When $\mathrm{CI}(\tilde{\boldsymbol{R}}'_t) \le \overline{\mathrm{CI}}$, IVFPR $\tilde{\boldsymbol{R}}'_t$ is acceptable geometric consistent, thus let $\tilde{\boldsymbol{R}}_t = \tilde{\boldsymbol{R}}'_t$. When $\mathrm{CI}(\tilde{\boldsymbol{R}}'_t) > \overline{\mathrm{CI}}$, IVFPR $\tilde{\boldsymbol{R}}'_t$ is unacceptable geometric consistent and add $\tilde{\boldsymbol{R}}'_t$ to the set $\boldsymbol{T}$. If all IVFPRs $\tilde{\boldsymbol{R}}'_t$ $(t = 1, 2, \ldots, m)$ are geometric consistent, i.e., $\boldsymbol{T}$ is an empty set, then go to Step 4; otherwise, go to the next step.

Step 3. Solving Eq. (13), the optimal solutions $\underline{r}_{ijt}$ and $\bar{r}_{ijt}$ $(i, j = 1, 2, \ldots, n,\ i < j)$ are derived from unacceptable geometric consistent IVFPRs $\tilde{\boldsymbol{R}}'_t$, $t \in \{l | \tilde{\boldsymbol{R}}'_l \in T\}$. By Eq. (14), the corresponding acceptable geometric consistent IVFPRs $\tilde{\boldsymbol{R}}_t$ $(t \in \{l | \tilde{\boldsymbol{R}}'_l \in T\})$ are generated.

Step 4. According to Eq. (21), determine the individual interval priority weight vectors $\tilde{\boldsymbol{w}}_t = (\tilde{w}_{1t}, \tilde{w}_{2t}, \ldots, \tilde{w}_{nt})^{\mathrm{T}}$ for DMs e_t $(t = 1, 2, \ldots, m)$.

Step 5. The similarities S_{tl} between DMs e_t and e_l $(t, l = 1, 2, \ldots, m)$ are computed by Eq. (23).

Step 6. The confidence degrees CS_t of DMs e_t $(t = 1, 2, \ldots, m)$ are calculated via Eq. (24).

Step 7. Using Eq. (25), DMs' weights ω_t $(t = 1, 2, \ldots, m)$ are determined.

Step 8. According to Eq. (30), the collective interval priority weights $\tilde{w}_i = [\underline{w}_i, \overline{w}_i]$ $(i = 1, 2, \ldots, n)$ are obtained.

Step 9. Based on Eq. (7), the likelihood matrix $\boldsymbol{L} = (l_{ih})_{n\times n}$ is constructed where

$$l_{ih} = l(\tilde{w}_i > \tilde{w}_h) = \max\left\{1 - \max\left\{\frac{\bar{w}_h - \underline{w}_i}{\bar{w}_h - \underline{w}_h + \bar{w}_i - \underline{w}_i}, 0\right\}, 0\right\}. \tag{31}$$

Utilizing Eq. (8), the priority weights v_i of interval priority weights $\tilde{w}_i$ $(i = 1, 2, \ldots, n)$ are obtained. Thus, the collective ranking order of alternatives can be generated by comparing the priority weights v_i $(i = 1, 2, \ldots, n)$.

The above process for GDM with IVFPRs is depicted in Fig. 2.

Remark 2. The above method is proposed for solving GDM with IVFPRs. When solving a decision making problem with a single DM, Steps 5–8 that are used to derive the collective priority weights could be omitted. So the above method is exactly reduced to the method for individual decision making with IVFPR proposed in Section 4.2. Thus, the developed method can not only be applied to solve GDM with IVFPRs, but also be used to solve the individual decision making with IVFPR.

6. Application examples and comparative analyzes

In this section, some examples are applied to individual decision making with IVFPR and GDM with IVFPRs, respectively. Comparative analyzes with existing methods [4,11,21] are also performed.

6.1. Application to individual decision making with IVFPR

Firstly, two examples for individual decision making with IVFPR are given to interpret the advantages of the method for individual decision making with IVFPR proposed in Section 4.2.

Example 1. This example is taken from Genc et al. [4]. Suppose that a DM provides his/her preference information over a collection of alternatives x_i $(i = 1, 2, \ldots, 5)$ with the following IVFPR:

$$\tilde{\boldsymbol{R}}' = \begin{pmatrix} [0.5, 0.5] & [0.5, 0.6] & [0.2, 0.5] & [0.3, 0.7] & [0.3, 0.5] \\ [0.4, 0.5] & [0.5, 0.5] & [0.3, 0.6] & [0.6, 0.8] & [0.2, 0.4] \\ [0.5, 0.8] & [0.4, 0.7] & [0.5, 0.5] & [0.7, 0.8] & [0.4, 0.5] \\ [0.3, 0.7] & [0.2, 0.4] & [0.2, 0.3] & [0.5, 0.5] & [0.1, 0.4] \\ [0.5, 0.7] & [0.6, 0.8] & [0.5, 0.6] & [0.6, 0.9] & [0.5, 0.5] \end{pmatrix}$$

Considering that there is a single DM, the method for individual decision making with IVFPR proposed in Section 4.2 is applied to solve this example.

Step 1. Set $\delta = 0.5$ and $\overline{\mathrm{CI}} = 0.1$.

Step 2. By Eq. (11), calculate the geometric consistent index $\mathrm{CI}(\tilde{\boldsymbol{R}}') = 1.6895$. Since $\mathrm{CI}(\tilde{\boldsymbol{R}}') > \overline{\mathrm{CI}}$, IVFPR $\tilde{\boldsymbol{R}}'$ is unacceptable geometric consistent and go to step 3.

Step 3. Using Eq. (13), a goal programming model is constructed as follows:

$$\begin{array}{l}
\min\,(g^-_{12} + h^-_{12} + g^+_{12} + h^+_{12} + g^-_{13} + h^-_{13} + g^+_{13} + h^+_{13} + g^-_{14} + h^-_{14} + g^+_{14} + h^+_{14} + g^-_{15} + h^-_{15} + g^+_{15} + h^+_{15} + g^-_{23} + h^-_{23} + g^+_{23} + h^+_{23} + g^-_{24} + h^-_{24} + g^+_{24} \\
\quad + h^+_{24} + g^-_{25} + h^-_{25} + g^+_{25} + h^+_{25} + g^-_{34} + h^-_{34} + g^+_{34} + h^+_{34} + g^-_{35} + h^-_{35} + g^+_{35} + h^+_{35} + g^-_{45} + h^-_{45} + g^+_{45} + h^+_{45})
\end{array}$$

$$\left\{\begin{array}{l}
\frac{\delta}{10}(\varepsilon^-_{123} + \varepsilon^+_{123} + \varepsilon^-_{124} + \varepsilon^+_{124} + \varepsilon^-_{125} + \varepsilon^+_{125} + \varepsilon^-_{134} + \varepsilon^+_{134} + \varepsilon^-_{135} + \varepsilon^+_{135} + \varepsilon^-_{145} + \varepsilon^+_{145} + \varepsilon^-_{234} + \varepsilon^+_{234} + \varepsilon^-_{235} + \varepsilon^+_{235} + \varepsilon^-_{245} + \varepsilon^+_{245} + \\
\quad \varepsilon^-_{345} + \varepsilon^+_{345}) + (1-\delta)\lambda \le \overline{\mathrm{CI}} \\
\lambda \ge \varepsilon^-_{123} + \varepsilon^+_{123}, \lambda \ge \varepsilon^-_{124} + \varepsilon^+_{124}, \lambda \ge \varepsilon^-_{125} + \varepsilon^+_{125}, \lambda \ge \varepsilon^-_{134} + \varepsilon^+_{134}, \lambda \ge \varepsilon^-_{135} + \varepsilon^+_{135}, \\
\lambda \ge \varepsilon^-_{145} + \varepsilon^+_{145}, \lambda \ge \varepsilon^-_{234} + \varepsilon^+_{234}, \lambda \ge \varepsilon^-_{235} + \varepsilon^+_{235}, \lambda \ge \varepsilon^-_{245} + \varepsilon^+_{245}, \lambda \ge \varepsilon^-_{345} + \varepsilon^+_{345}, \\
g^-_{12} - h^-_{12} = 0.5 - \underline{r}_{12}, g^-_{13} - h^-_{13} = 0.2 - \underline{r}_{13}, g^-_{14} - h^-_{14} = 0.3 - \underline{r}_{14}, g^-_{15} - h^-_{15} = 0.3 - \underline{r}_{15}, g^-_{23} - h^-_{23} = 0.3 - \underline{r}_{23}, \\
g^-_{24} - h^-_{24} = 0.6 - \underline{r}_{24}, g^-_{25} - h^-_{25} = 0.2 - \underline{r}_{25}, g^-_{34} - h^-_{34} = 0.7 - \underline{r}_{34}, g^-_{35} - h^-_{35} = 0.4 - \underline{r}_{35}, g^-_{45} - h^-_{45} = 0.1 - \underline{r}_{45}, \\
g^+_{12} - h^+_{12} = 0.6 - \bar{r}_{12}, g^+_{13} - h^+_{13} = 0.5 - \bar{r}_{13}, g^+_{14} - h^+_{14} = 0.7 - \bar{r}_{14}, g^+_{15} - h^+_{15} = 0.5 - \bar{r}_{15}, g^+_{23} - h^+_{23} = 0.6 - \bar{r}_{23}, \\
g^+_{24} - h^+_{24} = 0.8 - \bar{r}_{24}, g^+_{25} - h^+_{25} = 0.4 - \bar{r}_{25}, g^+_{34} - h^+_{34} = 0.8 - \bar{r}_{34}, g^+_{35} - h^+_{35} = 0.5 - \bar{r}_{35}, g^+_{45} - h^+_{45} = 0.4 - \bar{r}_{45}, \\
[\ln \underline{r}_{12} + \ln \bar{r}_{12} + \ln \underline{r}_{23} + \ln \bar{r}_{23} + \ln(1-\underline{r}_{13}) + \ln(1-\bar{r}_{13})] - [\ln \underline{r}_{13} + \ln \bar{r}_{13} + \ln(1-\bar{r}_{23}) + \ln(1-\underline{r}_{23}) + \ln(1-\bar{r}_{12}) + \ln(1-\underline{r}_{12})] = \varepsilon^-_{123} - \varepsilon^+_{123} \\
[\ln \underline{r}_{12} + \ln \bar{r}_{12} + \ln \underline{r}_{24} + \ln \bar{r}_{24} + \ln(1-\underline{r}_{14}) + \ln(1-\bar{r}_{14})] - [\ln \underline{r}_{14} + \ln \bar{r}_{14} + \ln(1-\bar{r}_{24}) + \ln(1-\underline{r}_{24}) + \ln(1-\bar{r}_{12}) + \ln(1-\underline{r}_{12})] = \varepsilon^-_{124} - \varepsilon^+_{124} \\
[\ln \underline{r}_{12} + \ln \bar{r}_{12} + \ln \underline{r}_{25} + \ln \bar{r}_{25} + \ln(1-\underline{r}_{15}) + \ln(1-\bar{r}_{15})] - [\ln \underline{r}_{15} + \ln \bar{r}_{15} + \ln(1-\bar{r}_{25}) + \ln(1-\underline{r}_{25}) + \ln(1-\bar{r}_{12}) + \ln(1-\underline{r}_{12})] = \varepsilon^-_{125} - \varepsilon^+_{125} \\
[\ln \underline{r}_{13} + \ln \bar{r}_{13} + \ln \underline{r}_{34} + \ln \bar{r}_{34} + \ln(1-\underline{r}_{14}) + \ln(1-\bar{r}_{14})] - [\ln \underline{r}_{14} + \ln \bar{r}_{14} + \ln(1-\bar{r}_{34}) + \ln(1-\underline{r}_{34}) + \ln(1-\bar{r}_{13}) + \ln(1-\underline{r}_{13})] = \varepsilon^-_{134} - \varepsilon^+_{134} \\
[\ln \underline{r}_{13} + \ln \bar{r}_{13} + \ln \underline{r}_{35} + \ln \bar{r}_{35} + \ln(1-\underline{r}_{15}) + \ln(1-\bar{r}_{15})] - [\ln \underline{r}_{15} + \ln \bar{r}_{15} + \ln(1-\bar{r}_{35}) + \ln(1-\underline{r}_{35}) + \ln(1-\bar{r}_{13}) + \ln(1-\underline{r}_{13})] = \varepsilon^-_{135} - \varepsilon^+_{135} \\
[\ln \underline{r}_{14} + \ln \bar{r}_{14} + \ln \underline{r}_{45} + \ln \bar{r}_{45} + \ln(1-\underline{r}_{15}) + \ln(1-\bar{r}_{15})] - [\ln \underline{r}_{15} + \ln \bar{r}_{15} + \ln(1-\bar{r}_{45}) + \ln(1-\underline{r}_{45}) + \ln(1-\bar{r}_{14}) + \ln(1-\underline{r}_{14})] = \varepsilon^-_{145} - \varepsilon^+_{145} \\
[\ln \underline{r}_{23} + \ln \bar{r}_{23} + \ln \underline{r}_{34} + \ln \bar{r}_{34} + \ln(1-\underline{r}_{24}) + \ln(1-\bar{r}_{24})] - [\ln \underline{r}_{24} + \ln \bar{r}_{24} + \ln(1-\bar{r}_{34}) + \ln(1-\underline{r}_{34}) + \ln(1-\bar{r}_{23}) + \ln(1-\underline{r}_{23})] = \varepsilon^-_{234} - \varepsilon^+_{234} \\
[\ln \underline{r}_{23} + \ln \bar{r}_{23} + \ln \underline{r}_{35} + \ln \bar{r}_{35} + \ln(1-\underline{r}_{25}) + \ln(1-\bar{r}_{25})] - [\ln \underline{r}_{25} + \ln \bar{r}_{25} + \ln(1-\bar{r}_{35}) + \ln(1-\underline{r}_{35}) + \ln(1-\bar{r}_{23}) + \ln(1-\underline{r}_{23})] = \varepsilon^-_{235} - \varepsilon^+_{235} \\
[\ln \underline{r}_{24} + \ln \bar{r}_{24} + \ln \underline{r}_{45} + \ln \bar{r}_{45} + \ln(1-\underline{r}_{25}) + \ln(1-\bar{r}_{25})] - [\ln \underline{r}_{25} + \ln \bar{r}_{25} + \ln(1-\bar{r}_{45}) + \ln(1-\underline{r}_{45}) + \ln(1-\bar{r}_{24}) + \ln(1-\underline{r}_{24})] = \varepsilon^-_{245} - \varepsilon^+_{245} \\
[\ln \underline{r}_{34} + \ln \bar{r}_{34} + \ln \underline{r}_{45} + \ln \bar{r}_{45} + \ln(1-\underline{r}_{35}) + \ln(1-\bar{r}_{35})] - [\ln \underline{r}_{35} + \ln \bar{r}_{35} + \ln(1-\bar{r}_{45}) + \ln(1-\underline{r}_{45}) + \ln(1-\bar{r}_{34}) + \ln(1-\underline{r}_{34})] = \varepsilon^-_{345} - \varepsilon^+_{345} \\
0 \le \underline{r}_{12} \le \bar{r}_{12} \le 1, 0 \le \underline{r}_{13} \le \bar{r}_{13} \le 1, 0 \le \underline{r}_{14} \le \bar{r}_{14} \le 1, 0 \le \underline{r}_{15} \le \bar{r}_{15} \le 1, 0 \le \underline{r}_{23} \le \bar{r}_{23} \le 1, \\
0 \le \underline{r}_{24} \le \bar{r}_{24} \le 1, 0 \le \underline{r}_{25} \le \bar{r}_{25} \le 1, 0 \le \underline{r}_{34} \le \bar{r}_{34} \le 1, 0 \le \underline{r}_{35} \le \bar{r}_{35} \le 1, 0 \le \underline{r}_{45} \le \bar{r}_{45} \le 1.
\end{array}\right. \tag{32}$$

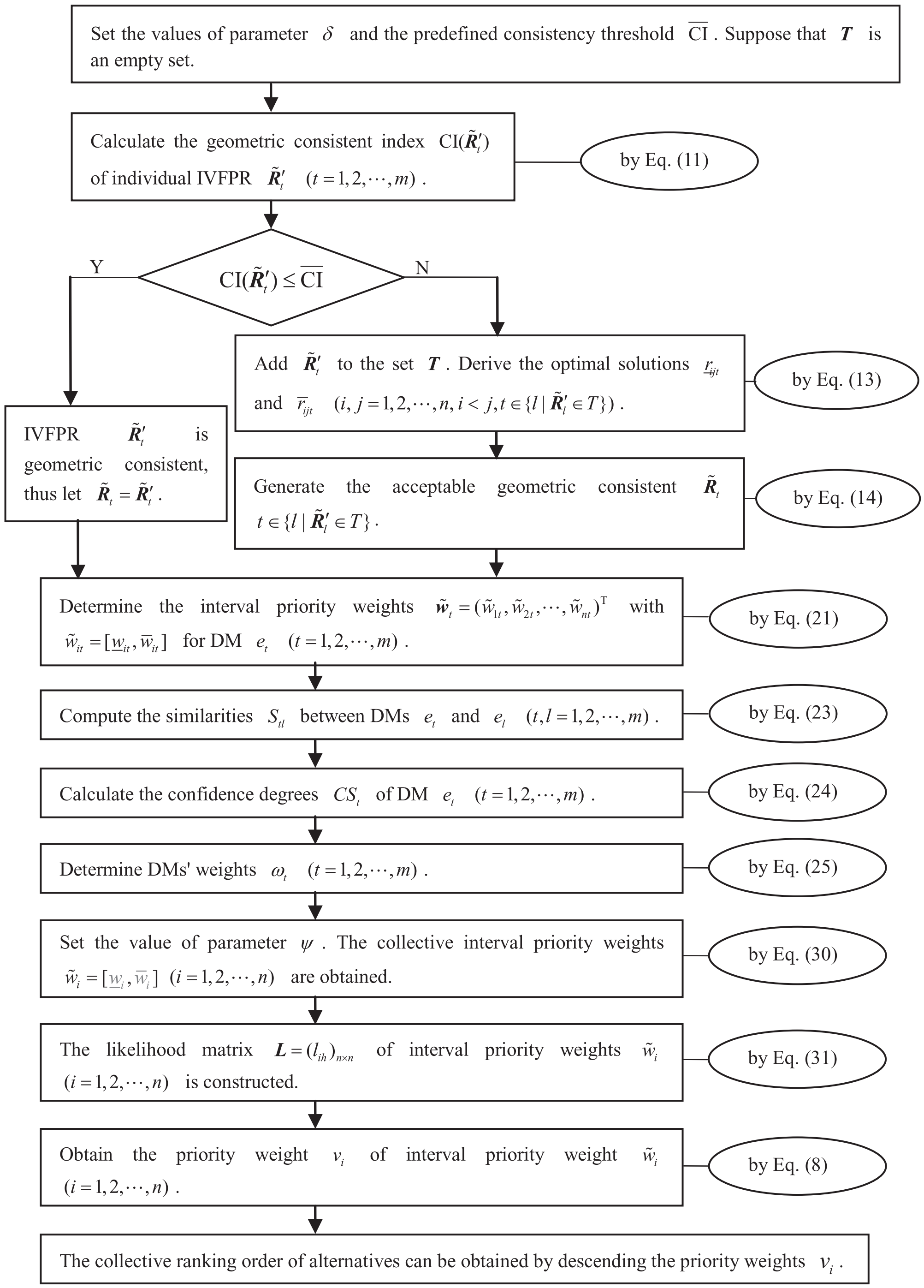

Fig. 2. Decision making process for GDM with IVFPRs.

Solving Eq. (32), the optimal solutions $\underline{r}_{ij}$ and $\bar{r}_{ij}$ for all $i, j = 1, 2, \ldots, 5$ and $i < j$ are derived.

Then the acceptable geometric consistent IVFPR $\tilde{\boldsymbol{R}}$ is generated by Eq. (14) as

$$\tilde{\boldsymbol{R}} = \begin{pmatrix} [0.5000, 0.5000] & [0.5000, 0.6000] & [0.3637, 0.5000] & [0.3000, 0.9256] & [0.3000, 0.5000] \\ [0.4000, 0.5000] & [0.5000, 0.5000] & [0.2222, 0.6000] & [0.6000, 0.7273] & [0.2759, 0.4000] \\ [0.5000, 0.6363] & [0.4000, 0.7778] & [0.5000, 0.5000] & [0.7000, 0.8000] & [0.4000, 0.5000] \\ [0.0744, 0.7000] & [0.2727, 0.4000] & [0.2000, 0.3000] & [0.5000, 0.5000] & [0.0968, 0.4000] \\ [0.5000, 0.7000] & [0.6000, 0.7241] & [0.5000, 0.6000] & [0.6000, 0.9032] & [0.5000, 0.5000] \end{pmatrix}$$

Step 4. Set $\theta_{ij}^{+} = \theta_{ij}^{-} = 0.5$ for $i, j = 1, 2, \ldots, 5$ and $i < j$. According to Eq. (21), a fuzzy logarithmic program is established as

Table 1
Ranking orders of alternatives with different parameter values in Example 1.

	The proposed method for individual decision making				Genc et al.'s method in [4]
	$\overline{\mathrm{CI}}=0.1,\delta=0$	$\overline{\mathrm{CI}}=0.1,\delta=0.5$	$\overline{\mathrm{CI}}=0.1,\delta=1$	$\overline{\mathrm{CI}}=0.01,\delta=0.5$	
$\tilde{w}_1$	[0.1386,0.3429]	[0.1374,0.3736]	[0.1398,0.3458]	[0.1445,0.3573]	[0.1364,0:2442]
$\tilde{w}_2$	[0.1386,0.1962]	[0.1510,0.2176]	[0.1398,0.2083]	[0.1445,0.1872]	[0.1111,0.2029]
$\tilde{w}_3$	[0.2080,0.2286]	[0.2266,0.2266]	[0.2097,0.2305]	[0.2058,0.2382]	[0.2029,0.3218]
$\tilde{w}_4$	[0.0404,0.2286]	[0.0495,0.2191]	[0.0950,0.2305]	[0.0426,0.2382]	[0.0662,0.1154]
$\tilde{w}_5$	[0.2080,0.2080]	[0.1993,0.1993]	[0.2097,0.2097]	[0.2167,0.2167]	[0.2442,0.3899]
Ranking order	$x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$	$x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$	$x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$	$x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$	$x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4$
Fitted error	0.2305	0.2422	0.2483	0.2267	0.3495

$$\max \beta$$
$$s.t.\begin{cases}
(\beta-1)\theta^+_{12} \le \ln\bar{w}_1 - \ln\underline{w}_2 + \ln(1-0.6) - \ln 0.6 + \alpha'_{12} \le (1-\beta)\theta^+_{12},\ (\beta-1)\theta^+_{13} \le \ln\bar{w}_1 - \ln\underline{w}_3 + \ln(1-0.5) - \ln 0.5 + \alpha'_{13} \le (1-\beta)\theta^+_{13},\\
(\beta-1)\theta^+_{14} \le \ln\bar{w}_1 - \ln\underline{w}_4 + \ln(1-0.9256) - \ln 0.9256 + \alpha'_{14} \le (1-\beta)\theta^+_{14},\ (\beta-1)\theta^+_{15} \le \ln\bar{w}_1 - \ln\underline{w}_5 + \ln(1-0.5) - \ln 0.5 + \alpha'_{15} \le (1-\beta)\theta^+_{15},\\
(\beta-1)\theta^+_{23} \le \ln\bar{w}_2 - \ln\underline{w}_3 + \ln(1-0.6) - \ln 0.6 + \alpha'_{23} \le (1-\beta)\theta^+_{23},\ (\beta-1)\theta^+_{24} \le \ln\bar{w}_2 - \ln\underline{w}_4 + \ln(1-0.7273) - \ln 0.7273 + \alpha'_{24} \le (1-\beta)\theta^+_{24},\\
(\beta-1)\theta^+_{25} \le \ln\bar{w}_2 - \ln\underline{w}_5 + \ln(1-0.4) - \ln 0.4 + \alpha'_{25} \le (1-\beta)\theta^+_{25},\ (\beta-1)\theta^+_{34} \le \ln\bar{w}_3 - \ln\underline{w}_4 + \ln(1-0.8) - \ln 0.8 + \alpha'_{34} \le (1-\beta)\theta^+_{34},\\
(\beta-1)\theta^+_{35} \le \ln\bar{w}_3 - \ln\underline{w}_5 + \ln(1-0.5) - \ln 0.5 + \alpha'_{35} \le (1-\beta)\theta^+_{35},\ (\beta-1)\theta^+_{45} \le \ln\bar{w}_4 - \ln\underline{w}_5 + \ln(1-0.4) - \ln 0.4 + \alpha'_{45} \le (1-\beta)\theta^+_{45},\\
(\beta-1)\theta^-_{12} \le \ln\bar{w}_1 - \ln\underline{w}_2 + \ln 0.5 - \ln(1-0.5) + \alpha'_{12} \le (1-\beta)\theta^-_{12},\ (\beta-1)\theta^-_{13} \le \ln\bar{w}_1 - \ln\underline{w}_3 + \ln 0.3637 - \ln(1-0.3637) + \alpha'_{13} \le (1-\beta)\theta^-_{13},\\
(\beta-1)\theta^-_{14} \le \ln\bar{w}_1 - \ln\underline{w}_4 + \ln 0.3 - \ln(1-0.3) + \alpha'_{14} \le (1-\beta)\theta^-_{14},\ (\beta-1)\theta^-_{15} \le \ln\bar{w}_1 - \ln\underline{w}_5 + \ln 0.3 - \ln(1-0.3) + \alpha'_{15} \le (1-\beta)\theta^-_{15},\\
(\beta-1)\theta^-_{23} \le \ln\bar{w}_2 - \ln\underline{w}_3 + \ln 0.2222 - \ln(1-0.2222) + \alpha'_{23} \le (1-\beta)\theta^-_{23},\ (\beta-1)\theta^-_{24} \le \ln\bar{w}_2 - \ln\underline{w}_4 + \ln 0.6 - \ln(1-0.6) + \alpha'_{24} \le (1-\beta)\theta^-_{24},\\
(\beta-1)\theta^-_{25} \le \ln\bar{w}_2 - \ln\underline{w}_5 + \ln 0.2759 - \ln(1-0.2759) + \alpha'_{25} \le (1-\beta)\theta^-_{25},\ (\beta-1)\theta^-_{34} \le \ln\bar{w}_3 - \ln\underline{w}_4 + \ln 0.7 - \ln(1-0.7) + \alpha'_{34} \le (1-\beta)\theta^-_{34},\\
(\beta-1)\theta^-_{35} \le \ln\bar{w}_3 - \ln\underline{w}_5 + \ln 0.4 - \ln(1-0.4) + \alpha'_{35} \le (1-\beta)\theta^-_{35},\ (\beta-1)\theta^-_{45} \le \ln\bar{w}_4 - \ln\underline{w}_5 + \ln 0.0968 - \ln(1-0.0968) + \alpha'_{45} \le (1-\beta)\theta^-_{45},\\
\ln\underline{w}_1 + \ln\underline{w}_2 - \ln\bar{w}_1 - \ln\bar{w}_2 \le 2\alpha'_{12} \le 0,\ \ln\underline{w}_1 + \ln\underline{w}_3 - \ln\bar{w}_1 - \ln\bar{w}_3 \le 2\alpha'_{13} \le 0,\\
\ln\underline{w}_1 + \ln\underline{w}_4 - \ln\bar{w}_1 - \ln\bar{w}_4 \le 2\alpha'_{14} \le 0,\ \ln\underline{w}_1 + \ln\underline{w}_5 - \ln\bar{w}_1 - \ln\bar{w}_5 \le 2\alpha'_{15} \le 0,\\
\ln\underline{w}_2 + \ln\underline{w}_3 - \ln\bar{w}_2 - \ln\bar{w}_3 \le 2\alpha'_{23} \le 0,\ \ln\underline{w}_2 + \ln\underline{w}_4 - \ln\bar{w}_2 - \ln\bar{w}_4 \le 2\alpha'_{24} \le 0,\\
\ln\underline{w}_2 + \ln\underline{w}_5 - \ln\bar{w}_2 - \ln\bar{w}_5 \le 2\alpha'_{25} \le 0,\ \ln\underline{w}_3 + \ln\underline{w}_4 - \ln\bar{w}_3 - \ln\bar{w}_4 \le 2\alpha'_{34} \le 0,\\
\ln\underline{w}_3 + \ln\underline{w}_5 - \ln\bar{w}_3 - \ln\bar{w}_5 \le 2\alpha'_{35} \le 0,\ \ln\underline{w}_4 + \ln\underline{w}_5 - \ln\bar{w}_4 - \ln\bar{w}_5 \le 2\alpha'_{45} \le 0,\\
\underline{w}_1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4 + \bar{w}_5 \ge 1,\ \bar{w}_1 + \underline{w}_2 + \bar{w}_3 + \bar{w}_4 + \bar{w}_5 \ge 1,\ \bar{w}_1 + \bar{w}_2 + \underline{w}_3 + \bar{w}_4 + \bar{w}_5 \ge 1,\ \bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \underline{w}_4 + \bar{w}_5 \ge 1,\ \bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4 + \underline{w}_5 \ge 1,\\
\bar{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4 + \underline{w}_5 \le 1,\ \underline{w}_1 + \bar{w}_2 + \underline{w}_3 + \underline{w}_4 + \underline{w}_5 \le 1,\ \underline{w}_1 + \underline{w}_2 + \bar{w}_3 + \underline{w}_4 + \underline{w}_5 \le 1,\ \underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \bar{w}_4 + \underline{w}_5 \le 1,\\
\underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4 + \bar{w}_5 \le 1,\ 0 \le \underline{w}_1 \le \bar{w}_1 \le 1,\ 0 \le \underline{w}_2 \le \bar{w}_2 \le 1,\ 0 \le \underline{w}_3 \le \bar{w}_3 \le 1,\ 0 \le \underline{w}_4 \le \bar{w}_4 \le 1,\ 0 \le \underline{w}_5 \le \bar{w}_5 \le 1,\ 0 \le \beta \le 1
\end{cases} \quad (33)$$

Solving Eq. (33), the interval priority weights are determined as $\tilde{w}_1 = [0.1374, 0.3736]$, $\tilde{w}_2 = [0.1511, 0.2176]$, $\tilde{w}_3 = [0.2266, 0.2266]$, $\tilde{w}_4 = [0.0495, 0.2191]$, $\tilde{w}_5 = [0.1993, 0.1993]$.

Step 5. Based on Eq. (22), the likelihood matrix $L = (l_{ih})_{5\times 5}$ is established as follows:

$$L = \begin{pmatrix} 0.5000 & 0.7352 & 0.6225 & 0.7987 & 0.7378 \\ 0.2648 & 0.5000 & 0 & 0.7118 & 0.2740 \\ 0.3775 & 1 & 0.5000 & 1 & 1 \\ 0.2013 & 0.2882 & 0 & 0.5000 & 0.1165 \\ 0.2622 & 0.7260 & 0 & 0.8838 & 0.5000 \end{pmatrix}$$

Then the priority weights are obtained by Eq. (8) as

$$v_1 = 0.2447, v_2 = 0.1625, v_3 = 0.2689,$$
$$v_4 = 0.1303, v_5 = 0.1936.$$

By descending the priority weights v_i $(i = 1, 2, \ldots, 5)$, the ranking order of alternatives is generated as $x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$.

With different values of parameters $\overline{\mathrm{CI}}$ and δ, the ranking results are derived and shown in Table 1.

It can be seen from Table 1 that the ranking order of alternatives is always $x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_2$ by the method for individual decision making with IVFPR proposed in this paper, which is different from that obtained by Genc et al. [4]. To compare method [4] with the proposed method effectively, we resort to the fitted error originated from Wang and Elhag [32].

Similar to Wang and Elhag [32], the fitted IVFPR $\bar{\tilde{\boldsymbol{R}}} = (\bar{\tilde{r}}_{ij})_{n\times n}$ is constructed by interval priority weights $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ $(i = 1, 2, \ldots, n)$, where

$$\bar{\tilde{r}}_{ij} = \left[r^-_{ij}, r^+_{ij}\right] = \begin{cases} [0.5, 0.5], & \text{if } i = j \\ \left[\frac{\underline{w}_i}{\underline{w}_i + \alpha_{ij}\bar{w}_j}, \frac{\alpha_{ij}\bar{w}_i}{\alpha_{ij}\bar{w}_i + \underline{w}_j}\right], & \text{if } i \ne j \end{cases} \quad (34)$$

Especially, since Genc et al. [4] proposed $\bar{\tilde{r}}_{ij} = [\frac{\underline{w}_i}{\underline{w}_i + \bar{w}_j}, \frac{\bar{w}_i}{\bar{w}_i + \underline{w}_j}]$ in Theorem 4, the fitted IVFPR $\bar{\tilde{\boldsymbol{R}}} = (\bar{\tilde{r}}_{ij})_{n\times n}$ is obtained by setting $\alpha_{ij} = 1$ in method [4].

Then the fitted error between the fitted IVFPR $\bar{\tilde{\boldsymbol{R}}} = (\bar{\tilde{r}}_{ij})_{n\times n}$ and the original IVFPR $\tilde{\boldsymbol{R}}'$ is defined as follows:

$$F(\bar{\tilde{\boldsymbol{R}}}, \tilde{\boldsymbol{R}}') = \frac{2}{n(n-1)} \sum\nolimits_{i=1}^{n-1} \sum\nolimits_{j=i+1}^{n} \left(|r^-_{ij} - \underline{r}_{ij}| + |r^+_{ij} - \bar{r}_{ij}|\right) \quad (35)$$

Apparently, $F(\bar{\tilde{\boldsymbol{R}}}, \tilde{\boldsymbol{R}}') \in [0, 1]$. The smaller the value of $F(\bar{\tilde{\boldsymbol{R}}}, \tilde{\boldsymbol{R}}')$, the more reliable the decision method.

Using Eqs. (34) and (35), the fitted errors derived from different interval priority weights are calculated and listed in the last line of Table 1. It is clear that all the fitted errors of the proposed method are smaller than that of method [4]. Thus, the proposed method could avoid information loss and contain more original information on IVFPR $\tilde{\boldsymbol{R}}'$, which verifies the effectiveness of the proposed method.

Table 2
Ranking orders of alternatives with different parameter values in Example 2.

	The proposed method for individual decision making					Wang and Chen's method in [11]
	$\overline{\mathrm{CI}}=0.1, \delta=0$	$\overline{\mathrm{CI}}=0.1, \delta=0.5$	$\overline{\mathrm{CI}}=0.1, \delta=1$	$\overline{\mathrm{CI}}=0.01, \delta=0.5$	$\overline{\mathrm{CI}}=0, \delta=0.5$	
$\tilde{w}_1$	[0.2831,0.2831]	[0.2325,0.2325]	[0.2325,0.2325]	[0.2889,0.2889]	[0.2404,0.2404]	[0.1580,0.3873]
$\tilde{w}_2$	[0.2831,0.4668]	[0.2234,0.3833]	[0.2234,0.3833]	[0.2889,0.4763]	[0.2253,0.3964]	[0.2338,0.5371]
$\tilde{w}_3$	[0.1589,0.1717]	[0.1580,0.3536]	[0.1580,0.3536]	[0.1498,0.1752]	[0.1634,0.3714]	[0.1310,0.3782]
$\tilde{w}_4$	[0.0911,0.2620]	[0.1410,0.2584]	[0.1410,0.2584]	[0.0850,0.2470]	[0.0953,0.1999]	[0.1793,0.4771]
Ranking order	$x_2 \succ x_1 \succ x_4 \succ x_3$	$x_2 \succ x_3 \succ x_1 \succ x_4$	$x_2 \succ x_3 \succ x_1 \succ x_4$	$x_2 \succ x_1 \succ x_4 \succ x_3$	$x_2 \succ x_3 \succ x_1 \succ x_4$	$x_2 \succ x_4 \succ x_1 \succ x_3$
Fitted error	0.2562	0.2328	0.2328	0.2586	0.2787	0.2817

Example 2. Consider the following IVFPR provided by method [11].

$$\tilde{\boldsymbol{R}}' = \begin{pmatrix} [0.5, 0.5] & [0.3, 0.4] & [0.5, 0.7] & [0.4, 0.5] \\ [0.6, 0.7] & [0.5, 0.5] & [0.6, 0.8] & [0.2, 0.6] \\ [0.3, 0.5] & [0.2, 0.4] & [0.5, 0.5] & [0.4, 0.8] \\ [0.5, 0.6] & [0.4, 0.8] & [0.2, 0.6] & [0.5, 0.5] \end{pmatrix}$$

When $\theta_{ij}^{+} = \theta_{ij}^{-} = 0.5$, the interval priority weights are derived by the method proposed in Section 4.2 with different values of parameters $\overline{\mathrm{CI}}$ and δ. Then the corresponding ranking orders of alternatives are generated. Moreover, the fitted errors derived by different interval priority weights are also calculated, respectively. Table 2 presents these computation results.

From Table 2, it can be found that the ranking orders obtained by the proposed method are diverse with different values of parameters $\overline{\mathrm{CI}}$ and δ. Moreover, although the best alternatives obtained by the proposed method and method [11] are the same x_2, the corresponding ranking orders are not identical. The proposed method can make DMs select different values of parameters to derive the interval priority weight vector according to the actual decision making, which embodies the flexibility of the proposed method. Additionally, the fitted errors of this paper are smaller than that of method [11]. Thus, the proposed method retains more original decision information than method [11] and can avoid the loss of information.

It can be seen from Table 2 that the ranking orders in these three cases are completely identical, i.e., $x_2 \succ x_3 \succ x_1 \succ x_4$. That is to say, the ranking orders derived from the two acceptable geometric consistent IVFPRs with high levels of consistency (i.e., the case $\overline{\mathrm{CI}}=0.1$ and $\delta=0.5$ and the case $\overline{\mathrm{CI}}=0.1$ and $\delta=1$ in Table 2) are completely the same as that derived from the completely geometric consistent IVFPR (i.e. the case $\overline{\mathrm{CI}}=0$ and $\delta=0.5$ in Table 2). Furthermore, the best alternatives derived from another two acceptable geometric consistent IVFPRs with high levels of consistency (i.e., the case $\overline{\mathrm{CI}}=0.1$ and $\delta=0$ and the case $\overline{\mathrm{CI}}=0.01$ and $\delta=0.5$ in Table 2) are completely the same as that derived from the completely geometric consistent IVFPR (i.e., the case $\overline{\mathrm{CI}}=0$ and $\delta=0.5$ in Table 2). These observations indicate that it is reasonable to obtain the priority weights from an inconsistent IVFPR with a high level of geometric consistency, i.e., an acceptable geometric consistent IVFPR. Moreover, the fitted error derived from completely geometric consistent IVFPR (i.e., the case of $\overline{\mathrm{CI}}=0$ and $\delta=0.5$ in Table 2) is larger than that obtained from the acceptable consistent IVFPRs (i.e., the former four cases in Table 2). Apparently, using the acceptable geometric consistent IVFPR can contain more original information during the decision making process. Thus, an inconsistent IVFPR can be accepted if it has a high level of consistency in some specified situations.

6.2. Application to GDM with IVFPRs

6.2.1. An enterprise resource planning system selection example

An example is provided to illustrate the effectiveness of the GDM method proposed in this paper.

Example 3. As the continual growth of business, a firm needs to install a new enterprise resource planning (ERP) system. After preliminary screening, four ERP systems, i.e., x_1, x_2, x_3 and x_4 are remained for further evaluation. Three DMs e_1, e_2 and e_3 are invited to form a decision group. DMs e_t $(t = 1, 2, 3)$ provide their preference information on alternatives expressed by intervals and elicit the corresponding IVFPRs $\tilde{\boldsymbol{R}}'_t = (\tilde{r}'_{ijt})_{4\times 4}$ as follows:

$$\tilde{\boldsymbol{R}}'_1 = \begin{pmatrix} [0.50, 0.50] & [0.65, 0.80] & [0.50, 0.60] & [0.30, 0.55] \\ [0.20, 0.35] & [0.50, 0.50] & [0.30, 0.55] & [0.50, 0.65] \\ [0.40, 0.50] & [0.45, 0.70] & [0.50, 0.50] & [0.55, 0.70] \\ [0.45, 0.70] & [0.35, 0.50] & [0.30, 0.45] & [0.50, 0.50] \end{pmatrix}$$

$$\tilde{\boldsymbol{R}}'_2 = \begin{pmatrix} [0.50, 0.50] & [0.65, 0.70] & [0.60, 0.80] & [0.75, 0.90] \\ [0.30, 0.35] & [0.50, 0.50] & [0.50, 0.60] & [0.70, 0.75] \\ [0.20, 0.40] & [0.40, 0.50] & [0.50, 0.50] & [0.60, 0.75] \\ [0.10, 0.25] & [0.25, 0.30] & [0.25, 0.40] & [0.50, 0.50] \end{pmatrix}$$

$$\tilde{\boldsymbol{R}}'_3 = \begin{pmatrix} [0.50, 0.50] & [0.45, 0.55] & [0.30, 0.65] & [0.50, 0.75] \\ [0.45, 0.55] & [0.50, 0.50] & [0.60, 0.85] & [0.25, 0.55] \\ [0.35, 0.70] & [0.15, 0.40] & [0.50, 0.50] & [0.35, 0.65] \\ [0.25, 0.50] & [0.45, 0.75] & [0.35, 0.65] & [0.50, 0.50] \end{pmatrix}$$

Step 1. Set parameter $\delta=0.5$ and the predefined consistency threshold $\overline{\mathrm{CI}}=0.1$. Suppose $\boldsymbol{T}$ is an empty set.

Step 2. By Eq. (11), calculate the geometric consistent indices for individual IVFPRs as follows:

$\mathrm{CI}(\tilde{\boldsymbol{R}}'_1) = 1.125$, $\mathrm{CI}(\tilde{\boldsymbol{R}}'_2) = 0.087$, $\mathrm{CI}(\tilde{\boldsymbol{R}}'_3) = 2.610$.

Since only $\mathrm{CI}(\tilde{\boldsymbol{R}}'_2) \le \overline{\mathrm{CI}}$, IVFPR $\tilde{\boldsymbol{R}}'_2$ is acceptable geometric consistent and let $\tilde{\boldsymbol{R}}_2 = \tilde{\boldsymbol{R}}'_2$. IVFPRs $\tilde{\boldsymbol{R}}'_1$ and $\tilde{\boldsymbol{R}}'_3$ are unacceptable geometric consistent. Thus one has $\boldsymbol{T} = \{\tilde{\boldsymbol{R}}'_1, \tilde{\boldsymbol{R}}'_3\}$ and then go to the next step.

Step 3. Solving Eq. (13), the optimal solutions $\underline{r}_{ijt}$ and $\bar{r}_{ijt}$ $(i, j = 1, 2, \ldots, n,\ i < j)$ are derived from unacceptable geometric consistent IVFPRs $\tilde{\boldsymbol{R}}'_t$ for $t \in \{1, 3\}$. By Eq. (14), the corresponding acceptable geometric consistent IVFPRs $\tilde{\boldsymbol{R}}_t$ $(t \in \{1, 3\})$ are generated as

$$\tilde{\boldsymbol{R}}_1 = \begin{pmatrix} [0.5000, 0.5000] & [0.6402, 0.6402] & [0.5000, 0.6000] & [0.3000, 0.9169] \\ [0.3598, 0.3598] & [0.5000, 0.5000] & [0.3000, 0.5500] & [0.5000, 0.6228] \\ [0.4000, 0.5000] & [0.4500, 0.7000] & [0.5000, 0.5000] & [0.5500, 0.7000] \\ [0.0831, 0.7000] & [0.3772, 0.5000] & [0.3000, 0.4500] & [0.5000, 0.5000] \end{pmatrix}$$

$$\tilde{\boldsymbol{R}}_3 = \begin{pmatrix} [0.5000, 0.5000] & [0.4500, 0.5500] & [0.3000, 0.8636] & [0.5000, 0.7500] \\ [0.4500, 0.5500] & [0.5000, 0.5000] & [0.6000, 0.6667] & [0.6223, 0.6223] \\ [0.1364, 0.7000] & [0.3333, 0.4000] & [0.5000, 0.5000] & [0.3500, 0.6500] \\ [0.2500, 0.5000] & [0.3777, 0.3777] & [0.3500, 0.6500] & [0.5000, 0.5000] \end{pmatrix}$$

Step 4. Set $\theta_{ij}^{+} = \theta_{ij}^{-} = 0.5$ $(i, j = 1, 2, 3, 4, i < j)$. According to Eq. (21), the individual interval priority weight vectors $\tilde{\boldsymbol{w}}_t$ $(t = 1, 2, 3)$ are obtained as

$\tilde{\boldsymbol{w}}_1$ =([0.2571,0.5178],[0.1872,0.1872],[0.2221,0.2646], [0.0730,0.2910])$^{\mathrm{T}}$,

$\tilde{\boldsymbol{w}}_2$ =([0.2513,0.5392],[0.2212,0.2723],[0.1659,0.2722], [0.0737,0.2042])$^{\mathrm{T}}$,

$\tilde{\boldsymbol{w}}_3$ =([0.2106,0.4794],[0.2600,0.2993],[0.1142,0.3178], [0.1204,0.1723])$^{\mathrm{T}}$.

Step 5. The similarities between DMs are computed by Eq. (23) as follows:

$S_{12} = S_{21} = 0.0372$, $S_{13} = S_{31} = 0.0746$, $S_{23} = S_{32} = 0.\ 0428$.

Step 6. The confidence degrees of DMs are calculated via Eq. (24):

$CS_1 = 0.1118$, $CS_2 = 0.0800$, $CS_3 = 0.1174$.

Step 7. Using Eq. (25), DMs' weights ω_t $(t = 1, 2, 3)$ are determined as

$\omega_1 = 0.3316$, $\omega_2 = 0.3372$, $\omega_3 = 0.3312$.

Step 8. Set $\psi = 0.5$. According to Eq. (30), the collective interval priority weights $\tilde{w}_i$ $(i = 1, 2, 3, 4)$ are derived as follows:

$\tilde{w}_1$ =[0.2513,0.5178], $\tilde{w}_2$ =[0.2212,0.2723], $\tilde{w}_3$ =[0.1659,0.2722], $\tilde{w}_4$ =[0.0737,0.2042].

Step 9. Utilizing Eq. (31), the likelihood matrix is constructed as

$$\boldsymbol{L} = \begin{pmatrix} 0.5 & 0.9339 & 0.9439 & 1 \\ 0.0661 & 0.5 & 0.6760 & 1 \\ 0.0561 & 0.3240 & 0.5 & 0.8383 \\ 0 & 0 & 0.1617 & 0.5 \end{pmatrix}$$

According to Eq. (8), the priority weights are obtained as

$$v_1 = 0.3648, v_2 = 0.2702, v_3 = 0.2265, v_4 = 0.1385.$$

By descending the priority weights v_i $(i = 1, 2, 3, 4)$, the collective ranking order of alternatives is generated as $x_1 \succ x_2 \succ x_3 \succ x_4$.

For different values of parameters $\overline{\mathrm{CI}}$, δ and ψ, the corresponding computation results are shown in Table 3. Table 3 shows that the ranking orders of alternatives are always $x_1 \succ x_2 \succ x_3 \succ x_4$ for different values of parameters. Especially, the ranking order derived from the completely geometric consistent IVFPR (i.e., the case $\overline{\mathrm{CI}} = 0$, $\delta = 0.5$, $\psi = 0.5$ in Table 3) is the same as that obtained from the acceptable geometric consistent IVFPRs (i.e., the other cases of the proposed GDM method in Table 3). Therefore, it may be useful and effective to directly apply the acceptable geometric consistent IVFPR for generating the priority weights in practice.

6.2.2. Fitted error analysis of the obtained results

To illustrate the effectiveness of the results obtained by the proposed method, the collective fitted error is defined. Similarly, the fitted collective IVFPR $\tilde{\boldsymbol{R}}^* = (\tilde{r}_{ij}^*)_{n\times n}$ with $\tilde{r}_{ij}^* = [\tilde{r}_{ij}^{*-}, \tilde{r}_{ij}^{*+}]$ is constructed by the derived collective interval priority weights $\tilde{w}_i = [\underline{w}_i, \overline{w}_i]$ $(i = 1, 2, \ldots, n)$, where

$$\tilde{r}_{ij}^* = \left[\tilde{r}_{ij}^{*-}, \tilde{r}_{ij}^{*+}\right] = \begin{cases} [0.5, 0.5], & \text{if } i = j \\ \left[\frac{\underline{w}_i}{\underline{w}_i + \alpha_{ij}\overline{w}_j}, \frac{\alpha_{ij}\overline{w}_i}{\alpha_{ij}\overline{w}_i + \underline{w}_j}\right], & \text{if } i \neq j \end{cases} \quad (36)$$

Considering that there exist several DMs in GDM, the individual fitted error between $\tilde{\boldsymbol{R}}^* = (\tilde{r}_{ij}^*)_{n\times n}$ and $\tilde{\boldsymbol{R}}_t$ for DM e_t is defined as follows:

$$F(\tilde{\boldsymbol{R}}^*, \tilde{\boldsymbol{R}}_t) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (|r_{ijt}^- - \tilde{r}_{ij}^{*-}| + |r_{ijt}^+ - \tilde{r}_{ij}^{*+}|). \quad (37)$$

Then the collective fitted error F^* for the group of DMs is calculated as

$$F^* = \frac{1}{m} \sum_{t=1}^{m} F(\tilde{\boldsymbol{R}}^*, \tilde{\boldsymbol{R}}_t). \quad (38)$$

Table 3
Ranking results of alternatives with different methods.

	The proposed GDM method							Zhang's method in [21]	
	$\overline{\mathrm{CI}}=0,\delta=0.5,\psi=0.5$	$\overline{\mathrm{CI}}=0.1,\delta=0,\psi=0$	$\overline{\mathrm{CI}}=0.1,\delta=0,\psi=0.5$	$\overline{\mathrm{CI}}=0.1,\delta=0,\psi=1$	$\overline{\mathrm{CI}}=0.1,\delta=0.5,\psi=0.5$	$\overline{\mathrm{CI}}=0.1,\delta=1,\psi=0.5$	$\overline{\mathrm{CI}}=0.01,\delta=0.5,\psi=0.5$		
$\boldsymbol{\omega}^{\mathrm{T}}$	(0.332,0.335,0.333)	(0.332,0.334,0.334)	(0.332,0.334,0.334)	(0.332,0.334,0.334)	(0.332,0.337,0.331)	(0.332,0.336,0.332)	(0.330,0.336,0.334)	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	(0.332,0.337,0.331)
$\tilde{w}_1$	[0.2074,0.5118]	[0.3098,0.4417]	[0.2257, 0.4794]	[0.2257,0.4794]	[0.2513,0.5178]	[0.2513,0.5043]	[0.2555,0.4896]	[0,0.5248]	[0,0.5248]
$\tilde{w}_2$	[0.2219,0.2647]	[0.1608,0.2701]	[0.2257, 0.2983]	[0.2257, 0.2983]	[0.2212,0.2723]	[0.2212,0.2723]	[0.2347,0.2655]	[0,0.5248]	[0,0.5248]
$\tilde{w}_3$	[0.1664,0.2989]	[0.1335,0.2653]	[0.1669,0.2701]	[0.1669,0.2701]	[0.1659,0.2722]	[0.1659,0.2723]	[0.1669,0.2726]	[0.2670,0.2670]	[0.2670,0.2670]
$\tilde{w}_4$	[0.0810,0.2290]	[0.1204,0.1548]	[0.1204,0.2059]	[0.1204,0.2059]	[0.0737,0.2042]	[0.0753,0.2042]	[0.0734,0.2065]	[0.2083,0.2083]	[0.2083,0.2083]
Ranking order	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_1\succ x_2\succ x_3\succ x_4$	$x_3\succ x_1\sim x_2\succ x_4$	$x_3\succ x_1\sim x_2\succ x_4$
$F(\tilde{\boldsymbol{R}}^*,\tilde{\boldsymbol{R}}_1)$	0.2628	0.2231	0.2401	0.2401	0.2649	0.2628	0.2668	0.5012	0.5012
$F(\tilde{\boldsymbol{R}}^*,\tilde{\boldsymbol{R}}_2)$	0.2146	0.1799	0.2135	0.2135	0.1725	0.1737	0.1708	0.6908	0.6908
$F(\tilde{\boldsymbol{R}}^*,\tilde{\boldsymbol{R}}_3)$	0.2642	0.2970	0.2328	0.2328	0.2946	0.2908	0.2903	0.5417	0.5417
F^*	0.2472	0.2333	0.2288	0.2288	0.2440	0.2424	0.2427	0.5779	0.5779

Table 4
Ranking results of two decomposed subproblems.

	Subproblem 1			Subproblem 2		
	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=0$	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=0.5$	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=1$	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=0$	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=0.5$	$\overline{\text{CI}}=0.1,\delta=0.5,\psi=1$
$\boldsymbol{\omega}^{\mathrm{T}}$	(0.3321,0.3368,0.3311)	(0.3321,0.3368,0.3311)	(0.3321,0.3368,0.3311)	(0.3347,0.3273,0.3380)	(0.3347,0.3273,0.3380)	(0.3347,0.3273,0.3380)
$\tilde{w}_1$	[0.3801,0.4892]	[0.3088,0.5505]	[0.3088,0.5505]	[0.3933,0.5150]	[0.3529,0.4698]	[0.3529,0.3933]
$\tilde{w}_2$	[0.2474,0.3562]	[0.2839.0.3562]	[0.2839.0.3562]	–	–	–
$\tilde{w}_3$	[0.1546,0.2637]	[0.1656,0.3416]	[0.1656,0.3416]	[0.2665,0.3883]	[0.2665,0.4280]	[0.2665,0.5045]
$\tilde{w}_4$	–	–	–	[0.1022,0.2184]	[0.1022,0.3402]	[0.1022,0.3402]
Ranking order	$x_1 \succ x_2 \succ x_3$	$x_1 \succ x_2 \succ x_3$	$x_1 \succ x_2 \succ x_3$	$x_1 \succ x_3 \succ x_4$	$x_1 \succ x_3 \succ x_4$	$x_1 \succ x_3 \succ x_4$

Apparently, $F^* \in [0, 1]$. The smaller the value of F^*, the more reliable the GDM method.

Using Eqs. (36)–(38), the individual and collective fitted errors derived from different interval priority weights are calculated and shown in Table 3.

From Table 3, it can be observed that the collective fitted error of the priority weights obtained from the completely geometric consistent IVFPR (i.e., case $\overline{\text{CI}}=0,\delta=0.5,\psi=0.5$ in Table 3) is larger than that derived from the acceptable geometric consistent IVFPRs (i.e., the other cases of the proposed GDM method in Table 3). This analysis again verifies that an inconsistent IVFPR that has a high level of geometric consistency (i.e., the acceptable geometric consistent IVFPR in Table 3) can be accepted during the decision making process as mentioned in Introduction.

To further verify the superiority of the proposed GDM method, rank reversal test is carried out in Section 6.2.3. In addition, we compare the results obtained by Zhang's method [21] with that obtained by the proposed method in Section 6.2.4.

6.2.3. Rank reversal test of the proposed GDM with IVFPRs

Since Wang and Luo [33] found that the rank reversal phenomenon occurs in many decision making methods, the rank reversal has become a common criterion to measure the performance of the decision making methods. A reasonable method should avoid the rank reversal by adding or deleting of an alternative. In other words, if an alternative is added or deleted from the problem, then any other two alternatives should keep the same ranking order. To test the rank reversal of the proposed GDM method, the original ERP system selection example is decomposed into Subproblem 1 with alternative set $\{x_1, x_2, x_3\}$ and Subproblem 2 with alternative set $\{x_1, x_3, x_4\}$, respectively. Subproblem 1 is obtained by deleting alternative x_4 and Subproblem 2 is obtained by deleting alternative x_2 from the original problem. Using the proposed GDM method, the corresponding computation results are respectively generated for different parameters and given in Table 4.

It is evident from Table 4 that the ranking order is $x_1 \succ x_2 \succ x_3$ for Subproblem 1 and $x_1 \succ x_3 \succ x_4$ for Subproblem 2, which are consistent with the ranking $x_1 \succ x_2 \succ x_3 \succ x_4$ for the original problem. The ranking for alternatives x_1, x_2 and x_3 is $x_1 \succ x_2 \succ x_3$ before alternative x_4 is introduced, keeps the same after x_4 is added. The same conclusion can be observed for alternatives x_1, x_3 and x_4. There is no any rank reversal phenomenon for any two alternatives by addition or deletion of an alternative. Thus, the proposed method can well avoid rank reversal. The examination of the rank reversal shows the validity and practicability of the GDM method proposed in this paper.

6.2.4. Comparison with Zhang's method

Based on the multiplicative consistency, Zhang [21] proposed a goal programming model to derive the priority weights for solving GDM with IVFPRs. Using Zhang's method to solve Example 3, the ranking orders of alternatives are generated for different DMs' weights. In particular, setting $\alpha_{ij}=1$, the collective fitted errors derived from the priority weights of method [21] are calculated by Eqs. (36)–(38). For comparison convenience, these computation results are also listed in Table 3.

It can be seen from Table 3 that the ranking order obtained by method [21] is different from that obtained by the proposed method. The primary reasons may come from two aspects:

(1) Zhang [21] gave DMs' weights a priori and ignored the determination of DMs' weights, which may lead to unreasonable decision results. By contrast, this paper determines DMs' weight by the similarity between DMs, which can avoid the subjective randomness and improve convincingness of decision results.
(2) The geometric consistent IVFPR in this paper considers the different values of parameter α_{ij}, while the multiplicative consistent IVFPR in Zhang [21] overlooked the parameter α_{ij} (i.e., set $\alpha_{ij}=1$ for i, j = 1,2,…,n). Hence, the later is a special case of the former and the former is a more general extension of the later.
(3) To obtain the interval weight vector, Zhang [21] constructed a goal programming model that only minimizes the deviation between the original IVFPR and the converted consistent one. In this paper, the geometric consistent index of IVFPR is introduced to derive the individual interval priority weights. To obtain the collective interval priority weights, a minimization deviation model is constructed and transformed into a parametric linear programming model to resolve considering DMs' preference principles. The proposed method is more flexible and comprehensive.
(4) The collective fitted errors obtained by the proposed method are smaller than those obtained by method [21]. Thus, the proposed method could retain more original information on the initial IVFPRs and is more reliable than method [21].

7. Conclusion

This paper aims to propose a GDM method with IVFPRs based on geometric consistency. Firstly, to measure the geometric consistency of IVFPR, the max-consistency index and the min-consistency index are defined, respectively. Combining these two indices, the geometric consistent index of IVFPR is presented by considering DM's risk attitude. Then a goal programming model is constructed to obtain the acceptable geometric consistent IVFPR from an unacceptable geometric consistent one. By constructing the membership function of the geometric consistent conditions for an IVFPR, a fuzzy logarithmic program is established to derive the interval priority weights. For GDM with IVFPRs, the individual acceptable geometric IVFPRs are derived from original IVFPRs and the individual interval priority weights are obtained by solving the corresponding fuzzy logarithmic programs. Through analyzing the similarity between DMs, DMs' weights are determined. A parametric linear programming model is developed to obtain the collective interval priority weights. Then the ranking order of alternatives is generated by the collective interval priority weights. Some examples are analyzed to illustrate the feasibility and effectiveness of the proposed method.

Further research is needed to address some significant issues. For instance, it is unclear how to deal with the consistency level and priority weights from the original incomplete IVFPRs provided by DMs. It is unknown to the determination of values of these parameters in the proposed method. After these issues are properly addressed, it would be worthwhile to investigate how the current framework can be adapted to handle these cases.

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