



Brief paper

Subsystem identification of multivariable feedback and feedforward systems[☆]

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ABSTRACT

We present a frequency-domain technique for identifying multivariable feedback and feedforward subsystems that are interconnected with a known subsystem. This subsystem identification algorithm uses closed-loop input–output data, but no other system signals are assumed to be measured. In particular, neither the feedback signal nor the outputs of the unknown subsystems are assumed to be measured. We use a candidate-pool approach to identify the feedback and feedforward transfer function matrices, while guaranteeing asymptotic stability of the identified closed-loop transfer function matrix. The main analytic result shows that if the data noise is sufficiently small and the candidate pool is sufficiently dense, then the parameters of the identified feedback and feedforward transfer function matrices are arbitrarily close to the true parameters.

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1. Introduction

Subsystem identification (SSID) is the process of building empirical models of unknown dynamic subsystems, which are interconnected with known dynamic subsystems. These connections can be series, parallel, or feedback. SSID relies on measured data to identify the unknown subsystems. However, not all input and output signals to the unknown subsystems are necessarily accessible, that is, available for measurement.

This paper is concerned with closed-loop SSID of unknown feedback and feedforward subsystems interconnected with a known subsystem as shown in Fig. 1. The exogenous input r and closed-loop output y are measured, whereas internal signals u and v are not assumed to be accessible. We note that closed-loop SSID is distinct from the well-studied problem of system identification in closed loop (Forsell & Ljung, 1999; Isermann & Münchhof, 2011; Van den Hof, 1998; Van den Hof & Schrama, 1995). Specifically, in SSID, the unknown subsystems have inputs or outputs that are inaccessible.

SSID has applications in biology and physics as well as human-in-the-loop systems. For example, many biological systems are

modeled by the interconnection of subsystems, which may be unknown and have inaccessible inputs and outputs (Roth, Sponberg, & Cowan, 2014). Similarly, physical systems are often modeled by a composition of subsystems, which are based on either physical laws or empirical information. For example, in D'Amato, Ridley, and Bernstein (2011), a large-scale physics-based model of the global ionosphere–thermosphere is improved by using measured data to estimate thermal conductivity, which can be regarded as an unknown feedback subsystem. In this application, the output of the unknown subsystem is inaccessible.

SSID also has application to modeling human behavior. For example, there is interest in modeling human-in-the-loop behavior for applications such as aircraft (Itoh & Suzuki, 2005; Nieuwenhuizen, Beykirch, Mulder, & Bülthoff, 2007; Nieuwenhuizen & Bülthoff, 2013; Olivari, Nieuwenhuizen, Venrooij, Bülthoff, & Pollini, 2012) and automobiles (Hellstrom & Jankovic, 2015; Macadam, 2003; Steen, Damveld, Happee, van Paassen, & Mulder, 2011). In addition, SSID methods can be used to model human behavior in motor control experiments, which study human learning (Drop, Pool, Damveld, van Paassen, & Mulder, 2013; Kiemel, Zhang, & Jeka, 2011; Laurence, Pool, Damveld, van Paassen, & Mulder, 2015; Zhang & Hoagg, 2016).

Closed-loop SSID of feedback and feedforward models is considered in D'Amato et al. (2011), Gillijns and De Moor (2006), Morozov et al. (2011) and Palanthandalam-Madapusi, Gillijns, De Moor, and Bernstein (2006). However, the identified feedback and feedforward models obtained from the methods in D'Amato et al. (2011), Gillijns and De Moor (2006), Morozov et al. (2011)

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$N_{fb} \in \mathbb{R}^{m \times l}[z]$, and $D, D_{fb} \in \mathbb{R}^{m \times m}[z]$. Without loss of generality, we assume that D and D_{fb} are monic. Thus, (4) can be expressed as

$$\tilde{G}(z) = N_y(z)\tilde{D}^{-1}(z)[N_{fb}(z) + z^{-n_{ff}}D_{fb}(z)N_{ff}(z)],$$

where

$$\tilde{D} \triangleq D_{fb}D + N_{fb}N_v \in \mathbb{R}^{m \times m}[z].$$

Define $d \triangleq \deg D$, $d_{fb} \triangleq \deg D_{fb}$, $n_y \triangleq \deg N_y$, $n_v \triangleq \deg N_v$, and $n_{fb} \triangleq \deg N_{fb}$. We make the following assumptions:

(A1) $d + d_{fb} > n_v + n_{fb}$.

(A2) $N > n_y + (m - 1)(d + d_{fb}) + d_{fb} + n_{ff}$.

(A3) If $\lambda \in \mathbb{C}$ and $\det \tilde{D}(\lambda) = 0$, then $|\lambda| < 1$.

Assumption (A1) states that $G_{fb}G_v$ is strictly proper. Assumption (A2) requires that the number N of frequency response data points is sufficiently large. Assumption (A3) implies that \tilde{G} is asymptotically stable. We also assume that n_{ff} , d_{fb} , and n_{fb} are known.

To formulate the SSID problem, define $a \triangleq l(n_{fb} + 1) + md_{fb}$ and $b \triangleq m(n_{ff} + 1)$, and consider the functions $\mathcal{N}_{ff}: \mathbb{C} \times \mathbb{R}^{b \times l} \rightarrow \mathbb{C}^{m \times l}$, $\mathcal{N}_{fb}: \mathbb{C} \times \mathbb{R}^{a \times m} \rightarrow \mathbb{C}^{m \times l}$, $\mathcal{D}_{fb}: \mathbb{C} \times \mathbb{R}^{a \times m} \rightarrow \mathbb{C}^{m \times m}$ given by

$$\mathcal{N}_{ff}(z, \beta) \triangleq ([z^{n_{ff}} \ \cdots \ z \ 1] \otimes I_m)\beta,$$

$$\mathcal{N}_{fb}(z, \phi) \triangleq \phi^T \begin{bmatrix} I_l \otimes [z^{n_{fb}} \ \cdots \ z \ 1]^T \\ 0_{md_{fb} \times l} \end{bmatrix},$$

$$\mathcal{D}_{fb}(z, \phi) \triangleq z^{d_{fb}}I_m + \phi^T \begin{bmatrix} 0_{l(m_{fb}+1) \times m} \\ I_m \otimes [z^{d_{fb}-1} \ \cdots \ z \ 1]^T \end{bmatrix},$$

where $\beta \in \mathbb{R}^{b \times l}$ contains the unknown parameters of \mathcal{N}_{ff} , and $\phi \in \mathbb{R}^{a \times m}$ contains the unknown parameters of \mathcal{N}_{fb} and \mathcal{D}_{fb} . Consider $\mathcal{G}_{ff}: \mathbb{C} \times \mathbb{R}^{b \times l} \rightarrow \mathbb{C}^{m \times l}$ and $\mathcal{G}_{fb}: \mathbb{C} \times \mathbb{R}^{a \times m} \rightarrow \mathbb{C}^{m \times l}$ given by

$$\mathcal{G}_{ff}(z, \beta) \triangleq z^{-n_{ff}}\mathcal{N}_{ff}(z, \beta),$$

$$\mathcal{G}_{fb}(z, \phi) \triangleq \mathcal{D}_{fb}^{-1}(z, \phi)\mathcal{N}_{fb}(z, \phi),$$

which, for each $\beta \in \mathbb{R}^{b \times l}$ and $\phi \in \mathbb{R}^{a \times m}$, are real rational transfer function matrices.

Let $\beta_* \in \mathbb{R}^{b \times l}$ and $\phi_* \in \mathbb{R}^{a \times m}$ be such that,

$$N_{ff}(z) \equiv \mathcal{N}_{ff}(z, \beta_*),$$

$$N_{fb}(z) \equiv \mathcal{N}_{fb}(z, \phi_*),$$

$$D_{fb}(z) \equiv \mathcal{D}_{fb}(z, \phi_*).$$

Thus, $\mathcal{G}_{ff}(z, \beta_*) \equiv G_{ff}(z)$ and $\mathcal{G}_{fb}(z, \phi_*) \equiv G_{fb}(z)$. Consider $\tilde{\mathcal{G}}: \mathbb{C} \times \mathbb{R}^{b \times l} \times \mathbb{R}^{a \times m} \rightarrow \mathbb{C}^{n \times l}$ given by

$$\tilde{\mathcal{G}}(z, \beta, \phi) \triangleq N_y(z)\tilde{D}^{-1}(z, \phi)[\mathcal{N}_{fb}(z, \phi) + z^{-n_{ff}}\mathcal{D}_{fb}(z, \phi)\mathcal{N}_{ff}(z, \beta)], \quad (6)$$

where

$$\tilde{D}(z, \phi) \triangleq \mathcal{D}_{fb}(z, \phi)D(z) + \mathcal{N}_{fb}(z, \phi)N_v(z). \quad (7)$$

Note that $\tilde{\mathcal{G}}(z, \beta, \phi)$ is the closed-loop transfer function obtained from β and ϕ . Thus, $\tilde{\mathcal{G}}(z, \beta_*, \phi_*) \equiv \tilde{G}(z)$.

Our objective is to determine β and ϕ such that \mathcal{G}_{ff} and \mathcal{G}_{fb} approximate G_{ff} and G_{fb} , respectively. To achieve this objective, we seek to minimize

$$J(\beta, \phi) = \sum_{k=1}^N \|\tilde{\mathcal{G}}(e^{j\theta_k}, \beta, \phi) - H(\theta_k)\|_F^2, \quad (8)$$

subject to the constraint that $\tilde{D}(z, \phi)$ is asymptotically stable, that is, $\phi \in \mathcal{S}$, where

$$\mathcal{S} \triangleq \{\phi \in \mathbb{R}^{a \times m} : \text{if } \lambda \in \mathbb{C} \text{ and } \det \tilde{D}(\lambda, \phi) = 0, \text{ then } |\lambda| < 1\}.$$

The cost (8) measures the difference between the data $\{H(\theta_k)\}_{k=1}^N$ and the closed-loop transfer function matrix obtained from the estimates \mathcal{G}_{ff} and \mathcal{G}_{fb} . The cost (8) and constraint $\phi \in \mathcal{S}$ are nonlinear and nonconvex in (β, ϕ) . If $\Gamma(e^{j\theta_k}) \equiv 0$, then $J(\beta_*, \phi_*) = 0$.

4. Subsystem identification algorithm

We now develop an SSID algorithm to estimate G_{ff} and G_{fb} . For each $k \in \mathcal{N}$, define $\sigma_k \triangleq e^{j\theta_k}$, and define

$$\mathcal{A}_k(\phi) \triangleq \sigma_k^{-n_{ff}}N_y(\sigma_k)\tilde{D}^{-1}(\sigma_k, \phi)\mathcal{D}_{fb}(\sigma_k, \phi)v(\sigma_k), \quad (9)$$

$$\mathcal{B}_k(\phi) \triangleq N_y(\sigma_k)\tilde{D}^{-1}(\sigma_k, \phi)\mathcal{N}_{fb}(\sigma_k, \phi) - H(\theta_k), \quad (10)$$

where $v(z) \triangleq [z^{n_{ff}} \ z^{n_{ff}-1} \ \cdots \ z \ 1] \otimes I_m$. Since $\text{vec}(AB) = (I \otimes A)\text{vec} B$, it follows from (6)–(10) that

$$\begin{aligned} J(\beta, \phi) &= \sum_{k=1}^N \|\mathcal{A}_k(\phi)\beta + \mathcal{B}_k(\phi)\|_F^2 \\ &= \sum_{k=1}^N \|[I_l \otimes \mathcal{A}_k(\phi)]\text{vec} \beta + \text{vec} \mathcal{B}_k(\phi)\|_2^2 \\ &= [\text{vec} \beta]^T \Omega_2(\phi)\text{vec} \beta + \Omega_1^T(\phi)\text{vec} \beta + \Omega_0(\phi), \end{aligned} \quad (11)$$

where

$$\Omega_0(\phi) \triangleq \sum_{k=1}^N \|\mathcal{B}_k(\phi)\|_F^2 \in \mathbb{R}, \quad (12)$$

$$\Omega_1(\phi) \triangleq 2\text{Re} \sum_{k=1}^N \text{vec}(\mathcal{A}_k^*(\phi)\mathcal{B}_k(\phi)) \in \mathbb{R}^{lb}, \quad (13)$$

$$\Omega_2(\phi) \triangleq I_l \otimes \text{Re} \sum_{k=1}^N \mathcal{A}_k^*(\phi)\mathcal{A}_k(\phi) \in \mathbb{R}^{lb \times lb}. \quad (14)$$

For the remaining of this paper, we assume that for all $\phi \in \mathcal{S}$, $\Omega_2(\phi)$ is positive definite. The following result provides a sufficient condition such that $\Omega_2(\phi)$ is positive definite. The proof is in [Appendix A](#).

Proposition 1. Consider Ω_2 given by (14), where (A1) and (A2) are satisfied. Assume that $\max_{z \in \mathbb{C}} \text{rank} N_y(z) = m \leq n$. Then, for all $\phi \in \mathcal{S}$, $\Omega_2(\phi)$ is positive definite.

Proposition 1 implies that if G_y is SISO (i.e., $m = n = 1$), then for all $\phi \in \mathcal{S}$, $\Omega_2(\phi)$ is positive definite. **Proposition 1** also implies that if the number m of measurements does not exceed the number n of controls and G_y has full normal rank, then for all $\phi \in \mathcal{S}$, $\Omega_2(\phi)$ is positive definite.

The next result shows that for each $\phi \in \mathcal{S}$, $J(\beta, \phi)$ has a unique global minimizer ([Sundaram, 1996, Chap. 1](#)).

Proposition 2. Consider J given by (11), where (A1) and (A2) are satisfied. Let $\phi \in \mathcal{S}$ and define

$$\beta_{\min} \triangleq -\frac{1}{2}\text{vec}^{-1}[\Omega_2^{-1}(\phi)\Omega_1(\phi)] \in \mathbb{R}^{b \times l}.$$

Let $\beta \in \mathbb{R}^{b \times l} \setminus \{\beta_{\min}\}$. Then, $J(\beta_{\min}, \phi) < J(\beta, \phi)$.

Let $\Phi \subseteq \mathcal{S}$ be a set with M elements. We call Φ the candidate pool. We now create a candidate sequence using the M elements in Φ . For all $i, j \in \mathcal{M} \triangleq \{1, 2, \dots, M\}$, let $\phi_i, \phi_j \in \Phi$ be such that if $i \neq j$, then $\phi_i \neq \phi_j$. Now, for all $i \in \mathcal{M}$, define the quadratic cost function

$$\mathcal{J}_i(\beta) \triangleq J(\beta, \phi_i).$$

Since $\phi_1, \dots, \phi_M \in \Phi \subseteq \mathcal{S}$, it follows that $\Omega_2(\phi_1), \dots, \Omega_2(\phi_M)$ are positive definite. Then, for each $i \in \mathcal{M}$, define

$$\beta_i \triangleq -\frac{1}{2}\text{vec}^{-1}[\Omega_2^{-1}(\phi_i)\Omega_1(\phi_i)] \in \mathbb{R}^{b \times l},$$

and it follows from **Proposition 2** that β_i is the unique global minimizer of \mathcal{J}_i . Next, let $\ell \in \mathcal{M}$ be the smallest integer such that

$\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$. Thus, the identified parameters are $\beta^+ \triangleq \beta_\ell$ and $\phi^+ \triangleq \phi_\ell$, and the identified transfer function matrices are

$$G_{\text{ff}}^+(z) \triangleq \mathcal{G}_{\text{ff}}(z, \beta^+) = z^{-n_{\text{ff}}} \mathcal{N}_{\text{ff}}(z, \beta^+),$$

$$G_{\text{fb}}^+(z) \triangleq \mathcal{G}_{\text{fb}}(z, \phi^+) = \mathcal{D}_{\text{fb}}^{-1}(z, \phi^+) \mathcal{N}_{\text{fb}}(z, \phi^+).$$

Note that $\arg \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$ is not necessarily unique. In this case, $\ell \in \mathcal{M}$ is the smallest integer such that $\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$. In practice, $\arg \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$ is generally unique. We now summarize this SSID method.

Algorithm 1. Consider the closed-loop transfer function matrix (4), where G_y , G_v , and $\{H(\theta_k)\}_{k=1}^N$ are known, and (A1)–(A3) are satisfied. Then, the subsystem identification algorithm is as follows:

- Step 1. Generate the candidate pool $\Phi \subseteq \mathcal{S}$ and candidate sequence $\{\phi_i\}_{i=1}^M$.
- Step 2. For each $i \in \mathcal{M}$, find the global minimizer of \mathcal{J}_i , which is $\beta_i = -\frac{1}{2} \text{vec}^{-1}[\Omega_2^{-1}(\phi_i) \Omega_1(\phi_i)] \in \mathbb{R}^{b \times l}$.
- Step 3. Find the smallest integer $\ell \in \mathcal{M}$ such that $\mathcal{J}_\ell(\beta_\ell) = \min_{i \in \mathcal{M}} \mathcal{J}_i(\beta_i)$.
- Step 4. The identified parameters are $\beta^+ = \beta_\ell$ and $\phi^+ = \phi_\ell$, and the identified transfer function matrices are $G_{\text{ff}}^+(z) = \mathcal{G}_{\text{ff}}(z, \beta^+)$, and $G_{\text{fb}}^+(z) = \mathcal{G}_{\text{fb}}(z, \phi^+)$.

5. Analysis of Algorithm 1

We assume N_{fb} and D_{fb} are left coprime, and impose the following assumption, which is stronger than (A2):

$$(A4) \quad N > n_y + (2m - 1)(d + d_{\text{fb}}) + n_{\text{ff}} + \max\{d_{\text{fb}}, n_{\text{fb}}\}.$$

Assumption (A4) requires that the number N of frequency response data points is sufficiently large. This assumption is used in the next result to obtain sufficient conditions on $\beta \in \mathbb{R}^{b \times l}$ and $\phi \in \mathbb{R}^{a \times m}$ such that $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$. The proof is in Appendix A.

Proposition 3. Let $\beta \in \mathbb{R}^{b \times l}$ and $\phi \in \mathbb{R}^{a \times m}$, and assume (A4) is satisfied. Then, $\sum_{k=1}^N \|\tilde{\mathcal{G}}(\sigma_k, \beta, \phi) - \tilde{G}(\sigma_k)\|_{\text{F}} = 0$ if and only if $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$.

The conditions in Proposition 3 are not sufficient to conclude that $\beta = \beta_*$ and $\phi = \phi_*$. The following example demonstrates this scenario.

Example 1. Let

$$G_y(z) = G_v(z) = \frac{1}{z - 0.7}, \quad (15)$$

$$G_{\text{ff}}(z) = -0.4, \quad G_{\text{fb}}(z) = \frac{1.2z + 0.2}{z - 1}, \quad (16)$$

and note that $\beta_* = -0.4$ and $\phi_* = [1.2 \quad 0.2 \quad -1]^T$. The closed-loop transfer function (4) is $\tilde{G}(z) = (0.8z + 0.6)/(z^2 - 0.5z + 0.9)$. Let $\beta = 0.3 \neq \beta_*$ and $\phi = [0.5 \quad 0.69 \quad -0.3]^T \neq \phi_*$, and it follows that $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$. \triangle

In Example 1, the SSID problem is not well posed, because (β_*, ϕ_*) cannot be uniquely determined from the noiseless frequency response data. See Isermann and Münchhof (2011, Chap. 13) for more details in the case with feedback only. We now impose an additional assumption to ensure that $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$ if and only if $\beta = \beta_*$ and $\phi = \phi_*$.

Let $\Psi \subseteq \mathbb{R}^{a \times m}$ be a compact set with no isolated points such that $\phi_* \in \Psi$. In practice, Ψ is used to generate the candidate pool. We assume Ψ is known, and

(A5) If $\beta \in \mathbb{R}^{b \times l}$, $\phi \in \Psi \cap \mathcal{S}$, and $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$, then $\beta = \beta_*$ and $\phi = \phi_*$.

Example 2. Consider G_y , G_v , G_{ff} , and G_{fb} given by (15) and (16), and note that $\beta_* = -0.4$ and $\phi_* = [1.2 \quad 0.2 \quad -1]^T$. Example 1 shows that $(\beta, \phi) = (\beta_*, \phi_*)$ is not the only point in $\mathbb{R} \times \mathcal{S}$ such that $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$. We now define $\Psi \triangleq [0, 2] \times [0, 2] \times [-1.5, -0.5]$, and note that $\phi_* \in \Psi$. Furthermore, it can be confirmed that $(\beta, \phi) = (\beta_*, \phi_*)$ is the only point in $\mathbb{R} \times (\mathcal{S} \cap \Psi)$ such that $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{G}(z)$. Therefore, (A5) is satisfied. \triangle

The following result addresses the case where ϕ_* is in the candidate pool Φ . This result demonstrates that a sufficiently small noise $\|\eta_*\|$ yields identified parameters β^+ and ϕ^+ such that $\phi^+ = \phi_*$ and β^+ is arbitrarily close to β_* . The proof is in Appendix B.

Theorem 1. Assume (A1)–(A5) are satisfied. Let $\Phi \subseteq (\Psi \cap \mathcal{S})$, and assume $\phi_* \in \Phi$. Let β^+ and ϕ^+ denote the identified parameters obtained from Algorithm 1 with the candidate pool Φ . Then, the following statements hold:

- (i) There exists $\delta_0 > 0$ such that if $\|\eta_*\| < \delta_0$, then $\phi^+ = \phi_*$. Moreover, for all $\epsilon > 0$, there exists $\delta \in (0, \delta_0)$ such that if $\|\eta_*\| < \delta$, then $\beta^+ \in \mathbb{B}_\epsilon(\beta_*)$.
- (ii) If $\eta_* = 0$, then $\beta^+ = \beta_*$ and $\phi^+ = \phi_*$.

We now extend the analysis to address the case where ϕ_* is not necessarily in the candidate pool Φ . Let $\rho \in (0, 1)$ be such that if $\lambda \in \mathbb{C}$ and $\det \tilde{\mathcal{D}}(\lambda, \phi_*) = 0$, then $|\lambda| < \rho$, and define

$$\mathcal{S}_\rho \triangleq \{\phi \in \mathbb{R}^{a \times m} : \text{if } \lambda \in \mathbb{C} \text{ and } \det \tilde{\mathcal{D}}(\lambda, \phi) = 0, \text{ then } |\lambda| < \rho\}.$$

Note that as ρ approaches 1, \mathcal{S}_ρ approaches the empty set. In practice, \mathcal{S}_ρ is used to generate the candidate pool, and ρ can be selected sufficiently closed to 1 to ensure that $\phi_* \in \mathcal{S}_\rho$.

In the following result, we consider Algorithm 1 with a sequence of candidate pools that converge to $\Psi \cap \mathcal{S}_\rho$, which is bounded and contains no isolated points (Zhang & Hoagg, 2016, Prop. 7). This result demonstrates that a sufficiently dense candidate pool and sufficiently small noise $\|\eta_*\|$ yield identified parameters β^+ and ϕ^+ that are arbitrarily close to β_* and ϕ_* . The proof is in Appendix B.

Theorem 2. Assume (A1)–(A5) are satisfied. For all $j \in \mathbb{Z}^+$, let $\Lambda_j \subseteq (\Psi \cap \mathcal{S}_\rho)$ be a finite set such that $\{\Lambda_j\}_{j=1}^\infty$ converges to $\Psi \cap \mathcal{S}_\rho$. For each $j \in \mathbb{Z}^+$, let β_j^+ and ϕ_j^+ denote the identified parameters obtained from Algorithm 1 with the candidate pool $\Phi = \Lambda_j$. Then, for all $\epsilon > 0$, there exist $\delta > 0$ and $L \in \mathbb{Z}^+$ such that if $\|\eta_*\| < \delta$ and $j > L$, then $\beta_j^+ \in \mathbb{B}_\epsilon(\beta_*)$ and $\phi_j^+ \in \mathbb{B}_\epsilon(\phi_*)$.

6. Numerical examples

We present examples, where $m = l = n = 2$. For all examples, let

$$D(z) = \text{diag}(z + 0.1, z - 0.2), \quad D_{\text{fb}}(z) = I_2$$

$$N_y(z) = N_v(z) = \text{diag}(1, -1),$$

and

$$N_{\text{fb}}(z) = \begin{bmatrix} 0.3 & 1 \\ 0.2 & 0.4 \end{bmatrix}, \quad N_{\text{ff}}(z) = \begin{bmatrix} z - 0.3 & z + 0.4 \\ 0.5z & 1 \end{bmatrix}.$$

Note that

$$\beta_* = \begin{bmatrix} 1 & 0.5 & -0.3 & 0 \\ 1 & 0 & 0.4 & 1 \end{bmatrix}^T, \quad \phi_* = \begin{bmatrix} 0.3 & 1 \\ 0.2 & 0.4 \end{bmatrix}^T.$$

Let $N = 20$, and for $k \in \mathcal{N}$, let $\theta_k = 0.02\pi k$. This example satisfies (A1)–(A4), and for any compact set $\Psi \subseteq \mathbb{R}^{a \times m}$ containing ϕ_* , (A5) is satisfied.

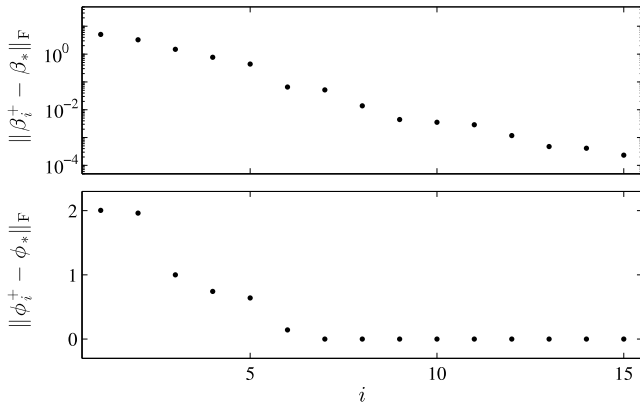


Fig. 3. Noisy data and $\phi_* \in \Phi$. For $i = 1, \dots, 15$, [Algorithm 1](#) is used with the candidate pool Λ_0 and data $\{H_i(\theta_k)\}_{k=1}^N$ to obtain β_i^+ and ϕ_i^+ . For $i \geq 7$, $\phi_i^+ = \phi_*$, and for sufficiently large i , $\|\beta_i^+ - \beta_*\|_F$ is arbitrarily small.

Example 3. Consider the case with $\phi_* \in \Phi$ and noiseless data. Define the candidate pool $\Lambda_0 \triangleq \{\phi \in \mathbb{R}^{2 \times 2} : \text{for } i, j \in \{1, 2\}, \phi_{(i,j)} \in \{-0.5 + 0.1k\}_{k=0}^{20} \cap \mathcal{S}\}$, and note that $\phi_* \in \Lambda_0$. [Algorithm 1](#) is used with the candidate pool $\Phi = \Lambda_0$ to obtain $\beta^+ = \beta_*$ and $\phi^+ = \phi_*$, which agrees with (ii) of [Theorem 1](#). \triangle

Example 4. Consider the case with $\phi_* \in \Phi$ and noisy data. For $i = 1, \dots, 15$, let $\Gamma_i(z) \in \mathbb{C}^{2 \times 2}$ be the noise, and define the noise-to-signal ratio $R_i \triangleq \frac{1}{N} \sum_{k=1}^N \|\Gamma_i(\sigma_k)\|_F / \|\tilde{G}(\sigma_k)\|_F$. For $i = 1, \dots, 15$, the frequency response data is $H_i(\theta_k) \triangleq \tilde{G}(\sigma_k) + \Gamma_i(\sigma_k)$. In this example, $\Gamma_1, \dots, \Gamma_{15}$ are randomly generated such that $R_1 > R_2 > \dots > R_{15}$. Specifically, $R_1 = 2.39$, $R_2 = 1.29$, $R_4 = 0.28$, $R_7 = 3.76 \times 10^{-2}$, and $R_{15} = 1.42 \times 10^{-4}$. For $i = 1, \dots, 15$, [Algorithm 1](#) is used with the candidate pool $\Phi = \Lambda_0$ and data $\{H_i(\theta_k)\}_{k=1}^N$ to obtain the identified parameters β_i^+ and ϕ_i^+ . [Fig. 3](#) shows that for $i \geq 7$, $\phi_i^+ = \phi_*$, and for sufficiently large i , $\|\beta_i^+ - \beta_*\|_F$ is arbitrarily small, which agrees with (i) of [Theorem 1](#). \triangle

Example 5. Consider the case with $\phi_* \notin \Phi$ and noisy data. For $j = 1, \dots, 18$, define the candidate pool $\Lambda_j \triangleq \{\phi \in \mathbb{R}^{2 \times 2} : \text{for } i, h \in \{1, 2\}, \phi_{(i,h)} \in \{-0.5 + 2k/(1+j)\}_{k=0}^{1+j} \cap \mathcal{S}_\rho\}$, where $\rho = 0.99$, and note that for $j = 1, \dots, 18$, $\phi_* \notin \Lambda_j$. For $i = 1, \dots, 15$ and $j = 1, \dots, 18$, [Algorithm 1](#) is used with the candidate pool $\Phi = \Lambda_j$ and data $\{H_i(\theta_k)\}_{k=1}^N$ to obtain the identified parameters $\beta_{j,i}^+$ and $\phi_{j,i}^+$. [Fig. 4](#) shows that for sufficient large j and i , $\|\beta_{j,i}^+ - \beta_*\|_F$ and $\|\phi_{j,i}^+ - \phi_*\|_F$ are arbitrarily small, which agrees with [Theorem 2](#). \triangle

7. Computational complexity

The computational complexity of [Algorithm 1](#) is dominated by Step 2, where a quadratic minimization problem is solved by M times. We use the modified Gram Schmidt method ([Ford, 2014, Chap. 14](#)) to perform this quadratic minimization. It follows from [Ford \(2014, Chap. 14\)](#) that the computational complexity of [Algorithm 1](#) is $C \triangleq M[4Nnm^2l^3(n_{\text{ff}} + 1)^2 + 2Nnm^2(n_{\text{ff}} + 1)]$ flops.

We compare the computational complexity of [Algorithm 1](#) with the SSID algorithm in [Zhang and Hoagg \(2016\)](#). The SSID method in [Zhang and Hoagg \(2016\)](#) also uses a candidate pool approach; however, \mathcal{G}_{ff} is parameterized as an IIR transfer function, and the method only applies to subsystems that are SISO (i.e., $v = y$ and $l = m = n = 1$). For SISO subsystems, the computational complexity of [Algorithm 1](#) is $C = M[4N(n_{\text{ff}} + 1)^2 + 2N(n_{\text{ff}} + 1)]$ flops.

Next, consider ([Zhang & Hoagg, 2016, Alg. 1](#)), where \hat{N} denotes the number of frequency response data points, M_{fb} denotes the

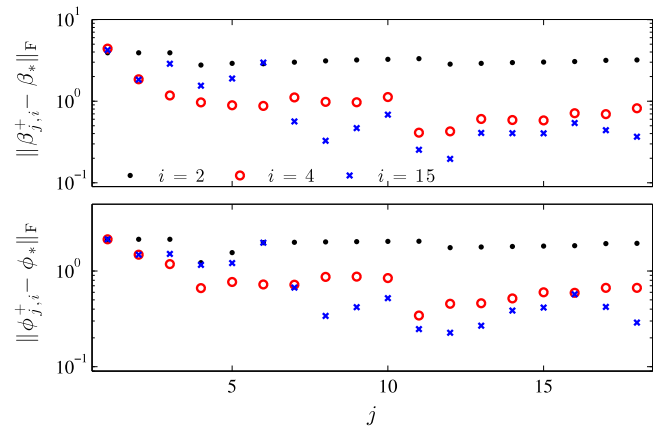


Fig. 4. Noisy data and $\phi_* \notin \Phi$. For $i = 2, 4, 15$, and $j = 1, \dots, 18$, [Algorithm 1](#) is used with the candidate pool Λ_j and data $\{H_i(\theta_k)\}_{k=1}^N$ to obtain $\beta_{j,i}^+$ and $\phi_{j,i}^+$. For sufficient large j and i , $\|\beta_{j,i}^+ - \beta_*\|_F$ and $\|\phi_{j,i}^+ - \phi_*\|_F$ are arbitrarily small.

number of elements in the feedback candidate pool, and M_{ff} denotes the number of elements in the feedforward candidate pool. Let \hat{n}_{ff} denote the degree of numerator of \mathcal{G}_{ff} . The computational complexity of ([Zhang & Hoagg, 2016, Alg. 1](#)) is $\hat{C} = M_{\text{ff}}M_{\text{fb}}[4\hat{N}(\hat{n}_{\text{ff}} + 1)^2 + 2\hat{N}(\hat{n}_{\text{ff}} + 1)]$ flops.

To compare the computational complexities of these algorithms, we assume these algorithms use the same frequency response data and feedback candidate pool, which implies that $N = \hat{N}$ and $M = M_{\text{fb}}$. Thus, the computational complexity ratio is

$$\mathcal{R} \triangleq \frac{C}{\hat{C}} = \frac{2(n_{\text{ff}} + 1)^2 + (n_{\text{ff}} + 1)}{M_{\text{ff}}[2(\hat{n}_{\text{ff}} + 1)^2 + (\hat{n}_{\text{ff}} + 1)]}.$$

In general, $\hat{n}_{\text{ff}} \leq n_{\text{ff}}$ and $M_{\text{ff}} \gg 1$, which result $\mathcal{R} < 1$.

Example 6. Let

$$G_y(z) = G_v(z) = \frac{0.4}{z - 0.9},$$

$$G_{\text{ff}}(z) = \frac{0.2(z - 0.77)}{(z - 0.83)(z - 0.89)}, \quad G_{\text{fb}}(z) = \frac{0.32}{z - 0.81}.$$

Let $N = 31$, and for $k \in \mathcal{N}$, let $\theta_k = 0.008\pi(k - 1)$. Consider the candidate pool $\Lambda_0 \triangleq \{\phi \in \mathbb{R}^2 : \phi_{(1)} \in \{-2 + 0.05k\}_{k=0}^{40}, \phi_{(2)} \in \{-2 + 0.05k\}_{k=0}^{80}\} \cap \mathcal{S}$, which contains 297 elements.

First, we use [Algorithm 1](#) with the candidate pool Λ_0 . For $n_{\text{ff}} = 1, 2, \dots, 17$, we identify feedforward and feedback controllers, where \mathcal{G}_{ff} is FIR even though G_{ff} is IIR.

Next, consider ([Zhang & Hoagg, 2016, Alg. 1](#)), where \mathcal{G}_{ff} is parameterized as IIR with $\hat{n}_{\text{ff}} = 1$. Consider feedforward candidate pool $F = \{[x_1 + x_2 \quad x_1 x_2]^T \in \mathbb{R}^2 : x_1, x_2 \in \{-0.05k\}_{k=0}^{19}\}$, which contains $M_{\text{ff}} = 210$ elements. We use ([Zhang & Hoagg, 2016, Alg. 1](#)) with the candidate pool $F \times \Lambda_0$ to identify feedback and feedforward controllers.

[Fig. 5](#) shows the identification errors

$$\mathcal{E}_{\text{fb}} \triangleq \int_{\theta_1}^{\theta_N} |G_{\text{fb}}^+(e^{j\theta}) - G_{\text{fb}}(e^{j\theta})| d\theta,$$

$$\mathcal{E}_{\text{ff}} \triangleq \int_{\theta_1}^{\theta_N} |G_{\text{ff}}^+(e^{j\theta}) - G_{\text{ff}}(e^{j\theta})| d\theta,$$

for each algorithm. In this example, for $n_{\text{ff}} = 14, \dots, 17$, [Algorithm 1](#) yields errors \mathcal{E}_{fb} and \mathcal{E}_{ff} that are comparable to those obtained from ([Zhang & Hoagg, 2016, Alg. 1](#)). [Fig. 5](#) also shows that for $n_{\text{ff}} = 1, 2, \dots, 17$, $\mathcal{R} < 0.32$. \triangle

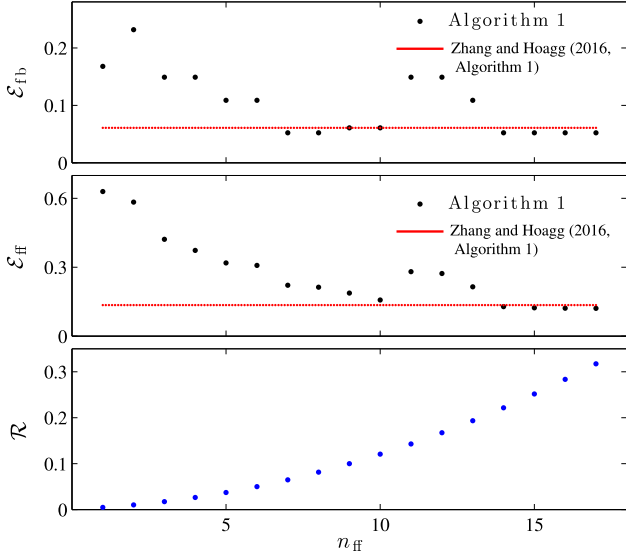


Fig. 5. Computation complexity. For $n_{\text{ff}} = 14, \dots, 17$, Algorithm 1 yields errors ε_{fb} and ε_{ff} that are comparable to those obtained from (Zhang & Hoagg, 2016, Alg. 1). For $n_{\text{ff}} = 1, 2, \dots, 17$, $\mathcal{R} < 0.32$.

8. Conclusion

We presented a discrete-time SSID algorithm for identifying MIMO feedback and feedforward subsystems. This SSID algorithm uses closed-loop input–output data but does not rely on the measurement of any internal signals. The method ensures asymptotic stability of the identified closed-loop transfer function matrix. The analytic results are Theorems 1 and 2, which provide properties of the SSID algorithm. In particular, Theorem 2 shows that the parameters of the identified feedback and feedforward transfer function matrices are arbitrarily close to the true parameters if the candidate pool is sufficiently dense and the data noise is sufficiently small.

Appendix A. Proofs of Propositions 1 and 3

Proof of Proposition 1. Let $\phi \in \mathcal{S}$. Then, for all $k \in \mathcal{N}$, $\tilde{\mathcal{D}}(\sigma_k, \phi)$ is nonsingular, and thus, it follows from (9) and (14) that $\Omega_2(\phi)$ is well defined and positive semidefinite. Assume for contradiction that there exists $x \in \mathbb{R}^b \setminus \{0\}$ such that $x^T [\text{Re} \sum_{k=1}^N \mathcal{A}_k^*(\phi) \mathcal{A}_k(\phi)] x = 0$. Let $k \in \mathcal{N}$, and it follows that $\mathcal{A}_k(\phi)x = 0$. Define $\mathcal{E}(z) \triangleq N_y(z) \tilde{\mathcal{D}}^A(z, \phi) \mathcal{D}_{\text{fb}}(z, \phi) \nu(z)x \in \mathbb{R}^n[z]$. Thus, (9) implies that $0 = \mathcal{E}(\sigma_k) / [z_k^{n_{\text{ff}}} \det \tilde{\mathcal{D}}(\sigma_k, \phi)]$, which implies that $\mathcal{E}(\sigma_k) = 0$. Since $\deg \tilde{\mathcal{D}} = d + d_{\text{fb}}$, it follows that $\deg \tilde{\mathcal{D}}^A \leq (m-1)(d + d_{\text{fb}})$. Since, in addition, $\deg \nu(z)x \leq n_{\text{ff}}$, it follows from (A2) that $\deg \mathcal{E} \leq n_y + (m-1)(d + d_{\text{fb}}) + d_{\text{fb}} + n_{\text{ff}} < N$. Since $\mathcal{E}(\sigma_1) = \dots = \mathcal{E}(\sigma_N) = 0$ and $\deg \mathcal{E} < N$, it follows that $\mathcal{E} = 0$.

Define $\mathcal{T} \triangleq \{z \in \mathbb{C} : \min\{\text{rank } N_y(z), \text{rank } \tilde{\mathcal{D}}^A(z, \phi), \text{rank } \mathcal{D}_{\text{fb}}(z, \phi)\} < m\}$. Since, for all $z \in \mathbb{C} \setminus \mathcal{T}$, $\text{rank } N_y(z) = m \leq n$, it follows that N_y has full column rank, which implies that for all $z \in \mathbb{C} \setminus \mathcal{T}$, $N_y(z)$ is left invertible. Thus, for all $z \in \mathbb{C} \setminus \mathcal{T}$, $\tilde{\mathcal{D}}^A(z, \phi) \mathcal{D}_{\text{fb}}(z, \phi) \nu(z)x = 0$. Since for all $z \in \mathbb{C} \setminus \mathcal{T}$, $\tilde{\mathcal{D}}^A(z, \phi)$ and $\mathcal{D}_{\text{fb}}(z, \phi)$ are nonsingular, it follows that for all $z \in \mathbb{C} \setminus \mathcal{T}$, $\nu(z)x = 0$. Finally, the structure of ν implies that $x = 0$, which is a contradiction. Thus, $\text{Re} \sum_{k=1}^N \mathcal{A}_k^*(\phi) \mathcal{A}_k(\phi)$ is positive definite. Therefore, it follows from (14) that $\Omega_2(\phi)$ is positive definite. \square

Proof of Proposition 3. Let $\beta \in \mathbb{R}^{b \times l}$ and $\phi \in \mathbb{R}^{m \times a}$, and define $\vartheta: \mathbb{C} \rightarrow \mathbb{C}^{n \times l}$ by $\vartheta(z) \triangleq \tilde{\mathcal{G}}(z, \beta, \phi) - \tilde{\mathcal{G}}(z)$. Next, define $\mathcal{P}(z) \triangleq N_y(z) \tilde{\mathcal{D}}^A(z, \phi) [z^{n_{\text{ff}}} \mathcal{N}_{\text{fb}}(z, \phi) + \mathcal{D}_{\text{fb}}(z, \phi) \mathcal{N}_{\text{ff}}(z, \beta)] \in \mathbb{R}^{n \times l}[z]$, $\mathcal{Q}(z) \triangleq N_y(z) \tilde{\mathcal{D}}^A(z, \phi_*) \times [z^{n_{\text{ff}}} \mathcal{N}_{\text{fb}}(z, \phi_*) + \mathcal{D}_{\text{fb}}(z, \phi_*) \mathcal{N}_{\text{ff}}(z, \beta_*)] \in$

$\mathbb{R}^{n \times l}[z]$, and $\mathcal{H}(z) \triangleq \mathcal{P}(z) \text{diag } \tilde{\mathcal{D}}(z, \phi_*) - \mathcal{Q}(z) \text{diag } \tilde{\mathcal{D}}(z, \phi) \in \mathbb{R}^{n \times l}[z]$. Note that $\vartheta(z) = \mathcal{H}(z) / [z^{n_{\text{ff}}} \det \tilde{\mathcal{D}}(z, \phi) \det \tilde{\mathcal{D}}(z, \phi_*)]$. Since $\sum_{k=1}^N \|\vartheta(\sigma_k)\|_F = 0$, it follows that for all $k \in \mathcal{N}$, $\vartheta(\sigma_k) = 0$, which implies that $\mathcal{H}(\sigma_k) = 0$. Since $\deg \det \tilde{\mathcal{D}}(z, \phi) = \deg \det \tilde{\mathcal{D}}(z, \phi_*) = m(d + d_{\text{fb}})$ and $\deg \mathcal{P}, \deg \mathcal{Q} \leq n_{\text{ff}} + n_y + (m-1)(d + d_{\text{fb}}) + \max\{d_{\text{fb}}, n_{\text{fb}}\}$, it follows that $\deg \mathcal{H} \leq n_y + (2m-1)(d + d_{\text{fb}}) + n_{\text{ff}} + \max\{d_{\text{fb}}, n_{\text{fb}}\}$. Since for all $k \in \mathcal{N}$, $\mathcal{H}(\sigma_k) = 0$, and (A4) implies that $\deg \mathcal{H} < N$, it follows that $\mathcal{H} = 0$, which implies that $\vartheta = 0$. Thus, $\tilde{\mathcal{G}}(z, \beta, \phi) \equiv \tilde{\mathcal{G}}(z)$. \square

Appendix B. Proofs of Theorems 1 and 2

The following notation is needed in the proofs of Theorems 1 and 2. Define $\hat{\Omega}_1: \mathcal{S} \times \mathbb{C}^{n \times lN} \rightarrow \mathbb{R}^{lb}$ by $\hat{\Omega}_1(\phi, \eta) \triangleq \Omega_1(\phi) + 2\text{Re} \sum_{k=1}^N \text{vec } \mathcal{A}_k^*(\phi) [H(\theta_k) - \tilde{\mathcal{G}}(\sigma_k) - \eta_k]$, where $\eta_1, \dots, \eta_N \in \mathbb{C}^{n \times l}$ and $\eta \triangleq [\eta_1 \ \dots \ \eta_N] \in \mathbb{C}^{n \times lN}$. Note that $\hat{\Omega}_1(\phi, \eta_*) = \Omega_1(\phi)$. Thus, $\hat{\Omega}_1$ is a function not only of ϕ but also the noise η . Define $\hat{J}: \mathbb{R}^{b \times l} \times \mathcal{S} \times \mathbb{C}^{n \times lN} \rightarrow [0, \infty)$, $\hat{\theta}: \mathcal{S} \times \mathbb{C}^{n \times lN} \rightarrow \mathbb{R}^{b \times l}$, and $\hat{Q}: \mathcal{S} \times \mathbb{C}^{n \times lN} \rightarrow [0, \infty)$ by

$$\hat{J}(\beta, \phi, \eta) \triangleq \sum_{k=1}^N \|\tilde{\mathcal{G}}(\sigma_k, \beta, \phi) - \tilde{\mathcal{G}}(\sigma_k) - \eta_k\|_F^2 \in \mathbb{R}, \quad (17)$$

$$\hat{\theta}(\phi, \eta) \triangleq -\frac{1}{2} \text{vec}^{-1} \left[\Omega_2^{-1}(\phi) \hat{\Omega}_1(\phi, \eta) \right] \in \mathbb{R}^{b \times l}, \quad (18)$$

$$\hat{Q}(\phi, \eta) \triangleq \hat{J}(\hat{\theta}(\phi, \eta), \phi, \eta) \in \mathbb{R}. \quad (19)$$

Note that $\hat{J}(\beta, \phi, \eta_*) = J(\beta, \phi)$.

It follows from (6), (9), (10), and (17)–(19) that

$$\hat{\theta}(\phi_*, 0) = -\frac{1}{2} \text{vec}^{-1} \left[\Omega_2^{-1}(\phi_*) \hat{\Omega}_1(\phi_*, 0) \right] = \beta_*, \quad (20)$$

$$\hat{Q}(\phi_*, 0) = \hat{J}(\beta_*, \phi_*, 0) = 0. \quad (21)$$

Proof of Theorem 1. To prove (i), let $\phi \in \mathcal{P} \setminus \{\phi_*\}$, it follows from (19), Proposition 3 and (A5) that $\hat{Q}(\phi, 0) > 0$. Define $U \triangleq \min_{x \in \mathcal{P} \setminus \{\phi_*\}} \hat{Q}(x, 0) > 0$. It can be shown that \hat{Q} is continuous on $\mathcal{S} \times \mathbb{C}^{n \times lN}$. Thus, for each $j \in \mathcal{M}$, $\hat{Q}(\phi_j, \cdot)$ is continuous on $\mathbb{C}^{n \times lN}$, which implies that, for each $j \in \mathcal{M}$, there exists $\delta_j > 0$ such that for all $\eta \in \mathbb{B}_{\delta_j}(0)$, $|\hat{Q}(\phi_j, \eta) - \hat{Q}(\phi_j, 0)| < U/2$. Define $\delta_0 \triangleq \min_{j \in \mathcal{M}} \delta_j$, and assume that $\|\eta_*\| < \delta_0$. Since $\phi_* \in \mathcal{P}$, it follows that there exists $i \in \mathcal{M}$ such that $\phi_i = \phi_*$. Since $\hat{Q}(\phi_i, 0) = \hat{Q}(\phi_*, 0) = 0$, it follows that $\hat{Q}(\phi_i, \eta_*) = |\hat{Q}(\phi_i, \eta_*) - \hat{Q}(\phi_i, 0)| < U/2$. Let $j \in \mathcal{M} \setminus \{i\}$, and it follows that $-U/2 < \hat{Q}(\phi_j, \eta_*) - \hat{Q}(\phi_j, 0)$, which implies that $\hat{Q}(\phi_j, \eta_*) > \hat{Q}(\phi_j, 0) - U/2$. Since, in addition, $\hat{Q}(\phi_j, 0) \geq U$, it follows that $\hat{Q}(\phi_j, \eta_*) > U/2$. Thus, $\hat{Q}(\phi_i, \eta_*) < \hat{Q}(\phi_j, \eta_*)$, which implies that $\mathcal{J}_i(\beta_i) < \mathcal{J}_j(\beta_j)$ using (8) and (17)–(19). Therefore, it follows from Algorithm 1 that $\phi^+ = \phi_i = \phi_*$ and $\beta^+ = \beta_i = \hat{\theta}(\phi_*, \eta_*)$.

Let $\epsilon > 0$. It can be shown that $\hat{\theta}$ is continuous on $\mathcal{S} \times \mathbb{C}^{n \times lN}$. Thus, $\hat{\theta}(\phi_*, \cdot)$ is continuous on $\mathbb{C}^{n \times lN}$. Therefore, there exists $\delta \in (0, \delta_0)$ such that for all $\eta \in \mathbb{B}_\delta(0)$, $\hat{\theta}(\phi_*, \eta) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0))$. Finally, assume $\|\eta_*\| < \delta$. Since $\beta^+ = \hat{\theta}(\phi_*, \eta_*)$, it follows from (20) that $\beta^+ \in \mathbb{B}_\epsilon(\beta_*)$, which confirms (i).

To prove (ii), assume $\eta_* = 0$. Thus, $\|\eta_*\| = 0 < \delta_0$ and part (i) implies that $\phi^+ = \phi_*$. Since $\eta_* = 0$, it follows from (20) that $\beta^+ = \hat{\theta}(\phi_*, 0) = \beta_*$. \square

Proof of Theorem 2. Let $\epsilon > 0$. It can be shown that $\hat{\theta}$ is continuous on $\mathcal{S} \times \mathbb{C}^{n \times lN}$. Since, in addition, $\phi_* \in \mathcal{S}$, it follows that there exists $\delta_0 > 0$ such that for all $\phi \in \mathbb{B}_{\delta_0}(\phi_*)$ and all $\eta \in \mathbb{B}_{\delta_0}(0)$,

$$\hat{\theta}(\phi, \eta) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0)). \quad (22)$$

Define $\epsilon_1 \triangleq \min\{\epsilon, \delta_0\}$ and $\Lambda_c \triangleq \overline{\Psi \cap \mathcal{S}_\rho}$. Using the process in Zhang and Hoagg (2016, Prop. 8), it can be shown that $\Lambda_c \subseteq \mathcal{S}$ is compact. Since Λ_c is compact, and $\{x \in \mathbb{R}^{a \times m} : \|x - \phi_*\| \geq \epsilon_1\}$ is closed, it follows that $\Lambda_{\epsilon_1} \triangleq \Lambda_c \setminus \mathbb{B}_{\epsilon_1}(\phi_*) = \Lambda_c \cap \{x \in \mathbb{R}^{a \times m} : \|x - \phi_*\| \geq \epsilon_1\}$ is compact.

Let $c > \delta_0$, define $C \triangleq \{x \in \mathbb{C}^{n \times IN} : \|x\| \leq c\}$, and note that \hat{Q} is continuous on $\Lambda_{\epsilon_1} \times C$. Next, define $\Theta : C \rightarrow [0, \infty)$ by $\Theta(\eta) \triangleq \min_{\phi \in \Lambda_{\epsilon_1}} \hat{Q}(\phi, \eta)$, which exists because Λ_{ϵ_1} is compact and \hat{Q} is continuous on $\Lambda_{\epsilon_1} \times C$ (Beals, 2004, Thm. 7.7). It follows from Proposition 3 and (A5) that $\Theta(0) > 0$. Since \hat{Q} is continuous on $\Lambda_{\epsilon_1} \times C$, and Λ_{ϵ_1} and C are compact, it follows from Sundaram (1996, Thm. 9.14) that Θ is continuous on C . Furthermore, since \hat{Q} is continuous on $\mathcal{S} \times \mathbb{C}^{n \times IN}$, it follows that $\hat{Q}(\phi_*, \cdot)$ is continuous on C . Thus, $W : C \rightarrow \mathbb{R}$ defined by $W(\eta) \triangleq \Theta(\eta) - \hat{Q}(\phi_*, \eta)$ is continuous on C . Note that (21) implies that $W(0) = \Theta(0) - \hat{Q}(\phi_*, 0) = \Theta(0) > 0$. Therefore, it follows that there exists $\delta_1 \in (0, c)$ such that for all $\eta \in \mathbb{B}_{\delta_1}(0)$, $W(\eta) > 0$. Define $\delta \triangleq \min\{\delta_0, \delta_1\} > 0$ and assume $\|\eta_*\| < \delta$. Then, $W(\eta_*) > 0$.

Since $W(\eta_*) > 0$ and $\hat{Q}(\cdot, \eta_*)$ is continuous on Λ_c , it follows from the continuity of $\hat{Q}(\cdot, \eta_*)$ that there exists $\delta_2 > 0$ such that for all $\phi \in (\Lambda_c \cap \mathbb{B}_{\delta_2}(\phi_*))$, $|\hat{Q}(\phi, \eta_*) - \hat{Q}(\phi_*, \eta_*)| < W(\eta_*)$. Thus, for all $\phi \in (\Lambda_c \cap \mathbb{B}_{\delta_2}(\phi_*))$, $\hat{Q}(\phi, \eta_*) - \hat{Q}(\phi_*, \eta_*) \leq |\hat{Q}(\phi, \eta_*) - \hat{Q}(\phi_*, \eta_*)| < W(\eta_*) = \Theta(\eta_*) - \hat{Q}(\phi_*, \eta_*)$, which implies that $\hat{Q}(\phi, \eta_*) < \Theta(\eta_*)$. (23)

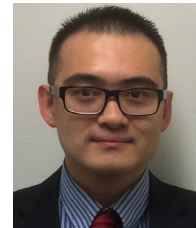
Since $\{A_j\}_{j=1}^\infty$ converges to $(\Psi \cap \mathcal{S}_\rho) \subseteq \Lambda_c$, it follows from Definition 1 that there exists a sequence $\{\phi_j : \phi_j \in A_j\}_{j=1}^\infty$ and $L \in \mathbb{Z}^+$ such that for all $j > L$, $\phi_j \in \mathbb{B}_{\min\{\epsilon_1, \delta_2\}}(\phi_*)$. Thus, it follows from (23) that for all $j > L$, $\hat{Q}(\phi_j, \eta_*) < \Theta(\eta_*)$.

Let $j \in \mathbb{Z}^+$ be such that $j > L$. It follows from Algorithm 1, (17)–(19), and (23) that $\hat{Q}(\phi_j^+, \eta_*) \leq \hat{Q}(\phi_j, \eta_*) < \Theta(\eta_*)$. Assume for contradiction that $\phi_j^+ \notin \mathbb{B}_{\epsilon_1}(\phi_*)$, which implies that $\phi_j^+ \in \Lambda_{\epsilon_1}$. Thus, $\Theta(\eta_*) = \min_{\phi \in \Lambda_{\epsilon_1}} \hat{Q}(\phi, \eta_*) \leq \hat{Q}(\phi_j^+, \eta_*) < \Theta(\eta_*)$, which is a contradiction. Therefore, $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_\epsilon(\phi_*)$. Since $\phi_j^+ \in \mathbb{B}_{\epsilon_1}(\phi_*) \subseteq \mathbb{B}_{\delta_0}(\phi_*)$, it follows from (20) and (22) that $\beta_j^+ = \hat{\theta}(\phi_j^+, \eta_*) \in \mathbb{B}_\epsilon(\hat{\theta}(\phi_*, 0)) = \mathbb{B}_\epsilon(\beta_*)$. \square

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