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MATHEMATICAL PHYSICS

# Quasi-Periodic and Chaotic Relaxation Oscillations in a Laser Model with Variable Delayed Optoelectronic Feedback

**E. V. Grigorieva[a,c] and S. A. Kashchenko[b,c*]**

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**Abstract**—We derive asymptotically discrete mappings which determine the dynamics of relaxation spikes in a model of laser with optoelectronic feedback in the pump circuit. The time delay in the feedback path varies periodically. Bifurcations of mapping's attractors correspond to the appearance of quasi-periodic and chaotic spiking with special properties.

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## 1. FORMULATION OF THE PROBLEM

Delayed feedback (FB) is widely used to control the dynamics of nonlinear optical and electro-optical systems [1, 2]. Recently, its modifications have been intensively studied, specifically, the method of high-frequency periodic modulation of the time delay in the FB loop aimed at the stabilization of an unstable equilibrium [3, 4]. On the other hand, time delay variations can be used to derive oscillation modes with prescribed characteristics and are of interest in the development of methods for optical data processing and coding, in optical vibrometry [5–8], and in other applications.

In this paper, we study the relaxation spikes in the model of semiconductor laser dynamics with optoelectronic FB proposed in [9]. Additionally, we assume that the time delay varies periodically, namely,

$$\frac{du}{dt} = vu(y-1), \quad \frac{dy}{dt} = q + \gamma u(g(t)) - y - yu, \qquad (1)$$

where $u$ and $y$ are proportional to the photon density and the population inversion, respectively; $v$ is the ratio of the photon decay rate in the resonator to the population relaxation rate; the resonator loss is normalized to unity; $q$ is the constant pumping rate; and $t$ is the current time normalized by the relaxation time of the population inversion. The optoelectronic FB is given by the term $\gamma u(g(t))$, where $\gamma$ is the feedback factor; the delayed argument has the form $g(t) = t - (\tau_0 + B\cos\Phi(t))$; $\Phi(t) = \omega t + \varphi$ is the modulator phase; $\tau_0$ is the constant time of radiation transformation in the FB loop; $B$ and $\omega$ are the amplitude and frequency of the time delay modulation, respectively; and $\varphi$ is the initial phase of the modulating signal. In what follows, we make the natural assumptions that $B < tau_0$ and $\omega < \frac{1}{B}$.

Importantly, for semiconductor lasers (and other lasers of class B), the parameter $v \sim 10^3$ is large, while the other parameter values are on the order of 1. Accordingly, the problem arises of asymptotic analysis of relaxation oscillations in the system as $v \to \infty$. The duration of pulses of the function $u(t)$ (generation spikes) tends to zero, while their amplitude tends to infinity. A feature of system (1) is that the standard research techniques for singularly perturbed systems are not applicable directly, since the corresponding degenerate system (at $v^{-1} = 0$) yields no information on the solutions for $v \gg 1$. Here, we propose a special method for reducing differential-difference equations to discrete mappings that determines the dynamics of oscillation spikes in the original system.

Previously, the method was used to study system (1) with a constant time delay [10]. It was shown that the following solutions are possible in this system:

(i) slowly oscillating (SO) relaxation solutions, in which the pulse ratio is greater than the time delay;

(ii) rapidly oscillating (RO) solutions, in which the pulse ratio is less than the time delay;

[a] Belarus State Economic University, Minsk, 220070 Belarus
[b] Yaroslavl State University, Yaroslavl, 150000 Russia
[c] National Research Nuclear University "MEPhI," Moscow, 115409 Russia
*e-mail: kasch@uniyar.ac.ru

(iii) mixed SO and RO solutions.

Below, we obtain asymptotics of such regimes and corresponding mappings in the case of a periodically varying time delay.

## 2. SLOWLY OSCILLATING SOLUTIONS

In SO solutions, the pulse ratio is greater than the time delay. The set of initial conditions is defined as

$$S(c,\varphi) = \{y(0) = c, \Phi(0) = \varphi,\\ u(s) = \psi(s), s \in [-\tau_0 - B, 0]\}, \quad (2)$$

where $c \in (1,q]$, $\varphi \in [0,2\pi]$, and the functions $\psi(s) \in S_0$ have the following properties:

$$S_0 = \left\{\psi(s) : 0 \le \psi(s) \ll 1, \psi(0) = 1, \int_{-\tau_0-B}^{0} \psi(s)ds \le v^{-1/2}\right\}.$$

This set corresponds to the beginning of a pulse chosen as an initial moment, when the radiation density is $u(0) = 1$, while on the preceding interval, it takes small values. Note that no particular form of the function $\psi(s) \in S_0$ is indicated, so the set $S(c,\varphi)$ is rather large.

For notational simplicity, the solutions $u(t,c,\varphi,\psi)$, $y(t,c,\varphi,\psi)$ with initial conditions from $S$ are hereafter designated as $u(t), y(t)$. Let $t_0, t_1, t_2, \ldots$ denote the consecutive positive roots of the equation $u(t_i) = 1$, $i = 0,1,2,\ldots$. Specifically, $t_0 = 0$. Obviously, $t_0, t_2, \ldots, t_{2k}, \ldots$ are the beginning times of the radiation pulses, while $t_1, t_3, \ldots, t_{2k+1}$, ... are the ending times of the pulses. Additionally, we consider the sequence $\tilde{t}_0(\varphi), \tilde{t}_1(\varphi), \tilde{t}_2(\varphi)$, ... of positive roots of the equation $g(\tilde{t}_i) = \tilde{t}_i$. Specifically, the first positive root $\tilde{t}_0 = \tilde{t}_0(\varphi)$ of the equation

$$\tau_0 + B\cos(\omega\tilde{t}_0 + \varphi) = \tilde{t}_0 \quad (3)$$

corresponds to the time when the delayed argument is zero.

System (1) is integrated sequentially on the intervals $[t_i, t_{i+1}]$ with the use of estimates for $u(t)$ and $u(g(t))$ (as $v \to \infty$). The terms with values on the order of $v^{-1/2}$ are included in $o(1)$. The solutions obtained on consecutive intervals are matched, by using them as initial conditions on the subsequent interval.

(i) On the pulse interval $t \in (0, t_1)$, we have $u(t) \gg 1$ and $u(g(t)) \ll 1$. Moreover, $t_1 \to 0$ as $v \to \infty$, and, at the endpoints of the interval,

$$u(0) = 1, \quad u_{\max} = v(c - 1 - \ln c) + 1 + o(1),\\ u(t_1) = 1,\\ y(0) = c, \quad y(t_1) = c - p + o(1), \quad (4)$$

where $p = \int_0^{t_1} u(t)dt$ characterizes the pulse energy, which is found as the positive root $p = p(c)$ of the equation $c - p = ce^{-p}$.

(ii) On the interval $t \in (t_1, \tilde{t}_0)$, we have $u(t) \ll 1$, $u(g(t)) \ll 1$, $u(t_1) = 1$, and $y(t_1) = c - p$. Integrating system (1) with these conditions yields

$$y(t) = q + (c - p - q)e^{-t} + o(1),\\ u(t) = \exp[va(t)(1 + o(1))], \quad (5)$$

where $a(t) = (q-1)t + (c-p-q)(1-e^{-t})$. Note that the used estimate $u(t) \ll 1$ holds if

$$a(\tilde{t}_0) < 0. \quad (6)$$

(iii) On the interval $t \in (\tilde{t}_0, \tilde{t}_1)$, we have the estimates $u(t) \ll 1$, $u(g(t)) \gg 1$, and $\tilde{t}_1 = \tilde{t}_0 + o(1)$. Therefore,

$$u(\tilde{t}_1) = u(\tilde{t}_0)(1 + o(1)),\\ y(\tilde{t}_1) = y(\tilde{t}_0) + \gamma\int_{\tilde{t}_0}^{\tilde{t}_1} u(g(s))ds + o(1).$$

Making the change of variable $\tilde{s} = g(s)$ under the integral sign and taking into account $t_1 = t_0 + o(1)$ yields

$$\int_{\tilde{t}_0}^{\tilde{t}_1} u(g(s))ds = \int_{t_0}^{t_1} \frac{u(\tilde{s})}{g'(s)}d\tilde{s} = \frac{p}{g'(\tilde{t}_0)}.$$

(iv) On the interval $t \in (\tilde{t}_1, t_2)$, we have $u(t) \ll 1$ and $u(g(t)) \ll 1$ and the solution becomes

$$y(t) = q + [y(\tilde{t}_1) - q]e^{\tilde{t}_1 - t} + o(1),\\ u(t) = \exp[vA(t)(1 + o(1))], \quad (7)$$

where $A(t) = a(\tilde{t}_1) + (q-1)(t-\tilde{t}_1) + [y(\tilde{t}_1) - q](1 - e^{\tilde{t}_1 - t})$ if $A(t,c,\varphi) < 0$. At the right endpoint of the interval, $u(t_2) = 1$ and $\dot{u}(t_2) > 0$, which corresponds to the start of a new pulse. The condition $A(t_2) = 0$ yields the time $t_2 = T + o(1)$, where $T = T(c,\varphi)$ is the fist positive root of the equation

$$(q-1)T + (c-p-q)(1-e^{-T})\\ + \gamma\frac{p}{g'(\tilde{t}_0)}(1 - e^{-(T-\tilde{t}_0)}) = 0. \quad (8)$$

Note that the problem of further integration of system (1) over $t > t_2$ returns to the original problem with initial conditions $u(s + t_2) = \bar{\psi}(s), s \in [-\tau_0, 0]$; moreover, under condition (6), we have $\bar{\psi}(s) \in S_0$, $y(t_2) = \bar{c} + o(1)$, and $\Phi(t_2) = \bar{\varphi}$, where

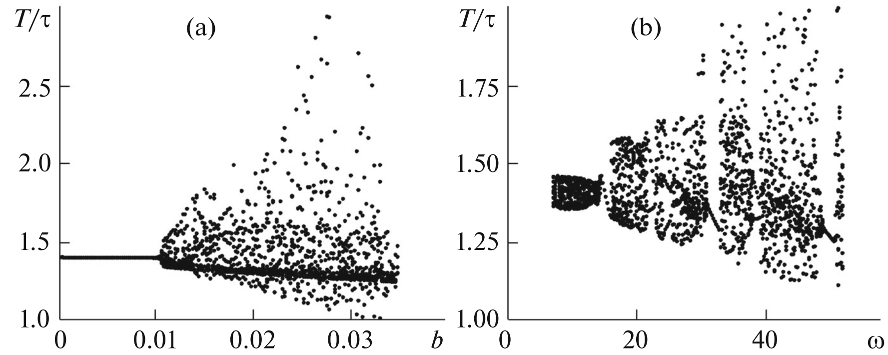


**Fig. 1.** Pulse ratio $\frac{T}{\tau}$ as a function of (a) modulation amplitude at $\omega = 29.8$ and (b) modulation frequency at $B = 0.015$. The other parameters of the mapping are $q = 1.5$, $\gamma = 0.3$, and $\tau_0 = 0.3$.

$$\bar{c} = q + (c - p - q)e^{-T} + \gamma \frac{p}{g'(\tilde{t}_0)} e^{-(T - \tilde{t}_0)},$$
$$\bar{\varphi} = \omega T + \varphi, \quad \mathrm{mod}\, 2\pi. \tag{9}$$

Here, $\tilde{t}_0 = \tilde{t}_0(\phi)$, $p = p(c)$, and $T = T(c, \varphi)$ are defined in (3), (4), and (8).

On trajectories, we introduce the shift operator $\Pi(c, \varphi, \psi(s)) = (y(t_2), \omega t_2 + \varphi, u(s + t_2))$, $s \in [-\tau_0 - B, 0]$, which takes each element from the set of initial conditions $S$ (with the help of the obtained solutions) to an element of the same set. It is naturally expected that the evolution of the operator is determined by iterations of the two-dimensional mapping (9). Specifically, a fixed point $(c_0, \varphi_0)$ (if any) is associated with a fixed point of the operator $\Pi$, and the latter, with a periodic SO solution of the original system, which is described by (4)–(7) and has the pulse period $T_0 = T(c_0, \varphi_0) > (\tau_0 + B)$. Periodic cycles of the mapping are associated with SO quasi-periodic solutions of the original system.

Inspecting the dynamics of the mapping, we can keep track of bifurcations in SO spikes. Figure 1 presents the pulse ratio $\frac{T}{\tau}$ in the steady-state mapping as a function of (a) modulation amplitude and (b) frequency of the time delay in the FB loop as computed with initial conditions $c = 1.25$, $\varphi = 4.0$. For sufficiently small modulation amplitudes $0 < B < 0.012$, the mapping has a stable point (a stable cycle with a period longer than the time delay). For $0.012 < B < 0.04$, we can observe cycles and chaotic attractors (quasi-periodic and chaotic SO oscillations). For $B > 0.04$ (which corresponds to the violation of the condition $B\omega < 1$), the system passes to oscillations with pulse ratios greater and less than the time delay. A cascade of bifurcations and synchronization windows are also observed in the case of a varying modulation frequency (see Fig. 1b). Thus, we have obtained the FB parameters for which chaotic SO oscillations occur in the system. This conclusion is confirmed by numerical computations of the original differential-difference system (1) for $v = 10^3$.

As $\tau_0$ is increased, the following two scenarios are possible: (i) for $\gamma > 0$, condition (6) is violated and we observe the transition to RO solutions with a single pulse on the delay interval and, later, to $RO^m$ solutions; (ii) for $\gamma < 0$, the stable and unstable fixed points merge (saddle-node bifurcation) and we observe alternating SO and RO solutions. The corresponding mappings are presented below.

## 3. RAPIDLY OSCILLATING SOLUTIONS

In RO solutions, the pulse ratio is less than the time delay. To construct $RO^1$ solutions with a single pulse on the delay interval, we introduce the set of initial conditions

$$S(c, \varphi\xi, p_1) = \{y(0) = c, \Phi(0) = \varphi, u(s) = \psi_1(s), \; s \in [-\tau_0 - B, 0]\}, \tag{10}$$

where $c \in (1, q]$, $\varphi \in [0, 2\pi]$, $\psi_1(s) \in S_1(\xi, p_1)$, and

$$S_1(\xi, p_1) = \left\{ \psi_1(s) \in C_{[-\tau_0 - B, 0]} : \psi_1(0) = 1, \int\limits_{-\tau+\xi}^{-\tau+\xi+\delta_1} \psi_1(s)ds = p_1, \int\limits_{-\tau}^{-\tau+\xi} \psi_1(s)ds + \int\limits_{-\tau+\xi+\delta_1}^{0} \psi_1(s)ds < v^{-1/2} \right\}.$$

The parameter $\xi$ determines the beginning moment the pulse on the delay interval preceding $t = 0$, while the parameter $p_1 > 0$ determines its energy. In the intervals between pulses, the values of $\psi_1(s)$ are asymptotically small. The pulse shape is not defined, but it is important that the pulse width $\delta_1 \to 0$ as $v \to \infty$.

Integrating the system sequentially with allowance for asymptotic estimates for $u(t)$ and $u(g(t))$, we find that, at the time $t_2 = T + o(1)$, the resulting solutions belong to the initial set $S$ with parameters $c, \xi, p_1, \varphi$ replaced by $\overline{c}, \overline{\xi}, \overline{p}_1, \overline{\varphi}$, where

$$\overline{c} = q + (c - p - q)e^{-T} + \gamma \frac{p_1}{g'(\tilde{\xi})} e^{\tilde{\xi} - T},$$
$$\overline{p}_1 = p, \qquad (11)$$
$$\overline{\xi} = \tau_0 + B\cos\varphi - T,$$
$$\overline{\varphi} = \omega T + \varphi, \quad \mathrm{mod} 2\pi.$$

Here, $p(c)$ is a positive root of the equation $c - p = c\exp(-p)$, $\tilde{\xi}(\xi, \varphi)$ is a root of the equation $\tilde{\xi} - B\cos(\omega\tilde{\xi} + \varphi) = \xi - B\cos\varphi$, and $T(c, \varphi, p_1, \xi)$ is a root of the equation

$$(q-1)T + (c - p - q)(1 - e^{-T}) + \gamma \frac{p_1}{g'(\tilde{\xi})}(1 - e^{\tilde{\xi}_1 - T}) = 0.$$

Mapping (11) describes the dynamics of the RO[1] solution with a single pulse on the delay interval if every iteration of the mapping satisfies the inequalities

$$a(\tilde{\xi}) < 0, \quad T < \tau_0 - B.$$

Numerical computations of the mapping confirm the existence of attractors for $\gamma > 0$. Moreover, the domain of parameters where they exist intersect the domain of SO solutions. Thus, we can indicate parameters for which quasi-periodic and chaotic oscillations with pulse ratios greater or less than the time delay coexist (but they are reached under different initial conditions).

In a similar nanner, $(2m + 2)$-dimensional mappings responsible for the dynamics of $\mathrm{RO}^m$ solutions with $m$ pulses on the delay interval can be constructed.

## 4. MIXED RAPIDLY AND SLOWLY OSCILLATING SOLUTIONS

Various regular and irregular combinations of SO and RO time structures are possible. Here, we present a mapping in which pulse ratios that are greater and less than the time delay strictly alternate.

The set $S(c, \varphi)$ of initial conditions is defined by (2), i.e., in the same manner as for SO solutions. Integrating system (1) with such initial conditions, we take into account the violation of condition (6), which ensures the SO structure of the solution. Therefore, at the time $t_2 = \xi + o(1) < \tau$, there appears a new pulse with energy $p_1$. Continuing the integration, after a time interval longer than the time delay, at $t_4 = T + o(1) > \xi + \tau_0$, we obtain solutions that belong to the original set of initial conditions with parameters $(c, \varphi)$ replaced by $(\overline{c}, \overline{\varphi})$, where

$$\overline{c} = q + (c - p - q)e^{-T} - p_1 e^{-(T-\xi)} + \gamma \frac{p}{g'(\tilde{t}_0)} e^{-(T-\tilde{t}_0)} + \frac{\gamma p_1}{g'(\tilde{\xi})} e^{-(T-\tilde{t}_0+\xi)},$$
$$\overline{\varphi} = \omega T + \varphi, \quad \mathrm{mod} 2\pi.$$

Here, the energy $p$ of the first pulse on the interval $t \in [0, t_1]$ is determined as the root of the equation $c - p = c\exp(-p)$, the energy $p_1$ of the second pulse on the interval $t \in [t_2, t_3]$ is determined as the root of the equation $[q + (c - p - q)e^{-\xi}] - p_1 = [q + (c - p - q)e^{-\xi}]\exp(-p_1)$, $t_2 = \xi + o(1)$, the shortest interval $\xi < \tau$ between the pulses can be found by solving the equation $a(\xi) = 0$, $\tilde{t}_0 = \tilde{t}_0(\varphi)$ is a positive root of Eq. (3), and $T$ is the root of the equation

$$(q-1)T + (c - p - q)(1 - e^{-T}) - p_1(1 - e^{-(T-\xi)}) + \frac{\gamma p}{g'(\tilde{t}_0)}(1 - e^{-(T-\tilde{t}_0)}) + \frac{\gamma p_1}{g'(\tilde{\xi})}(1 - e^{-(T-\tilde{t}_0+\xi)}) = 0.$$

Assuming that every iteration of the mapping satisfies the inequalities

$$\xi < \tau_0, \quad T - \xi > \tau_0,$$

the attractors of the mapping are associated with spiking oscillations occurring at intervals strictly greater and less than the time delay. The numerical computation of the mapping confirms that such attractors exist for $\gamma < 0$.

To conclude, we note that the dynamics of spiking oscillations in the laser model with a variable time delay can be adequately described by the resulting mappings. It was shown that, with suitably chosen delay modulation amplitudes and frequencies, we can obtain oscillation modes, including chaotic ones, with special properties, namely, slowly and rapidly oscillating solutions and mixed-type modes. For each type of attractors, we described the corresponding domains of initial conditions, which can be used in the development of methods for switching between modes with the help of external signals.

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