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Waiting for public transport services: Queueing analysis with balking and reneging behaviors of impatient passengers

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ABSTRACT

Queues of batch arrivals and bulk service including balking and reneging behaviors of customers are commonly observed in real life. This study formulates queues of this type using compound Poisson processes and determines some key probabilistic measures. Analytical investigation is undertaken yielding a range of mathematical results. The developed mathematical model and approaches apply to a variety of practical queueing processes that are featured with bulk queues, balking, and reneging. A bus bridging response to rail disruption is considered as an application example. And large-scale Monte-Carlo simulations are conducted to demonstrate the mathematical results.

1. Introduction

Queueing or waiting for services is one of the disagreeable but necessary experiences of life. To a large extent, queueing theory originated and has grown from the study of such experiences. There is a vast amount of human-queueing phenomena among which those associated with public transport services are commonly observed on a daily basis (Ceder, 2007; Higgins and Kozan, 1998; Huisman and Boucherie, 2001; Marguier and Ceder, 1984; Trietsch, 1993;Vansteenwegen and Van Oudheusden, 2007). This study addresses a special class of queueing problems with an orientation to public transport services.

Queueing theory characterizes queueing systems according to (see e.g. Allen, 1990; Gross et al., 2008; Kleinrock, 1975): (a) arrival patterns of customers (e.g. Poisson/Erlang/general); (b) service patterns; (c) queue discipline (e.g. first-come-first-served, priority-based); (d) the number of servers provided; (e) the maximum queue length allowed; (f) configuration of servers (e.g. in series/in parallel/mixed). This paper is primarily concerned with the arrival and service patterns, the behavior of impatient customers, and the impact of such behavior on queueing. More precisely, the focus is on queueing processes of the following features:

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(i) *batch arrival*: customers arrive in teams rather than individually;
(ii) *bulk service*: customers are served by the server (or each server in the case of multiple servers) in teams rather than individually;
(iii) *balking*: some customers choose not to join a queue upon their arrivals, normally because of too long a queue ahead;
(iv) *reneging*: some other customers first choose to join a queue, but gradually lose their patience, and eventually leave the queue before receiving service in case of intolerable waiting.

Features (i) and (ii) fall in the categories of arrival and service patterns, while features (iii) and (iv) are mostly concerned with the queueing psychology.[2] Queueing of the above features are ubiquitous in the real world, and typically seen in the sectors of public services, transportation, manufacturing, and telecommunication. Although this paper concentrates on queueing of these features in public transport services, a brief survey is first presented on bulk queues with balking and reneging in a much broader sense.

Queueing with *batch arrivals* and *bulk service* are common (Claeys et al., 2011; Powell, 1985; Sikdar and Gupta, 2008). A server of a certain capacity becomes available after a random amount of time to serve a pool of customers. If the capacity is less than the number of customers waiting, the server leaves behind some customers (Kahraman and Gosavi, 2011). Elevators in buildings form a common example for this type of systems. Other examples arise in various settings. In transport and freight systems, queues of this type are found with airport buses/metros/taxis, urban buses/trains/metros/trams, people-movers (e.g. cable cars in amusement parks), cargo-delivering airplanes/ships, etc. In the setting of manufacturing, machines may serve several units at the same time. For instance, equipment for heat treatment can usually handle a number of parts simultaneously. Automated guided vehicles to deliver jobs from one site to another, which are used in both freight (e.g. harbor-related) and manufacturing settings, often involve bulk queues.[3] In the setting of information technologies, individual information packets are grouped in larger entities for transmission. In addition, the operation of online reservation systems is generally related to bulk queues. Besides the above examples concerning real-time operations, bulk queues may also be noticed over a larger scale of time (e.g. days/months), e.g. the ordering of some special goods or service. Despite a body of literature on bulk queueing systems, the classics of queueing theory (e.g. Allen, 1990; Gross et al., 2008; Kleinrock, 1975) focuses on queueing systems of single arrivals and service, and takes bulk queues as a special case. Specifically for bulk queues in public transportation systems, only limited work has been published (Powell, 1983, 1985; Rapoport et al., 2010; Sim and Templeton, 1982; Selvi, 1983).

Customers are often discouraged by long queues. They usually tend to join a queue only when a short wait is expected or first join it but depart if a further wait would be intolerable. This leads to two actions: *balking* (the refusal of an arriving customer to join a queue); *reneging* (the departure of a queueing customer before obtaining aimed service). Although the phenomenon that customers are "lost" through balking and reneging are widespread in real life, the classical queueing theory is primarily concerned with queues in which customers are all patient and eventually get served. The simplest balking phenomenon is observed in the loss system where arrivals do not enter the system when all servers are found busy. The study of the loss system can be traced back to 1917 when the Danish mathematician A. K. Erlang, a pioneer in queueing theory, considered the calls lost by a busy telephone exchange and derived the renowned Erlang's loss formula (Allen, 1990; Gross et al., 2008; Kleinrock, 1975). A balking behavior may generally depend on the queue length, while the period that a customer stays in line before reneging is usually modeled as a random variable. Readers are referred to (Al-Seedya et al., 2009; Barrer, 1957; Blackburn, 1972; Rao, 1965; Stanford, 1979; Ziya et al., 2006) for queueing theory with balking and reneging. The late renowned transport researcher Frank Haight was among the earliest group of researchers who studied reneging (Haight, 1959).

It is not rare to see queues with features (i)–(iv) in practice, e.g. queueing at the entry of a popular restaurant, where customers arrive by groups/families, served by tables, and balking and reneging certainly happen. Such queues relating to public transport services are observed at:

- an outbound bus/metro/taxis station at an airport (where passengers arrive in teams from landed flights and may be served in teams, involved with balking and reneging);
- an urban bus/train/tram/metro station;
- an entry to a people-mover (e.g. cable-car) in an amusement park.

[2] Limited attention was given to this regard (Maister, 1985; Larson, 1987). More precisely, human factors and the psychology of queueing customers played little role in queueing theory and therefore were not regarded as one specific aspect of queueing processes. Nevertheless, according to Allen (1990), Gross et al. (2008), and Kleinrock (1975), the impacts of customer psychology such as balking and reneging are attributed to the category of arrival patterns. It should also be pointed out that whenever an imposed maximum queue length is reached (see (e) in the proceeding text), balking happens to any subsequently arriving customer until after the queue length becomes lower than the maximum permissible limit.

[3] In this paper a bulk queue refers to a queue with batch arrivals and/or bulk service.

Limited attention has been paid to such queueing systems (see e.g. El-Paoumy and Ismail, 2009). Specifically in transportation research and traffic engineering, queueing models with balking and reneging remain an important gap in knowledge.[4] For instance, only one paper was found in this journal that mentions the balking and reneging behavior of customers (Rapoport et al., 2010).

This paper addresses queueing systems of all features (i)–(iv) in the field of public transport services. Since the complete treatment of this type of queueing problem is still an ongoing task of queueing theorists, this paper focuses on the probabilistic nature of balking and reneging behavior of impatient passengers as well as the impact of such behavior on queueing. To this end, the theory of compound Poisson processes is employed to establish a stochastic model that applies to a broad class of bulk queueing systems with balking and reneging. Two key measures of queueing systems, the mean and variance of queue lengths, as well as other indices of interest are analytically explored in this modeling framework. This study is of general significance to the planning, allocation and management of resources in many fields including public transport services. This can be elaborated with the following example. In cities like Melbourne, Australia, suburban public transport services are provided by trains and buses in parallel. Time schedules for bus and train operations are often associated to facilitate passenger transfers. As such, buses feed a primary portion of passenger demand for trains and vice versa; a train and a bus can both serve a number of passengers at one time. In the case of rail/bus service disruptions, passengers arriving in consecutive buses/trains can accumulate at affected railway/bus stations. Depending on the extent of delay, passenger balking and reneging may gradually take place, thus leading to a queueing process of features (i)–(iv). It is crucial to estimate the mean number of possibly accumulated passengers and its variance as well as their temporal dynamics in this circumstance or other similar situations so as to meet a number of needs, e.g. the determination of shelter size of a stop/station or waiting-plaza size of an interchange/terminal. This will be further explained in Sections 2 and 3 based on a mathematical model, while a similar but more comprehensive application example is presented in Section 4.

The contributions of this work are as follows.

(a) This is the first study to explore analytically the bulk queueing problem involving balking and reneging with special orientation to public transport services.
(b) The theory of compound Poisson processes is introduced as a principal mathematical tool to deal with the targeted queueing problem. This is a new analytical conceptualization of the problem for transportation research.
(c) The study solves a challenging task of determining the mean and variance of the length of a queue subject to the balking and reneging actions of impatient customers. The reached mathematical conclusions are of some general significance for wider applications.
(d) Simulations are conducted to evaluate the obtained theoretical results. It is the first time to emulate compound Poisson processes with balking and reneging using large-scale Monte-Carlo simulations in the context of public transport services.

The remainder of this paper is organized as follows. Section 2 introduces the theory of compound Poisson processes and applies it to the mathematical modeling of a broad class of bulk queueing systems involving balking and reneging. Section 3 explores analytically the probabilistic nature of such systems and delivers a comprehensive set of mathematical results. Section 4 presents a bus bridging example, its modeling in compound Poisson processes, and the numerical evaluation of the modeling results in large-scale Monte-Carlo simulations. The theoretical exploration and simulation evaluation are supplemented with further discussions in Section 5. The paper is finalized in Section 6 with some conclusive remarks given.

2. Compound Poisson processes with balking and reneging

A stochastic process $\{S_1(t), t \geqslant 0\}$ is referred to as a compound Poisson process if it can be represented by

$$S_1(t) = \sum_{i=1}^{N(t)} M_i \tag{1}$$

where $\{N(t), t \geqslant 0\}$ is a Poisson process of constant intensity λ, and $M_1, M_2, \ldots$ are independent and identically distributed non-negative random variables that are also independent of the process $\{N(t)\}$. As displayed in Fig. 1, M_i is the change in the value of $S_1(t)$ at the ith occurrence instant t_i of the Poisson process $\{N(t)\}$, $1 \leqslant i \leqslant N(t)$. It is postulated for an ordinary Poisson

[4] Some explanation may be needed. Queues in many fields (e.g. information technologies) are highly stochastic with random arrivals and service. However, queues in transportation often tend to be more deterministic and predictable. First, this is because travel of people and goods often generates demands of repetitive patterns. Second, in transportation, queues caused by random variations in inter-arrival and service times are often deemed to be secondary relative to queues caused by predictable demand patterns (Hall, 2003; May and Keller, 1967; Newell, 1982). Distinct mathematical models have been developed to address stochastic and deterministic queueing problems. More specifically, various methods presented in e.g. Allen (1990), Gross et al. (2008), and Kleinrock (1975) are typically used to analyze stochastic queueing problem, while cumulative diagrams are used as a principal tool to analyze deterministic transport queueing problems (Hall, 2003; May and Keller, 1967; Newell, 1982). Balking and reneging, which are of random nature, have been studied only along the stochastic line. This could partially explain why the phenomena of balking and reneging, despite widespread in public transport services, have not been much addressed by transportation researchers. In this paper, balking and reneging arising in public transport services are studied along the stochastic line of thoughts.

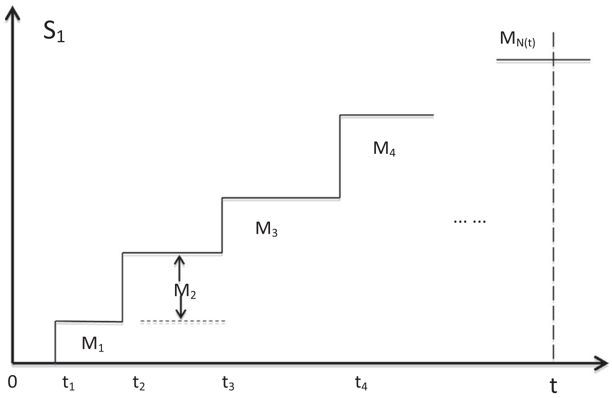

Fig. 1. The stochastic process $S_1(t)$.

process that at most one event can occur at any time. The compound Poisson process $S_1(t)$ can then be interpreted as follows: a random number M_i of events occur simultaneously as one cluster at instant t_i such that the total number of clusters in time t constitutes an ordinary Poisson process $\{N(t)\}$ while the total number of events having occurred up to t is described by $S_1(t)$. Moreover, the independent property stated above indicates that:

$$Pr\{S_1(t_1)=k_1,S_1(t_2)=k_2,\ldots,S_1(t_n)=k_n\}=Pr\{S_1(t_1)=k_1\}Pr\{S_1(t_2-t_1)=k_2-k_1\}\cdots Pr\{S_1(t_n-t_{n-1})=k_n-k_{n-1}\}.$$

An interested reader is referred to Prazen (1967), Tijms (2003) for further details. It is noted that Fig. 1 illustrates the compound Poisson process $S_1(t)$ with only one specific realization (or sample path) of $S_1(t)$. Due to its stochastic nature, $S_1(t)$ can have virtually an infinite number of realizations; and Fig. 2 illustrates the case of 30 realizations.

This compound Poisson process model applies to a broad class of bulk queueing processes. Some typical examples are as follows:

- a number $S_1(t)$ of passengers arrive at a railway station over a period $[0,t]$, by a random number $N(t)$ of buses with a random number M_i of passengers in the ith bus;
- an accumulative demand $S_1(t)$ have been placed by a random number $N(t)$ of customers for a certain product up to time t, with an random amount M_i of the product ordered by the ith customer;
- a number $S_1(t)$ of persons injured in a random number $N(t)$ of accidents occurring over a period $[0,t]$ in a metropolitan city, with the ith accident yielding a random number M_i of injuries;
- a amount $S_1(t)$ of cash has been claimed by a random number $N(t)$ of customers against one insurance company by time t, with an random amount M_i claimed by the ith customer.

To incorporate the balking and reneging effects, more mathematical consideration is needed. Consider a number M_i of customers arriving at instant t_i and each customer may choose to balk in a probability of θ. Denote by M_{i1} the number of balking customers; thus $E[M_{i1}] = \theta \cdot M_i$. Denote also by M_{i2} the number of remaining customers (i.e. $M_{i2} = M_i - M_{i1}$). Furthermore, assume that any remaining customer stays in queue for an exponential time period of parameter γ and the service becomes available at t. Let $Q_i(t)$ represent the number of customers who are part of M_{i2} and eventually get served at t. Thus, $Q_i(t)$, $t \geqslant t_i$, is a function of M_i, θ, γ, t_i, and t, with the boundary conditions $Q_i(t_i) = M_{i2}$ (no reneging yet) and $Q_i(\infty) = 0$ (full reneging). Moreover, the number of reneging customers by t is represented by $M_{i2} - Q_i(t)$. Denote by $S_2(t)$ the total number of customers who are eventually served at t, we have,

$$S_2(t)=\sum_{i=1}^{N(t)}Q_i(t)=\sum_{i=1}^{N(t)}f(M_{i2},\gamma,t_i,t)=\sum_{i=1}^{N(t)}g(M_i,\theta,\gamma,t_i,t) \tag{2}$$

Fig. 3 compares $S_1(t)$ and $S_2(t)$, where the thick curve represents $S_1(t)$ (same as that in Fig. 1), the dash curve represents $S_2(t)$, and the thin curve displays their difference, which is the number of lost customers due to the effects of balking (taking place at each arrival moment t_i) and reneging (taking place continuously over time).[5] Fig. 3 depicts the case of only one realization. Like Fig. 2, multiple realizations can also be created, but omitted.

Both $S_1(t)$ and $S_2(t)$ are compound Poisson processes. More precisely, $S_1(t)=\{M_i\}_{i=1}^{N(t)}$ contains a Poisson number $N(t)$ of independent and identically distributed random variables M_i, and $S_2(t)=\{Q_i(t)\}_{i=1}^{N(t)}$ is a sequence of independent and semi-identically distributed random processes, $\{N(t),\ t \geqslant 0\}$ is independent of $\{Q_i(t)\}$. Table 1 summarizes the symbols defined above. It should be pointed out that, in a normal setting of a compound Poisson process (Prazen, 1967; Tijms,

[5] Fig. 3 has two y axes pointing to the opposite directions, both with no-negative values.

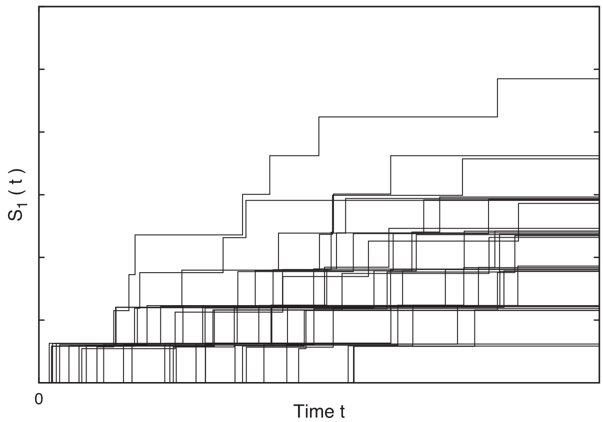

Fig. 2. Thirty realizations of $S_1(t)$.

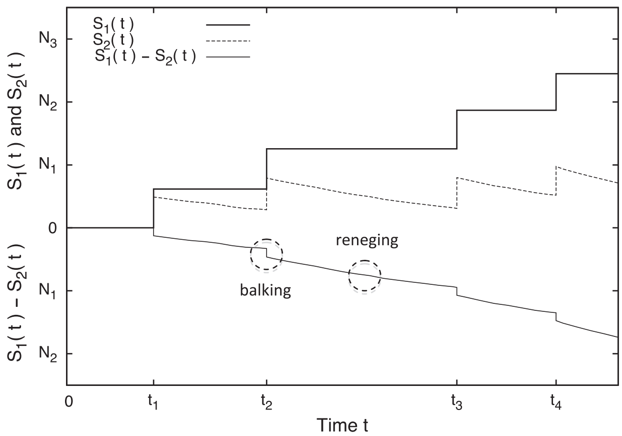

Fig. 3. $S_1(t)$, $S_2(t)$, and $S_1(t) - S_2(t)$.

Table 1
The definitions and probability characteristics of key variables or processes.

Variables or processes	Definitions	Probability characteristics
t	A time period of interest	Exponentially distributed with parameter μ
$N(t)$	The number of batches (of customers) that have joined the queue by t	A Poisson process of rate λ
t_i	The time instant that the ith batch has arrived.	$0 \leqslant t_1 < t_2 < \ldots < t_{N(t)} \leqslant t$. $t_{i+1} - t_i$ follows the exponential distribution of parameter λ
M_i	The number of customers in the ith batch ($M_i = M_{i1} + M_{i2}$)	Subject to a Poisson distribution of parameter C
$S_1(t)$	The total number of customers that have arrived by t in a number $N(t)$ of batches $S_1(t) = \sum_{i=1}^{N(t)} M_i$	A compound Poisson process
θ	The balking rate of any customer	Constant
M_{i1}	The balking portion of M_i	Subject to a Poisson distribution of parameter $C\theta$[a]
M_{i2}	The non-balking portion of M_i	Subject to a Poisson distribution of parameter $C(1\text{-}\theta)$[a]
γ	The reneging rate concerning M_{i2}	Constant
$Q_i(t)$	The number of customers that have arrived in the ith batch and remained in queue by t $Q_i(t) = f(M_i, t_i, \theta, \gamma, t)$	Poisson distributed with parameter $C(1-\theta)e^{-\gamma(t-t_i)}$, given t_i and t[a]
$S_2(t)$	The total number of customers that have arrived in a number $N(t)$ of batches and remained in queue by t $S_2(t) = \sum_{i=1}^{N(t)} Q_i(t)$	A compound Poisson process

[a] Proved in Theorem 2.

2003), only $\{N(t)\}$ and M_i ($1 \leqslant i \leqslant N(t)$) are stochastic or random while t is considered deterministic or given. This paper focuses however on the cases (with special orientation to public transport services), in which t is often found random. Since both $S_1(t)$ and $S_2(t)$ are stochastic processes, it is hardly possible to predict their exact values over time. Hence, of practical interest are their probabilistic measures such as means and variances. As remarked in Section 1, the determination of these probabilistic measures are crucial for needs in public transport services such as the estimation of shelter size of a stop/station or waiting-plaza size of an interchange/terminal. In addition, only a few papers were published in this journal in past decade concerning the compound Poisson process (Cetinkayaa and Bookbinderb, 2003; Ebbena et al., 2004; Gillen and Hasheminia, 2013).

3. Mathematical results

Three theorems are presented in this section and the proof is found in Appendices A, B, C, D. Throughout the paper, $E[\cdot]$ and $\mathrm{Var}[\cdot]$ denote mean and variance, respectively.

Theorem 1. The probabilistic characterization of S_1 (without involving balking and reneging)
Given a fixed time period τ,

(a)

$$E[N(\tau)] = \lambda\tau$$
$$\mathrm{Var}[N(\tau)] = \lambda\tau$$
$$E[S_1(\tau)] = \lambda C\tau$$
$$\mathrm{Var}[S_1(\tau)] = \lambda(C^2 + C)\tau$$

For a random time period t,
(b)

$$E[N(t)] = \frac{\lambda}{\mu}$$
$$\mathrm{Var}[N(t)] = \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}$$
$$E[S_1(t)] = \frac{\lambda C}{\mu}$$
$$\mathrm{Var}[S_1(t)] = \frac{\lambda}{\mu}[C + C^2] + \frac{\lambda^2 C^2}{\mu^2}$$

The probability distribution of $S_1(t)$ is determined by

$$Pr\{S_1(t) = k\} = \left(\frac{\mu}{\lambda + \mu}\right)\sum_{n=0}^{\infty}\frac{(Cn)^k e^{-Cn}}{k!}\left(\frac{\lambda}{\lambda + \mu}\right)^n$$

The proof is presented in Appendix B.

Remark 1. The results on $E[S_1(t)]$ and $\mathrm{Var}[S_1(t)]$ in Theorem 1(b) are derived along the line of compound Poisson processes, but may also be verified using Theorem 1(c), see Appendix B for the details.

Theorem 2. Effects of balking and reneging
Focus on the ith batch of M_i customers joining the queue at t_i,

(a) M_{i1} and M_{i2} are Poisson distributed with parameters $C\theta$ and $C(1-\theta)$, respectively.
(b) Given τ, $Q_i(\tau) = f(M_i, t_i, \theta, \gamma, \tau)$ is Poisson distributed with parameter $C(1-\theta)e^{-\gamma(\tau - t_i)}$.
(c) Considering random t, $E[Q_i(t)] = C(1-\theta)\frac{\mu}{\gamma+\mu}$.
(d) Denote by T_1 the moment the first customer loses his/her patience to quite from the queue. $E[T_1|M_{i2} > 0] > \frac{1-e^{-C(1-\theta)}}{\gamma C(1-\theta)}$.
(e) In case the service cannot be provided in time, all remaining customers may eventually run off by a certain moment T_2, then $\mathrm{E}[T_2|M_{i2} > 0] < \frac{1}{\gamma}\left[\ln\left[\frac{C(1-\theta)}{1-e^{-C(1-\theta)}}\right] + \varphi\right]$, where φ is Euler's constant.

The proof is presented in Appendix C.

Remark 2. Theorem 2(d) and (e) tell that the most impatient customer would on average wait at least for a period of $\frac{1-e^{-C(1-\theta)}}{\gamma C(1-\theta)}$ and the most patient customer would not on average wait longer than $\frac{1}{\gamma}\left[\ln\left[\frac{C(1-\theta)}{1-e^{-C(1-\theta)}}\right] + \varphi\right]$.

Theorem 3. The probabilistic characterization of S_2 (involving balking and reneging)

Given a fixed time period τ,

(a)

$$E[S_2(\tau)] = \frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}]$$
$$\mathrm{Var}[S_2(\tau)] = \frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}] + \frac{\lambda C^2(1-\theta)^2}{2\gamma}[1-e^{-2\gamma\tau}]$$

For a random time period t,

(b)

$$E[S_2(t)] = \frac{\lambda C(1-\theta)}{\mu+\gamma}$$
$$\mathrm{Var}[S_2(t)] = \frac{\lambda C(1-\theta)}{\mu+\gamma} + \frac{\lambda C^2(1-\theta)^2}{\mu+2\gamma} + \frac{\mu\lambda^2 C^2(1-\theta)^2}{(\mu+2\gamma)(\mu+\gamma)^2}$$

The proof is presented in Appendix D.

Remark 3. Theorem 3(a) and (b) can be compared with Theorem 1(a) and (b). Note that balking and reneging that happens to M_i has no impact on $N(t)$. Therefore, the conclusions of $N(\tau)$ and $N(t)$ presented in Theorem 1 still hold in the current case.

Remark 4. Theorem 3(b) states that the average total number of customers who are eventually served is $\frac{\lambda C(1-\theta)}{\mu+\gamma}$. Recall with Theorem 2($c$) that the average number of customers who arrive in one batch and are eventually served is $C(1-\theta)\frac{\mu}{\mu+\gamma}$. Based on Theorem 1($b$), on average λ/μ batches arrive over the period of t. Thus, $\frac{\lambda C(1-\theta)}{\mu+\gamma}$ can be verified with the product of λ/μ and $C(1-\theta)\frac{\mu}{\mu+\gamma}$, while a rigorous proof is presented in Appendix D.

4. An application example of bus bridging

4.1. A bus bridging issue

An application example is presented in this section to demonstrate the obtained analytical results. This example is concerned with bus bridging. Urban rail networks constitute an important part of the overall transport network in many cities (Ceder, 2007; De-Los-Santos et al., 2012). Railway operation is highly dependent on technology and infrastructure. Any infrastructure failure, system malfunction, or incident can lead to major service disruptions (Kepaptsoglou and Karlaftis, 2009; Pender et al., 2012, 2013). In response to rail service disruptions, it is of great significance to provide quick and efficient substitution of service so as to ensure network credibility. This includes, among other options, bridging the impacted and disconnected stations with temporary bus routes (Pender et al., 2012, 2013; Boyd et al., 1998) (see Fig. 4). The selection of railway stations to provide bus bridging services depends on the availability of track crossovers (Fig. 5), which enable trains to turn back to the incoming directions. As depicted in Fig. 4, an accident has occurred somewhere along a railway line;

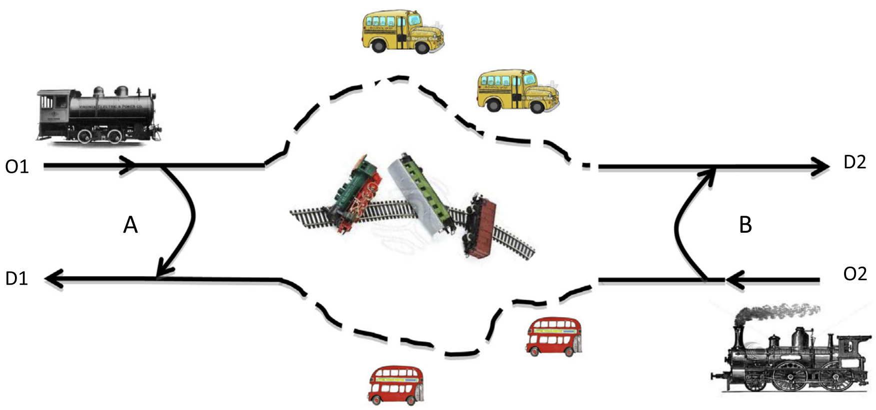

Fig. 4. Railway disruption, crossovers, and bus bridging.

stations A and B, both equipped with track crossovers, are selected as the bridging stations. During the disruption period, bus bridging services are run to restore the connectivity between A and B, while the railway services remain operational between O1 and D1 via A and O2 and D2 via B. Bus bridging can be handled by use of reserve buses located at depots, or buses retracted from operational bus lines, or both (Kepaptsoglou and Karlaftis, 2009; Pender et al., 2013). As the capacity of a train is much bigger than that of a bus, bridging services may need to be provided simultaneously by a platoon of buses. In addition, the headway between two bus platoons can be so substantial that several trains may arrive at a bridging station over the headway period.

Bus bridging is commonly applied in developed countries in response to severe railway disruptions. For instance, 15,549 unplanned disruptions happened to the metropolitan rail services in Melbourne, Australia in the first half of 2011, which range from small delays to full service closures. Among those disruptions, 47 disruptions were addressed through bus bridging, suggesting an average of 8 per month. Moreover, 1712 passengers and 42 separate trains were affected by those bus-replacement incidents. More details can be found in Powell et al. (2012). Research on bus bridging is scarce. Operators and authorities mainly rely on their experience to deliver bus bridging services in an ad hoc manner. Thus, there is a need to develop more rigorous and comprehensive approaches to bus bridging planning and management.

4.2. The application of the theoretical results

Given a rail disruption, it is highly desirable to learn at an early stage the incident severity and to determine the scale of bus bridging service that would be required to accommodate affected passengers. Thus, accurate demand modeling of affected passengers is essential for the quick planning and management of bus bridging resources and for efficient provision of bus replacement service. More precisely, the following issues are of much interest to passenger demand modeling:

(1) If bus bridging service can be managed so well that its starting time can be accurately predicted (e.g. assuming urban traffic conditions are trivial), probabilistically how many affected passengers would need to be accommodated by bridging service?
(2) In a less fortunate but more realistic case, the starting time of bridging service is uncertain. What is the answer to the above question?
(3) Consider the balking and reneging of affected passengers and focus on those who initially choose to wait for bridging services. Probabilistically, when would the first (i.e. the most impatient) passenger lose his or her patience, and in case bridging service cannot be provided in time (due to e.g. traffic jams), when would all remaining passengers run off?
(4) Taking into account balking and reneging effects in (1) and (2) above, how many passengers would on average be served eventually by bridging buses?

The bus bridging problem is exactly a queueing problem of batch arrivals and bulk service involving balking and reneging. More precisely,

- *Batch arrival*: each affected train brings a random number of passengers to a bridging station at one time.
- *Bulk service*: each platoon of bridging buses can serve at one time a (random) number of affected passengers that is upper bounded by the platoon size and bus capacity.

Fig. 5. Track crossovers.

- *Balking*: some passengers, upon reaching a bridging station, may choose to leave immediately without waiting for bridging buses.
- *Reneging*: some of the remaining passengers may gradually lose their patience and eventually depart before bridging service starts.

Therefore, the passenger demand can be modeled in a compound Poisson process, whereby the train arrivals are modeled as a Poisson process $\{N(t), t \geqslant 0\}$ and each train carries a number M_i $(1 \leqslant i \leqslant N(t))$ of passengers. The probabilistic dynamics of balking and reneging must be considered for impacted passengers in order to determine a more appropriate platoon size of bridging buses. We specify $t = 0$ the instant the railway disruption occurs. Furthermore, the second column of Table 1 can be specified as in Table 2. Fig. 6 displays 30 realizations of $S_1(t)$ in the context of bus bridging, and τ represents the time instant the bridging service starts.

Questions (1)–(4) previously proposed in the context of bus bridging can be addressed by Theorems 1–3 in Section 3 as follows:

- **Question (1)**: Assume that the time instant τ is known or can be accurately predicted. Due to the stochastic nature of $S_1(\tau)$, the value of $S_1(\tau)$ is uncertain (see e.g. Fig. 6). Thus, of primary interest in this case are the mean and variance of $S_1(\tau)$, to which Theorem 1(a) gives the answer.
- **Question (2)**: In case the start time of bridging services is uncertain; in other words, the dash vertical line in Fig. 6 may be moved randomly and horizontally along the time axis. Thus, the mean and variance of $S_1(t)$ also depends on t. Theorem 1(b) addresses this case exactly.
- **Question (3)**: This is answered by Theorem 2(d) and (e).
- **Question (4)**: Theorems 3(a) and (b) deal with this question, with balking and reneging taken into account.

4.3. Simulation evaluation

4.3.1. A numerical example

With reference to Fig. 1 and Table 2, consider the eastbound trains arriving at the bridging station A at a rate of 6 trains per hour ($\lambda = 6$). Every train has 5 carriages, each carrying 120 passengers on average ($C = 600$). Upon reaching A, every affected passenger has a likelihood of 20% to leave the station immediately ($\theta = 20\%$), and any remaining passenger may stay on average for a period of 10 min ($\gamma = 6$). The time duration t between the moment an incident occurs and that the platoon of bridging buses arrives is on average 40 min ($\mu = 1.5$).

4.3.2. Algorithms for emulating compound Poisson processes

As shown in Table 2, $S_1(t)$ in (1) involves three random factors: t, $N(t)$, and M_i, and $S_2(t)$ in (2) involves two more: t_i and γ. The following algorithm is used to simulate $S_1(t)$:

- *Step 1*: A generator of the exponential distribution of parameter μ generates a realization of period t.
- *Step 2*: A generator of the Poisson distribution with parameter λ generates a number N of Poisson random points over a long interval L of, say, 300 min, which is probabilistically certainly much bigger than t. Based on Lemma 4 in Appendix A, this number N of points are uniformly distributed over L.
- *Step 3*: All points generated in Step 2 and falling into the range of t are regarded as the time instants of train arrivals. Steps 1–3 are shown in Fig. 7.
- *Step 4*: For each determined time instant t_i of train arrival, a Poisson number M_i is generated with the mean C to simulate the impacted passengers carried by a train arriving at t_i (see Fig. 1).

Table 2
Key variables and parameters for the bus bridging problem.

Variables or processes	Definitions
t	The time period between the occurrence of a railway disruption and the start of bus bridging services
$N(t)$	The number of trains that arrive at a bridging station over period t
t_i	The time instant that the ith train arrives at the station
M_i	The number of passengers carried by the ith train ($M_i = M_{i1} + M_{i2}$)
$S_1(t)$	The total number of impacted passengers (until bus bridging takes effect at t) $S_1(t) = \sum_{i=1}^{N(t)} M_i$
θ	The balking rate of any impacted passenger
M_{i1}	The number of balking passengers, who arrive with the ith train and choose to leave the station immediately
M_{i2}	The number of non-balking passengers, who arrive in the ith train but decide to stay at the station waiting for bridging buses
γ	The reneging rate of the number M_{i2} of passengers
$Q_i(t)$	The number of passengers who are part of M_{i2} and still remain at the station until bridging service starts at t $Q_i(t) = f(M_i, t_i, \theta, \gamma, t)$
$S_2(t)$	The total number of passengers who are eventually picked up by bridging buses $S_2(t) = \sum_{i=1}^{N(t)} Q_i(t)$

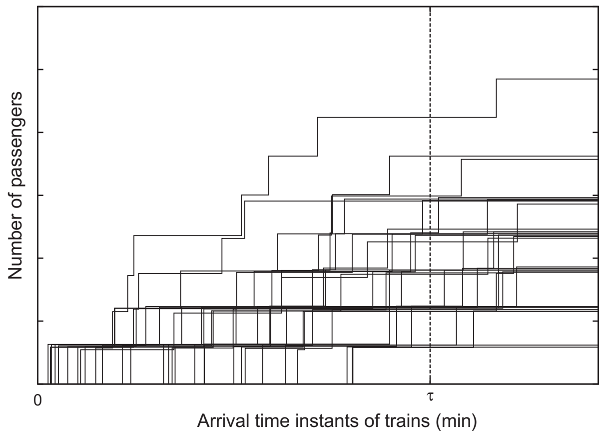

Fig. 6. 30 realizations of $S_1(t)$.

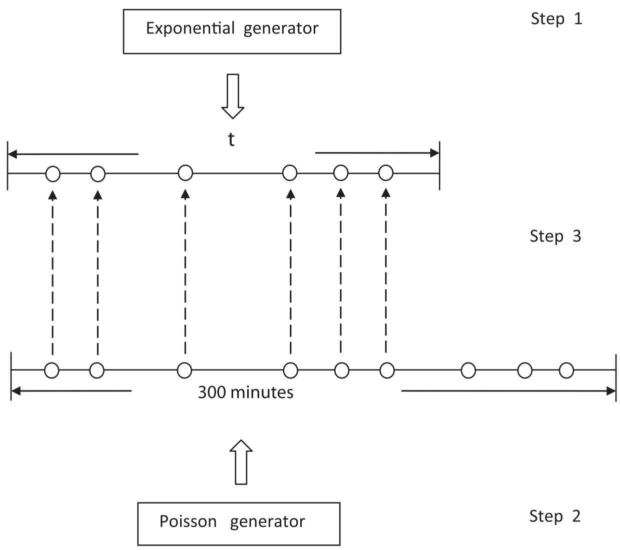

Fig. 7. Simulation setup for creating stochastic scenarios: Algorithm 1.

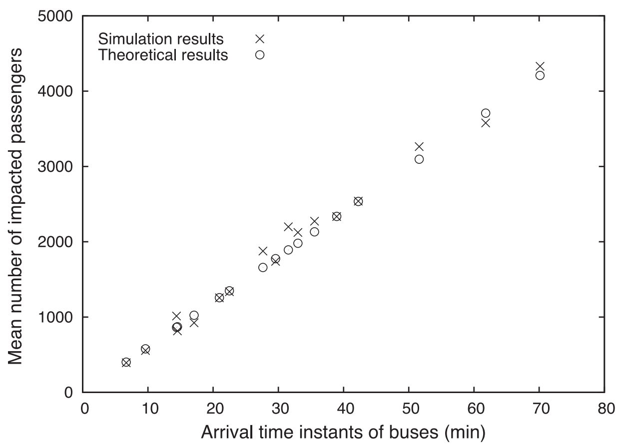

Fig. 8. Theorem 1(a): $E[S_1(\tau)]$.

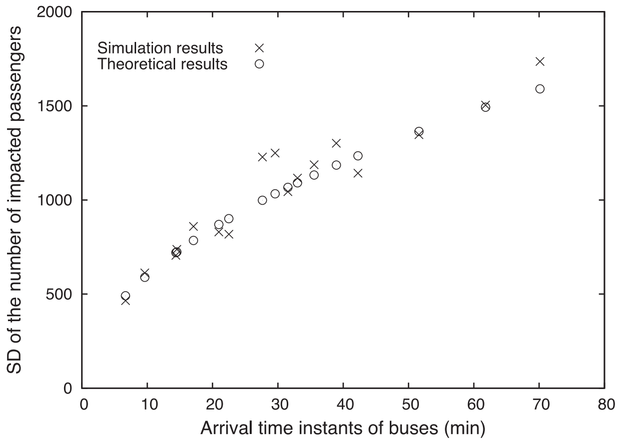

Fig. 9. Theorem 1(a): Var$[S_1(\tau)]$.

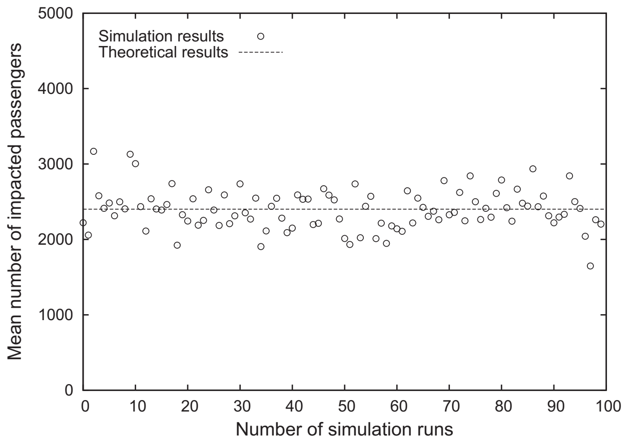

Fig. 10. Theorem 1(b): $E[S_1(t)]$.

To simulate $S_2(t)$, two more steps are needed:

- *Step 5*: Each passenger in M_i may initially choose to leave the station with the probability of θ. This is determined by a 0–1 sequence (0: leave; 1: stay) that is generated with a Bernoulli random variable with the mean likelihood of being "0" equal to θ.
- *Step 6*: Each remaining passenger in M_i may stay at the bridging station for an exponential-distributed period of γ^{-1} on average. Assume a train arrives at t_i and the first batch of bridging buses reaches at t. Then, only the passengers whose waiting times are larger than $t - t_i$ are picked up by the bridging buses, and the corresponding probability is $e^{-\gamma(t-t_i)}$. To this end, a 0–1 sequence (1: picked up by bridging buses) is generated using a Bernoulli random variable with the mean likelihood of being "1" equal to $e^{-\gamma(t-t_i)}$.

Steps 1–3 can alternatively be performed with another algorithm. A realization of t is first generated with μ, and a number of exponentially distributed headways are generated with λ. The headways are added up until the accumulative sum goes beyond the range of t. Then, the starting points of all headways within t can be regarded as the instants of train arrivals. This algorithm is equivalent to that in Fig. 7. In order to create an objective environment for evaluation, the simulation setup stated above is purely Monte-Carlo-based and independent of the conclusions of Theorems 1–3.

4.3.3. Evaluation results

Theorem 1(a) delivers the mean and variance of the number $S_1(\tau)$ of affected passengers accumulated until the first batch of bridging buses reaches at a given time instant τ. Figs. 8 and 9 evaluate Theorem 1(a) against simulation results. Either figure covers 17 different values of τ over a range of 80 min on the x-axis (with each τ value corresponding to a pair of circle and cross in the figures). More specifically, each run of Step 1 in Fig. 7 to generate a τ value is followed by the continuous run of Steps 2–4 for 30 times, and the pursued mean and variance results at this τ are produced from these 30 times of repetitive simulations and displayed as a cross in Figs. 8 and 9, respectively. On the other hand, the theoretical results from Theorem 1(a) are displayed as circles in the figures. The theoretical results in Fig. 8 are linear with the time axis. This can be checked with Theorem 1(a), and specifically the slope of the line (made up of all circles) is equal to λC = 3600 persons/h = 600 persons/10 min. For the convenience of comparison, Fig. 9 presents the results of standard deviation (SD) rather than variance, and that is why the theoretical result in the figure shows a parabolic tendency ($SD = \sqrt{\lambda[C + C^2]\tau}$). Clearly, the simulation (numerical) results demonstrate the results of Theorem 1(a).

Figs. 10–13 evaluate Theorem 1(b) against simulation results for the mean and standard deviation of $S_1(t)$, where t is a random variable. The horizontal lines in Figs. 10 and 11 represent the theoretical results, while each circle corresponds to 100 simulation samples. More precisely, each circle in Figs. 10 and 11 is delivered by running Steps 1–4 as a whole loop for 100 times. The horizontal line in Fig. 11 corresponds to $\lambda C/\mu$ (= 2400 persons), and in Fig. 12 it does to $\sqrt{\frac{\lambda}{\mu}[C + C^2] + \frac{\lambda^2 C^2}{\mu^2}}$ (= 2684 persons). Theorem 1(b) addresses the average results, and this is verified by the fact that the horizontal lines in Figs. 10 and 11 stay in the middle of the simulation results. Figs. 12 and 13 present the relative error resulting from Figs. 10 and 11, respectively. The absolute mean of the relative error with Fig. 12 is 8.5%, while it is 9.8% with Fig. 13.

Theorem 1 (a) delivers the mean and variance of $S_2(\tau)$. Figs. 14 and 15 evaluate Theorem 1 (a) against simulation results. The shown simulations results are produced by further running Steps 4–6 (Section 4.3.2), on the basis of Steps 1–3 (see Fig. 7). Again, repetitive simulations are conducted 30 times (for steps 2–6) at each time instant τ to deliver the simulated mean and variance (i.e. each cross in Figs. 14 and 15). It is noticed from Theorem 1 (a) that the impact of the exponential terms decays. For the current simulation example, the theoretical mean value approaches 480 persons ($\lambda C(1-\theta)/\gamma$ = 480) and the theoretical standard deviation tends to be 340 persons $\left(\sqrt{\frac{\lambda C(1-\theta)}{\gamma} + \frac{\lambda C^2(1-\theta)^2}{2\gamma}} = 340\right)$. Figs. 16–19 evaluate Theorem 1

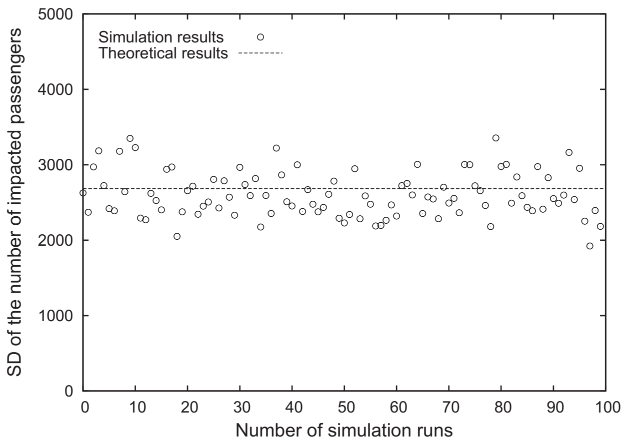

Fig. 11. Theorem 1(b): Var[$S_1(t)$].

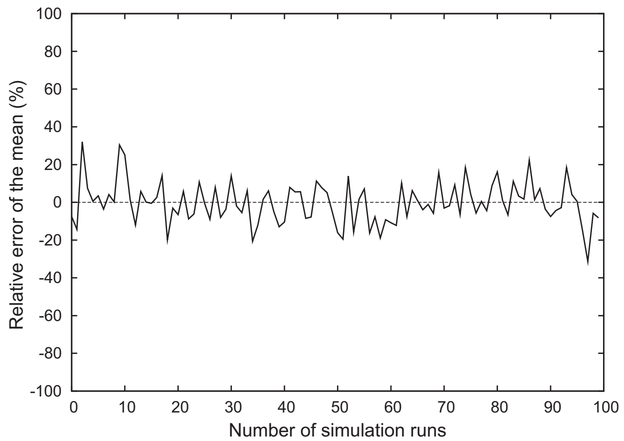

Fig. 12. Theorem 1(b): the simulation error of $E[S_1(t)]$ against the theoretical result.

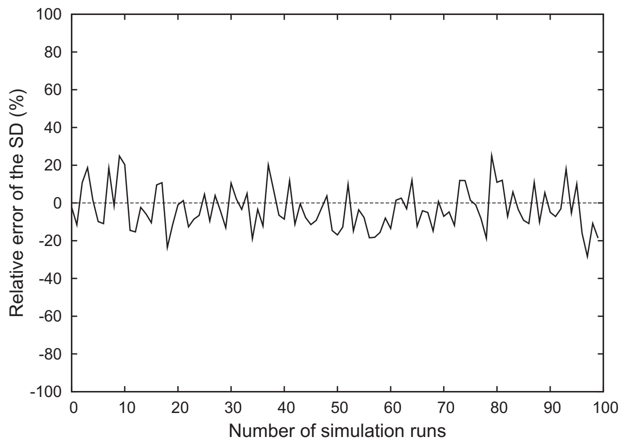

Fig. 13. Theorem 1(b): the simulation error of $\mathrm{Var}[S_1(t)]$ against the theoretical result.

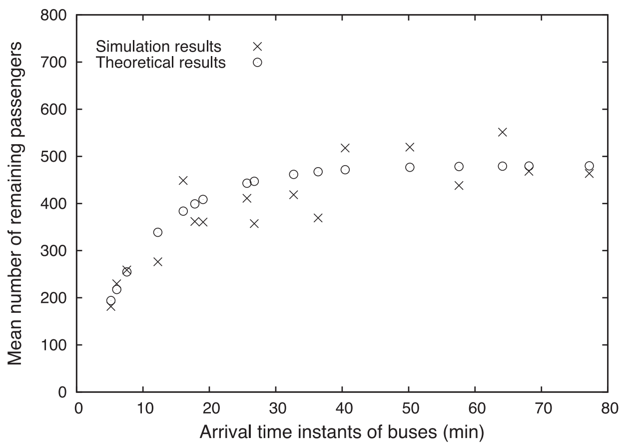

Fig. 14. Theorem 3(a): $E[S_2(\tau)]$.

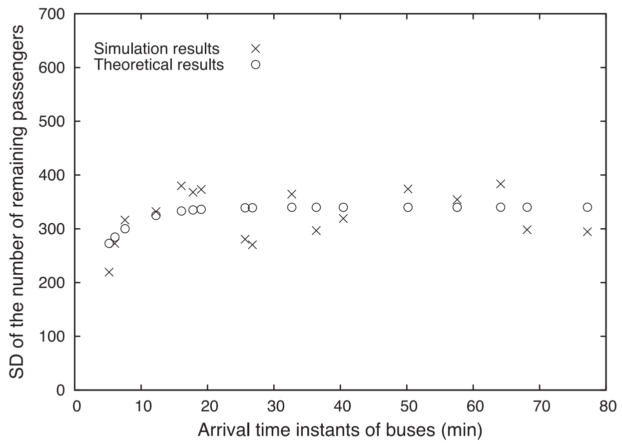

Fig. 15. Theorem 3(a): $\mathrm{Var}[S_2(\tau)]$.

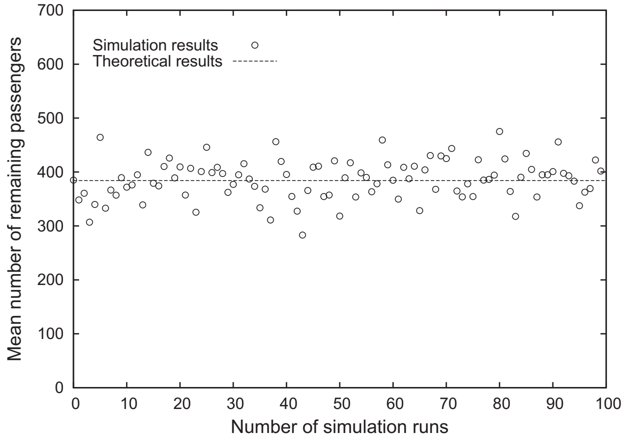

Fig. 16. Theorem 3(b): $E[S_2(t)]$.

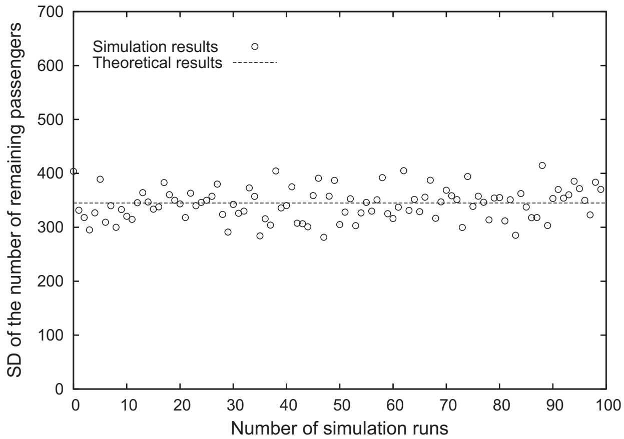

Fig. 17. Theorem 3(b): $\mathrm{Var}[S_2(t)]$.

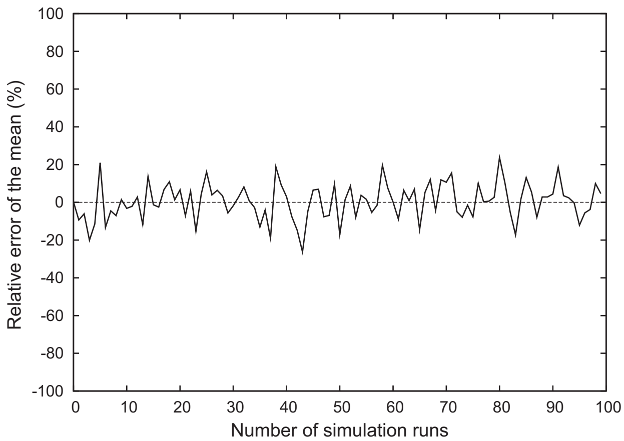

Fig. 18. Theorem 3(b): the simulation error of $E[S_2(t)]$ against the theoretical result.

(b) against the simulation results. In Figs. 16 and 17, the horizontal lines represent the theoretical results (384 and 345); each circle in either figure is delivered by running Steps 1–6 as a whole loop for 100 times. Figs. 18 and 19 present the relative error. The absolute mean of the relative error is 7.6% with Fig. 18 and 6.9% with Fig. 19. Clearly, the simulation results demonstrate the results of Theorem 3.

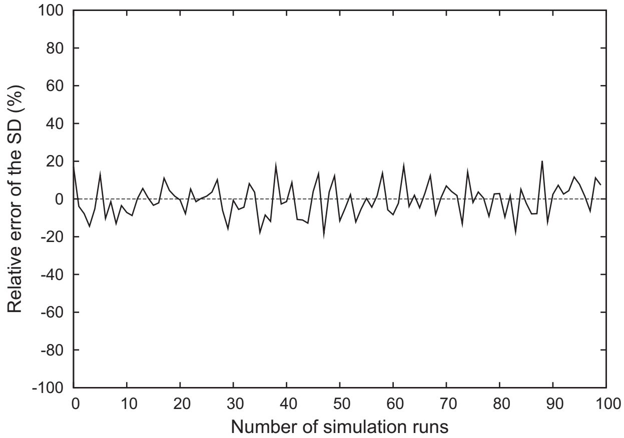

Fig. 19. Theorem 3(b): the simulation error of $\text{Var}[S_2(t)]$ against the theoretical result.

5. Further discussions

5.1. On stochastic assumptions involved

Theorems 1–3 involve three assumptions (Table 1):

(a) The arrival of batches of customers forms a Poisson process.
(b) Each batch contains a Poisson number of customers.
(c) Every non-balking customer stays in queue for an exponentially distributed period of time before reneging.

The bulk queueing problem with balking and reneging is of some general significance in practice (beyond the very scope of public transport services). Assumptions (a)–(c) are commonly considered in queueing theory. Specifically for the bus bridging problem, however, it is necessity to elaborate the consideration of these assumptions. First of all, (a) seems to be inconsistent with the practice in railway operation, whereby train arrivals are pre-scheduled with constant or semi-constant headways. Although the usage of (a) does facilitate the theoretical exploration as presented with Theorems 1–3, the replacement of (a) with a constant headway condition does not invalidate the main theoretical results obtained, see Appendix F for details.

As for the distribution of the number of passengers in a train carriage, very surprisingly, little information was found after a relatively thorough literature review. Nevertheless, it was noticed in some papers (e.g. Hickman, 2001) that the number of alighting passengers at a stop was assumed to follow a binomial distribution.[6] However, this does not help much in identifying the distribution of the number of passengers reaching each stop. We have proposed in the paper to use a Poisson distribution to this end, which can be reasoned as follows. Without loss of generality, passengers arriving at a station are assumed to follow a Poisson process or Poisson processes. Note that the sum of independent Poisson variables is still Poisson distributed. When a train arrives at a station, the accumulated number of passengers at the platform is therefore Poisson.[7] Assume each passenger has a same probability of stepping onto any carriage of the train. The number of boarding passengers for any carriage is Poisson distributed. Consider the train is empty before reaching the first station, and thus leave there with a Poisson number of boarding passengers in each carriage. At the second station, assume each passenger on board has a same probability to alight. Then, the number of alighting passenger is also Poisson. As such, the number of passengers carried by a carriage to any subsequent station is in fact the sum of a number of independent Bernoulli-sampled Poisson random variables, and hence is subject to the Poisson distribution.[8] As such, the total number of passengers carried by the whole train is also Poisson. It should also be emphasized the exact distribution form is actually not required for the main conclusions (Theorems 1 and 3). What is really of interest is the mean number C of passengers in the train (Table 1).

As stated in the introduction section, the balking and reneging behavior of passengers in public transport was not yet much studied by transport researchers. In fact, such behavior was only mentioned in one paper published by this journal

[6] Without tracing back to the origin of this proposal, we could understand it as follows. During a peak period, each carriage is supposed to be nearly full of passengers. At each stop, every passenger has a same probability p to alight, then the number of alighting passengers is subject to a binomial distribution $B(N,p)$, where N is the carriage capacity. If a carriage is far less full, however, this proposal may not hold strongly.

[7] The balking and reneging effects are negligible in this case.

[8] Due to the constraint of the carriage capacity, strictly speaking, the number of passengers each carriage may be subject to a truncated Poisson distribution; or when a carriage is nearly full, it may be more appropriate to use the binomial distribution instead, see the 6th.

(Rapoport et al., 2010). As such, there is no refereed evidence concerning possible distributions of the waiting period of a non-balking passenger before reneging. Nevertheless, it is routine practice in queueing-related study to model a (waiting) time period in an exponential distribution. This assumption virtually indicates that, due to the memorylessness, the time gap between the reneging actions of any two passengers (assuming at most one reneging action at any time instant) is also exponentially distributed. This further implies that the number of reneging passengers over a time period is Poisson distributed, which does not seem to be excessive.

5.2. No passenger left behind bridging services

As depicted in Fig. 1, $S_1(0) = 0$. This involves an assumption that the passengers remaining at the station when the bridging services start will all be picked up by bridging buses. Purely in terms of queueing theory, this may indicate that service capacity is nearly infinite, and hence may seem to be a bit idealized. Specifically for the practice of bus bridging, however, this may not be a concern. First of all, it is exactly the aim to serve all remaining passengers that we estimate via Theorem 3 the number of passengers so as to determine/recommend an appropriate size of bus platoons. Second, besides seats offered in buses, standees are allowed in order to avoid leaving any passengers not served. In practice, this is often adequate. Third, even if the leftover passengers are assumed, this would not add much difficulty to the theoretical exploration. This is because the passenger demand can be estimated and the capacity of a bus is known; thus the leftover demand can be estimated to determine $S_1(0)$ for the next round of bus bridging services. Except for this, there might not be any big change for the theoretical work presented with Theorems 1–3. Of course, the balking and reneging rates of the leftover passengers would be much higher than normal; see Section 5.3 for further discussion on the balking and reneging rates.

5.3. The balking and reneging rates

It is postulated in the paper that any queueing customer (or affected passenger in the case of bus bridging) makes decisions on balking and/or reneging randomly and independently. The genuine case in the circumstance of bus bridging could be much complicated. First, the balking rate may depend on a number of factors such as the locations of the bridging stations, the time of day, the population of affected passengers (initial queue length) and composition of these passengers. Second, the reneging rate is rather psychology-related. For a certain group of affected passengers, the longer to wait, the less patient they can be. However, it is also observed that the longer one waits the less is the propensity to renege (Maister, 1985; Larson, 1987). Quite likely, the distinction between the two groups is not fixed. A same passenger may fall in a different group, depending on specific cases. In addition, affected passengers arriving in different trains would be altogether at a bridging station, and the population of affected passengers may affect the decision of a single passenger. For some people, the more affected people to see, the more confident they may feel of waiting further; for some others, the converse may be perceived. In any case, the reneging rate is a function of at least two parameters: time spent in waiting (queuing time) and the total number of affected passengers to be with (queue length). Probably this reneging function in bus bridging is much different from that in supermarkets. To sum up, it is very hard to establish a deterministic model for balking and reneging phenomena, and therefore a simplified random model is adopted in this paper. The practical values (constant or varying) of the balking and reneging rates need to be identified with real data from traffic surveys. According to our literature review, no work was yet done along this line.

6. Conclusive remarks

Queueing or waiting for services is one of the unpleasant experiences of life ought to be better understood and modeled for a variety of economical, productivity and efficiency calculations. Queues of batch arrivals and service including balking and reneging behaviors of customers are commonly observed in public transport. This study formulates queues of this type using compound Poisson processes. Because of the complete treatment of this type of queueing problem is still an ongoing task of queueing theorists, this paper focuses on the probabilistic nature of balking and reneging behavior of impatient passengers as well as the impact of such behavior on queueing. The key probabilistic measures of such queues are determined via analytical study and evaluated using Monte-Carlo simulation. The contributions of this work are fourfold. First, the analytical exploration of the bulk queueing problem involving balking and reneging with special orientation to public transport services. Second, the introduction of the theory of compound Poisson processes as a new analytical conceptualization of the problem. Third, the determination of mean and variance for the length of a queue that is subject to the balking and reneging actions of impatient passengers. Fourth, the emulation of the compound Poisson processes with balking and reneging using large-scale Monte-Carlo simulations in the context of public transport services. The developed mathematical model, analytical approaches, and simulation methods can be applied to a variety of queueing processes of this type.

Acknowledgements

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Appendix A. Four Lemmas

Lemma 1. The law of iterated expectations and the law of total variance (Bertsekas and Tsitsiklis, 2008)

X and Y are two random variables, then

$$E[E[X|Y]] = E[X] \tag{3}$$
$$\mathrm{Var}[X] = E[\mathrm{Var}[X|Y]] + \mathrm{Var}[E[X|Y]] \tag{4}$$

Lemma 2. The mean and variance of a compound random variable (Tijms, 2003)

Let N be a non-negative and integer-valued random variable having finite first two moments. Let $\{M_i\}_{i=1}^N$ be a sequence of independent and identically distributed random variables with mean $E[M]$ and variance $Var[M]$. If N is independent of $M_1, M_2, \ldots$, then

$$E\left[\sum_{i=1}^{N} M_i\right] = E[N]E[M] \tag{5}$$
$$\mathrm{Var}\left[\sum_{i=1}^{N} M_i\right] = E[N]\mathrm{Var}[M] + \mathrm{Var}[N]E^2[M] \tag{6}$$

Lemma 2 can be derived with Lemma 1.

Lemma 3. Random sampling of a Poisson number (Prazen, 1967)

Let M_i be a Poisson random variable of parameter C. Let $\{B_j;\ j = 1, 2, \ldots\}$ be a Bernoulli process, independently of M_i, with probability p for success (i.e. $B_j = 1$) and $1 - p$ for failure (i.e. $B_j = 0$). Let M_{i1} be the total number of success among the number M_i of "tests", i.e. $M_{i1} = \sum_{j=1}^{M_i} B_j$. Then, M_{i1} and $M_{i2} = M_i - M_{i1}$ are independent Poisson variables, respectively, of parameters Cp and $C(1 - p)$.

Lemma 4. Campbell Theorem (Kulkarni, 2010)

Consider a Poisson counting process $N(t)$. Given that $N(t) = n$, i.e. n events have occurred in the interval $(0,t]$, the times of occurrence $0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant t$ have the same distribution as the ordered statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t]$.

Appendix B. Proof of Theorem 1

Theorem 1(a). *Given that $t = \tau$, $E[N(\tau)] = Var[N(\tau)] = \lambda\tau$ results immediately from the definition of $N(t)$. Based on this, it follows from Lemma 2 and the fact that $E[M_i] = Var[M_i] = C$,*

$$E[S_1(t)|t = \tau] = E\left[\sum_{i=1}^{N(\tau)} M_i\right] = \lambda C\tau \tag{7}$$
$$\mathrm{Var}[S_1(t)|t = \tau] = \mathrm{Var}\left[\sum_{i=1}^{N(\tau)} M_i\right] = \lambda\tau(C^2 + C) \qquad \square \tag{8}$$

Theorem 1(b). *It follows from $E[N(\tau)] = Var[N(\tau)] = \lambda\tau$ and Lemma 1 that*

$$E[N(t)] = \frac{\lambda}{\mu}, \quad \mathrm{Var}[N(t)] = \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 \tag{9}$$

Note that $E[N(t)] = \frac{\lambda}{\mu}$ may also be derived by use of some fundamental properties of the Poisson process (without using Lemma 1); see Appendix E. By Lemma 1 and Eq. (7),

$$E[S_1(t)] = E[E[S_1(t)|t = \tau]] = E[\lambda\tau C] = \frac{\lambda C}{\mu} \tag{10}$$

Again, from (7),

$$\mathrm{Var}[E[S_1(t)|t=\tau]] = \mathrm{Var}[\lambda\tau C] = \lambda^2 C^2 \frac{1}{\mu^2} \tag{11}$$

and with (8), we have,

$$E[\mathrm{Var}[S_1(t)|t=\tau]] = \frac{\lambda(C+C^2)}{\mu} \tag{12}$$

The use of Lemma 1 with (11) and (12) leads to

$$\mathrm{Var}[S_1(t)] = \frac{\lambda}{\mu}(C+C^2) + \frac{\lambda^2 C^2}{\mu^2} \tag{13}$$

□

Theorem 1(c).

$$Pr\{S_1(t)=k\} = E[Pr\{S_1(t)=k|t=\tau\}] = \int_0^\infty Pr\{S_1(\tau)=k\}\mu e^{-\mu\tau}d\tau \tag{14}$$

Note that the sum of a fixed number n of independent Poisson distributed variables each of parameter C is Poisson distributed with parameter nC,

$$Pr\{S_1(\tau)=k|N(\tau)=n\} = \frac{(nC)^k e^{-nC}}{k!}$$

Thus,

$$Pr\{S_1(\tau)=k\} = \sum_{n=0}^{\infty} Pr\{S_1(\tau)=k|N(\tau)=n\}Pr\{N(\tau)=n\} = \sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!} \tag{15}$$

Substituting (15) into (14) leads to

$$\begin{aligned}\int_0^\infty \sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!}\mu e^{-\mu\tau}d\tau &= \sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\int_0^\infty \frac{\mu(\lambda\tau)^n e^{-(\lambda+\mu)\tau}}{n!}d\tau \\ &= \sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\frac{\Gamma(n+1)}{n!}\left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^n \\ &= \left(\frac{\mu}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\left(\frac{\lambda}{\lambda+\mu}\right)^n \end{aligned} \tag{16}$$

Therefore,

$$\Pr\{S_1(t)=k\} = \left(\frac{\mu}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\left(\frac{\lambda}{\lambda+\mu}\right)^n \tag{17}$$

To verify (17), first the identity $\sum_{k=0}^{\infty} Pr\{S(t)=k\}=1$ holds.

$$\sum_{k=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!}\left(\frac{\lambda}{\lambda+\mu}\right)^n = \sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^n\sum_{k=0}^{\infty}\frac{(nC)^k e^{-nC}}{k!} = \frac{\mu}{\lambda+\mu}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+\mu}\right)^n = 1$$

Moreover, (10) derived using Lemmas 1 and 2 can also be reached through the definition of mathematic expectation using (17).

$$\begin{aligned}E[S_1(t)] &= \sum_{k=0}^{\infty} kPr\{S(t)=k\} = \sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^n\sum_{k=0}^{\infty}k\frac{(nC)^k e^{-nC}}{k!} = \sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^n nC \\ &= \frac{C\mu}{\lambda+\mu}\sum_{n=0}^{\infty}\left[(n+1)\left(\frac{\lambda}{\lambda+\mu}\right)^{(n+1)}\right] = \frac{\lambda C\mu}{(\lambda+\mu)^2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+\mu}\right)^n(n+1) = \lambda C\sum_{n=0}^{\infty}\frac{d}{d\lambda}\left[\left(\frac{\lambda}{\lambda+\mu}\right)^{n+1}\right] = \frac{\lambda C}{\mu}\end{aligned}$$

Note that (13) may also be derived from (17), albeit much more tedious. □

Appendix C. Proof of Theorem 2

Theorem 2(a). *It is straightforward with Lemma 3.* □

Theorem 2(b). *Any non-balking customers may choose to leave after a period of time that is exponentially distributed with parameter γ. Given a customer who arrives at t_i, the probability that he/she remains in queue when the first service starts at t is $e^{-\gamma(t-t_i)}$. Note that $t - t_i$ is still exponentially distributed (Papoulis and Pillai, 2002). As defined in Table 1, $Q_i(t)$ denote the part of M_{i2} that is eventually served by bridging buses. The proof is closed by following Theorem 2(a) and applying Lemma 3 to M_{i2} in consideration of the success rate $e^{-\gamma(t-t_i)}$.* □

Theorem 2(c). *By Theorem 2(b) and $\tau = t - t_i$,*

$$\int_0^\infty C(1-\theta)e^{-\gamma\tau}\mu e^{-\mu\tau}d\tau = C(1-\theta)\frac{\mu}{\mu+\gamma} \quad \square \tag{18}$$

Remarks: (18) may also be proved alternatively as follows:

The in-queue waiting time of a customer is an exponential variable (R_1) of parameter γ, and the duration $R_2 = t - t_i$ is exponentially distributed with parameter μ. The customers who are eventually served are those satisfying $R_1 \geqslant R_2$. Note that

$$Pr\{R_2 \leqslant R_1\} = \int_0^\infty Pr\{R_2 \leqslant R_1 | R_1 = x\}\gamma e^{-\gamma x}dx = \int_0^\infty [1-e^{-\mu x}]\gamma e^{-\gamma x}dx = \frac{\mu}{\mu+\gamma}$$

The use of Lemma 3 along with Theorem 2(a) and $Pr\{R_2 \leqslant R_1\}$ leads to (18). □

Theorem 2(d). *Note that $S_2(t_i) = M_{i2}$. $T_1 \triangleq \max\{t > t_i; Q_i(t) = M_{i2}\}$, $Pr\{T_1 > t\} = Pr\{Q_i(t) = M_{i2}\}$. Assume that every customer makes his/her decision independently, $Pr\{Q_i(t) = M_{i2}\} = [e^{-\gamma(t-t_i)}]^{M_{i2}}$. Thus,*

$$E[T_1|M_{i2} = m] = \int_{t_i}^\infty Pr\{T_1 > t | M_{i2} = m\}dt = \int_{t_i}^\infty e^{-m\gamma(t-t_i)}dt = \int_0^\infty e^{-m\gamma s}ds = \frac{1}{m\gamma}$$

M_{i2} is an integer. Based on the last equality above, it makes no sense if it equals 0. Therefore, by Lemma 1 and Theorem 2(a),

$$E[T_1|M_{i2} > 0] = E[E[T_1|M_{i2} = m]|m > 0] = \frac{1}{\gamma}E\left[\frac{1}{m}|m > 0\right] > \frac{1}{\gamma}\frac{1}{E[m|m > 0]}$$
$$= \frac{1}{\gamma}\left[\sum_{m=1}^\infty m\frac{[C(1-\theta)]^m e^{-C(1-\theta)}}{m!}\left(1-e^{-C(1-\theta)}\right)^{-1}\right]^{-1} = \frac{1}{\gamma}\left[\sum_{m=0}^\infty m\frac{[C(1-\theta)]^m e^{-C(1-\theta)}}{m!}\left(1-e^{-C(1-\theta)}\right)^{-1}\right]^{-1} = \frac{1-e^{-C(1-\theta)}}{\gamma C(1-\theta)}$$

The inequality above results from Jensen's inequality in the case of a convex function (Paolella, 2006). □

Theorem 2(e). *Note that $S_2(t_i) = M_{i2}$. $T_2 \triangleq \min\{t > t_i;\ Q_i(t) = 0\}$, $Pr\{T_2 \leqslant t\} = Pr\{Q_i(t) = 0\}$. Assume that every customer makes his/her decision independently, $Pr\{Q_i(t) = 0\} = [1 - e^{-\gamma(t-t_i)}]^{M_{i2}}$. Then,*

$$E[T_2|M_{i2} = m] = \int_{t_i}^\infty Pr\{T_2 > t | M_{i2} = m\}dt = \int_{t_i}^\infty \{1 - [1-e^{-\gamma(t-t_i)}]^m\}dt = \int_0^\infty \{1 - [1-e^{-\gamma s}]^m\}ds$$
$$= \int_0^1 (1-y^m)\frac{dy}{1-y}\gamma^{-1} = \frac{1}{\gamma}\int_0^1 \left(\frac{1-y^m}{1-y}\right)dy = \frac{1}{\gamma}\int_0^1 (1+y+y^2+\ldots+y^{m-1})dy = \frac{1}{\gamma}\sum_{i=1}^m \frac{1}{i} = \frac{1}{\gamma}[\ln(m)+\varphi+\varepsilon_m]$$

In the last equality, φ denotes Euler's constant and approximately equals 0.58 (Havil, 2009), ε_m approaches zero as m approaches infinite. Again, it makes no sense if M_{i2} equals zero. Thus,

$$E[T_2|M_{i2} > 0] = E[E[T_2|M_{i2} = m]|m > 0] = E\left[\frac{1}{\gamma}(\ln(m)+\varphi+\varepsilon_m)|m > 0\right] < \frac{1}{\gamma}\left[\ln\left[\frac{C(1-\theta)}{1-e^{-C(1-\theta)}}\right]+\varphi\right]$$

The inequality results from Jensen's inequality in the case of a concave function (Paolella, 2006), with the ε_m term omitted. □

Appendix D. Proof of Theorem 3

Theorem 3(a) (*Mean*).

By Lemma 1,

$$E[S_2(\tau)] = E[E[S_2(\tau)|N(\tau) = n]] = \sum_n E[S_2(\tau)|N(\tau) = n]Pr\{N(\tau) = n\} \tag{19}$$

Also, with (2), we have

$$E[S_2(\tau)|N(\tau) = n] = E\left[\sum_{i=1}^{N(\tau)} Q_i(\tau)|N(\tau) = n\right] = E\left[\sum_{i=1}^{N(\tau)} f(M_i, t_i, \theta, \gamma, \tau)|N(\tau) = n\right] = E\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)\right] \tag{20}$$

Note that the final equality of (20) results from Lemma 4. $u_{(1)}, u_{(2)}, \ldots, u_{(n)}$ are the ordered values of n independent random variables $u_1, u_2, \ldots, u_n$ that are uniformly distributed over $(0, \tau)$; the outcome of function f is a Poisson distributed. To continue with (20),

$$\begin{aligned} E\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)\right] &= E\left\{E\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)|u_{(i)}\right]\right\} \\ &= E\left[\sum_{i=1}^{n} C(1-\theta)e^{-\gamma(\tau-u_{(i)})}\right] \text{ (by Theorem 2(b))} \\ &= nC(1-\theta)E[e^{-\gamma(\tau-u_i)}] \\ &= nC(1-\theta)\tfrac{1}{\gamma\tau}[1-e^{-\gamma\tau}] \text{ (by Lemma 4)} \end{aligned} \tag{21}$$

Substituting (21) and (20) into (19), we have

$$E[S_2(\tau)] = \frac{C(1-\theta)}{\gamma\tau}[1-e^{-\gamma\tau}]E[N(\tau)] = \frac{C(1-\theta)}{\gamma\tau}[1-e^{-\gamma\tau}]\lambda\tau = \frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}] \qquad \square \tag{22}$$

Theorem 3(a) (*Variance*).

By Lemma 1,

$$\mathrm{Var}[S_2(\tau)] = E\{\mathrm{Var}[S_2(\tau)|N(\tau)=n]\} + \mathrm{Var}\{E[S_2(\tau)|N(\tau)=n]\} \tag{23}$$

Note that n in (21) is a random variable, then

$$\mathrm{Var}\{E[S_2(\tau)|N(\tau)=n]\} = C^2(1-\theta)^2\frac{1}{\gamma^2\tau^2}[1-e^{-\gamma\tau}]^2\lambda\tau \tag{24}$$

On the other hand,

$$E\{\mathrm{Var}[S_2(\tau)|N(\tau)=n]\} = E\left\{\mathrm{Var}\left[\sum_{i=1}^{N(t)} f(M_i, t_i, \theta, \gamma, \tau)|N(\tau)=n\right]\right\} = E\left\{\mathrm{Var}\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)\right]\right\} \text{ (By Lemma 4)} \tag{25}$$

In particular,

$$\begin{aligned} \mathrm{Var}\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)\right] &= \sum_{i=1}^{n} E\{\mathrm{Var}[f(M_i, u_{(i)}, \theta, \gamma, \tau)|u_{(i)}]\} \\ &\quad + \sum_{i=1}^{n} \mathrm{Var}\{E[f(M_i, u_{(i)}, \theta, \gamma, \tau)|u_{(i)}]\} \text{ (by Lemma 1)} \\ &= \sum_{i=1}^{n} E[C(1-\theta)e^{-\gamma(\tau-u_{(i)})}] \\ &\quad + \sum_{i=1}^{n} \mathrm{Var}[C(1-\theta)e^{-\gamma(\tau-u_{(i)})}] \text{ (by Theorem 2(b))} \end{aligned} \tag{26}$$

For the second term of (26), we have $\mathrm{Var}[e^{-\gamma(\tau-u_i)}] = \frac{1}{2\gamma\tau}[1-e^{-2\gamma\tau}] - \frac{1}{\gamma^2\tau^2}[1-e^{-\gamma\tau}]^2$, *because u_i is uniformly distributed over $(0,\tau)$. We then continue with (26) to get*

$$\mathrm{Var}\left[\sum_{i=1}^{n} f(M_i, u_{(i)}, \theta, \gamma, \tau)\right] = nC(1-\theta)\frac{1}{\gamma\tau}[1-e^{-\gamma\tau}] + nC^2(1-\theta)^2\left[\frac{1}{2\gamma\tau}[1-e^{-2\gamma\tau}] - \frac{1}{\gamma^2\tau^2}[1-e^{-\gamma\tau}]^2\right] \tag{27}$$

Substituting (27) into (25) leads to:

$$\begin{aligned} E\{\mathrm{Var}[S_2(\tau)|N(\tau)=n]\} &= \lambda\tau C(1-\theta)\frac{1}{\gamma\tau}[1-e^{-\gamma\tau}] + \lambda\tau C^2(1-\theta)^2\left[\frac{1}{2\gamma\tau}[1-e^{-2\gamma\tau}] - \frac{1}{\gamma^2\tau^2}[1-e^{-\gamma\tau}]^2\right] \\ &= \lambda C(1-\theta)\frac{1}{\gamma}[1-e^{-\gamma\tau}] + \lambda C^2(1-\theta)^2\left[\frac{1}{2\gamma}[1-e^{-2\gamma\tau}] - \frac{1}{\gamma^2\tau}[1-e^{-\gamma\tau}]^2\right] \end{aligned} \tag{28}$$

Substituting (28) and (24) into (23), we have

$$\begin{aligned} \mathrm{Var}[S_2(\tau)] &= \lambda C^2(1-\theta)^2\frac{1}{\gamma^2\tau}[1-e^{-\gamma\tau}]^2 + \lambda C(1-\theta)\frac{1}{\gamma}[1-e^{-\gamma\tau}] + \lambda C^2(1-\theta)^2\left[\frac{1}{2\gamma}[1-e^{-2\gamma\tau}] - \frac{1}{\gamma^2\tau}[1-e^{-\gamma\tau}]^2\right] \\ &= \frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}] + \frac{\lambda C^2(1-\theta)^2}{2\gamma}[1-e^{-2\gamma\tau}] \qquad \square \end{aligned} \tag{29}$$

Theorem 3(b) (*Mean*).

So far, we have derived with (22) *that* $E[S_2(\tau)] = \frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}]$. *Then, by* Lemma 1,

$$E[S_2(t)] = E[E[S_2(t)|t=\tau]] = E\left[\frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}]\right] = \frac{\lambda C(1-\theta)}{\gamma}\left[1-\int_0^\infty e^{-\gamma\tau}\mu e^{-\mu\tau}d\tau\right] = \frac{\lambda C(1-\theta)}{\mu+\gamma} \qquad \square$$

Theorem 3(b) (*Variance*).

It follows (22) *that*

$$\begin{aligned}\mathrm{Var}[E[S_2(t)|t=\tau]] &= \mathrm{Var}[E[S_2(\tau)]] = \mathrm{Var}\left[\frac{\lambda C(1-\theta)}{\gamma}[1-e^{-\gamma\tau}]\right] = \frac{\lambda^2C^2(1-\theta)^2}{\gamma^2}Var[1-e^{-\gamma\tau}]\\ &= \frac{\lambda^2C^2(1-\theta)^2}{\gamma^2}Var[e^{-\gamma\tau}] = \frac{\lambda^2C^2(1-\theta)^2}{\gamma^2}[E[e^{-2\gamma\tau}]-E^2[e^{-\gamma\tau}]] = \frac{\lambda^2C^2(1-\theta)^2}{\gamma^2}\left[\frac{\mu}{2\gamma+\mu}-\frac{\mu^2}{(\gamma+\mu)^2}\right]\\ &= \frac{\mu\lambda^2C^2(1-\theta)^2}{(2\gamma+\mu)(\gamma+\mu)^2}\end{aligned} \tag{30}$$

By (29),

$$\begin{aligned}E[\mathrm{Var}[S_2(t)|t=\tau]] &= E[\mathrm{Var}[S_2(\tau)]] = \frac{\lambda C(1-\theta)}{\gamma}\left[1-\int_0^\infty e^{-\gamma\tau}\mu e^{-\mu\tau}d\tau\right] + \frac{\lambda C^2(1-\theta)^2}{2\gamma}\left[1-\int_0^\infty e^{-2\gamma\tau}\mu e^{-\mu\tau}d\tau\right]\\ &= \frac{\lambda C(1-\theta)}{\mu+\gamma}+\frac{\lambda C^2(1-\theta)^2}{\mu+2\gamma}\end{aligned} \tag{31}$$

Finally, with (30) *and* (31),

$$\mathrm{Var}[S_2(t)] = E[\mathrm{Var}[S(t)|t=\tau]] + \mathrm{Var}[E[S(t)|t=\tau]] = \frac{\lambda C(1-\theta)}{\mu+\gamma}+\frac{\lambda C^2(1-\theta)^2}{\mu+2\gamma}+\frac{\mu\lambda^2C^2(1-\theta)^2}{(\mu+2\gamma)(\mu+\gamma)^2} \qquad \square$$

Appendix E

With reference to Fig. 1, let $x_1, x_2, \ldots, x_n$ represent consecutive inter-arrival times of customer batches over period t, that is, $x_1 = t_1$, $x_2 = t_2 - t_1, \ldots, x_n = t_n - t_{n-1}$. Thus,

$$Pr\{x_1+x_2+\cdots+x_n<t\} = Pr\{x_1+x_2+\cdots+x_n<t|x_1<t\}Pr\{x_1<t\}$$

From Table 1, t is exponentially distributed and x_i ($i = 1, 2$) are independent and identically distributed. By the lack-of-memory property,

$$Pr\{x_1+x_2+\cdots+x_n<t|x_1<t\} = Pr\{x_2+\cdots+x_n<t\}$$

and as such,

$$Pr\{x_1+x_2+\cdots+x_n<t\} = [Pr\{x<t\}]^n$$

Following the definition of mean,

$$E[N(t)] = \sum_{n=1}^{\infty}Pr\{N(t)\geqslant n\} = \sum_{n=1}^{\infty}Pr\{x_1+x_2+\cdots+x_n\leqslant t\} = \sum_{n=1}^{\infty}[Pr\{x<t\}]^n = \frac{Pr\{x<t\}}{1-Pr\{x<t\}}$$

Note that

$$Pr\{x<t\} = \int_0^\infty Pr\{x<t|x=\theta\}f(\theta)d\theta = \int_0^\infty e^{-\mu\theta}\lambda e^{-\lambda\theta}d\theta = \frac{\lambda}{\lambda+\mu}$$

Thus,

$$E[N(t)] = \frac{Pr\{x<t\}}{1-Pr\{x<t\}} = \frac{\lambda}{\mu}$$

Appendix F. Theorems 1–3 in the case of a constant train headway

In contrast to Figs. 6, 20 displays the case of a constant headway of trains, where $N(t)$ in Table 2 is exactly equal to λt, with λ^{-1} being the constant train headway. Accordingly, Theorems 1–3 are updated as follows:

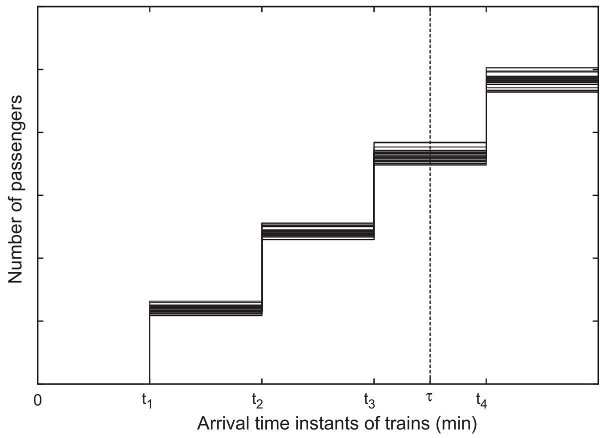

Fig. 20. Thirty realizations of $S_1(t)$ in the case of a constant train headway.

Theorem 1-1

(a)

$$E[N(\tau)] = \lambda\tau$$
$$\mathrm{Var}[N(\tau)] = 0$$
$$E[S_1(\tau)] = \lambda C\tau$$
$$\mathrm{Var}[S_1(\tau)] = \lambda C\tau$$

(b)

$$E[N(t)] = \frac{\lambda}{\mu}$$
$$\mathrm{Var}[N(t)] = \frac{\lambda^2}{\mu^2}$$
$$E[S_1(t)] = \frac{\lambda C}{\mu}$$
$$\mathrm{Var}[S_1(t)] = \frac{\lambda C}{\mu} + \frac{\lambda^2 C^2}{\mu^2}$$

(c)

$$Pr\{S_1(t) = k\} = \left(\frac{\mu}{\lambda C + \mu}\right)\left(\frac{\lambda C}{\lambda C + \mu}\right)^k$$

Compared to Theorem 1, it is not surprising that the use of a constant train headway leads to smaller variances.

Theorem 2-1. *It is the same as Theorem 2.*

Theorem 3-1

(a)

$$E[S_2(\tau)] = \frac{\lambda C(1-\theta)}{\gamma}[1 - e^{-\gamma\tau}]$$

$$\mathrm{Var}[S_2(\tau)] = \frac{\lambda C(1-\theta)}{\gamma}[1 - e^{-\gamma\tau}] + \frac{\lambda C^2(1-\theta)^2}{2\gamma}[1 - e^{-2\gamma\tau}] - \frac{\lambda C^2(1-\theta)^2}{\gamma^2\tau}[1 - e^{-\gamma\tau}]^2$$

(b)

$$E[S_2(t)] = \frac{\lambda C(1-\theta)}{\mu + \gamma}$$

$$\mathrm{Var}[S_2(t)] = \frac{\lambda C(1-\theta)}{\mu+\gamma} + \frac{\lambda C^2(1-\theta)^2}{\mu+2\gamma} + \frac{\mu\lambda^2 C^2(1-\theta)^2}{(\mu+2\gamma)(\mu+\gamma)^2} - \frac{\lambda C^2(1-\theta)^2\mu}{\gamma^2}\int_0^\infty \frac{e^{-\mu\tau}}{\tau}[1-e^{-\gamma\tau}]^2 d\tau$$

Compared to Theorem 3, the same means but smaller variances yield in the case of constant headways (see the additional terms of the negative sign appearing on the right-hand side of $\mathrm{Var}[S_2(\tau)]$ and of $\mathrm{Var}[S_2(t)]$). This also confirms our intuition concerning variances by comparing Fig. 6 and Fig. 21. Mathematically, this is due to the removal of the second term on the right-hand side of (4) when it is applied to a constant headway case. It is noticed that there is no compact analytical expression for the last term on the right-hand side of $\mathrm{Var}[S_2(t)]$. Nevertheless, $\int_0^\infty \frac{e^{-\mu\tau}}{\tau}[1-e^{-\gamma\tau}]^2 d\tau$ can be approximately expressed using the exponential integral $E_1(z) = \int_z^\infty \frac{e^{-\tau}}{\tau} d\tau$ (this is not an elementary function). Considering that τ in Theorem 3-1(b) must have a lower bound τ_0,

$$\int_0^\infty \frac{e^{-\mu\tau}}{\tau}[1-e^{-\gamma\tau}]^2 d\tau \approx \int_{\tau_0}^\infty \frac{e^{-\mu\tau}}{\tau}[1-e^{-\gamma\tau}]^2 d\tau = E_1(\mu\tau_0) - 2E_1((\mu+\gamma)\tau_0) + E_1((\mu+2\gamma)\tau_0)$$

A further estimate may be obtained using the relation $\frac{1}{2}e^{-z}ln(1+\frac{2}{z}) < E_1(z) < e^{-z}ln(1+\frac{1}{z})$. The proof of Theorems 1, $2-1$, and $3-1$ follows the same line of thoughts with Theorems 1–3. It is inspiring to see that the consideration of a constant headway does not necessarily lead to simpler conclusions than with an exponential headway. Exponential distributions (or sometimes its extension to the Gamma/Erlang distributions) are often used in the mathematics of stochastic processes, not only because of the general applicability of the distributions, but also the mathematical tractability enabled by the memoryless property of the distributions. Having this in mind, it is still thought-provoking that it is mathematically easier to handle the situation depicted in Fig. 6 than in Fig. 20.

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