Automatica 64 (2016) 174–181

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Mode-identifying time estimation and switching-delay tolerant control for switched systems: An elementary time unit approach☆

Lixian Zhang [a], Weiming Xiang [b,c]

[a] Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin, 150080, China
[b] School of Transportation and Logistics, Southwest Jiaotong University, Chengdu, 610031, China
[c] Department of Computer Science and Engineering, University of Texas at Arlington, Arlington, TX 76019, USA

ARTICLE INFO

Article history:
Received 2 July 2014
Received in revised form
28 September 2015
Accepted 21 October 2015
Available online 7 December 2015

Keywords:
Elementary time unit
Exponential stability
Mode identification
Switched system
Switching-delay tolerant control

ABSTRACT

In this paper, an elementary time unit (ETU) method is proposed to study the problem of estimating the admissible delay in the identification of the active mode in the analysis and design of switched systems. The activation interval of a subsystem is considered to consist of a finite-time number of segments called ETUs, by which a novel class of time-scheduled Lyapunov function is used to estimate the admissible delay in mode identification for switched systems. Further, the ETU method is applied for switching-delay tolerant control problem, and a class of time-scheduled state feedback controllers are designed to achieve the exponentially stability. Several numerical examples are presented to validate the theoretic findings.

1. Introduction

Switched systems have emerged as an important class of hybrid systems representing a very active research area in the field of systems and control (Chesi, Colaneri, Geromel, Middleton, & Shorten, 2012; Dehghan & Ong, 2012; Duan & Wu, 2014; Geromel & Colaneri, 2006; Hu, Shen, & Zhang, 2011; Lee & Dullerud, 2007; Lin & Antsaklis, 2007, 2009; Lu, Wu, & Kim, 2006; Margaliot, 2006; Shorten, Wirth, Mason, Wulff, & King, 2007; Zhang, Hu, & Abate, 2012). Switched system can be efficiently used to model many practical systems that are inherently multi-model in the sense that several dynamical subsystem models are required to describe their behaviors, e.g. see Morse (1996). A basic issue in control of switched systems is the mode identification to implement mode-dependent controllers. The mode identification is necessary in practice, and the process with multiple operating modes has to be controlled by a set of multiple controllers along with a mode estimator. Therefore, to determine the admissible mode-identifying time for concrete switched systems will be of great significance as *a priori* information to evaluate different identification methods. However, note that the problem still remains largely open in the area.

In most of real applications, the value of switching signal $\sigma_p(t)$ is unavailable once the switching of process occurs. One would naturally try to first identify the switching signal at the beginning of each time interval using a short time period $[t_k, t_k+\tau)$ (generally $\tau \ll t_{k+1} - t_k$), e.g. some design results for mode estimator in Baglietto, Battistelli, and Tesi (2013), Battistelli (2013), and then control the identified system in the rest of the time interval. As what has been suggested in numerous articles (Mahmoud & Shi, 2012; Vu & Morgansen, 2010; Xiang, Xiao, & Iqbal, 2011; Zhang & Gao, 2010; Zhang & Shi, 2009), the very first and important concern is that the inappropriately large (inadmissible) mode-identifying time τ would turn the stable closed loop to be unstable since the correct controller cannot be activated in time. Thus, the problem arises here:

- **Problem A.** Given a set of feedback controllers, how to estimate the admissible mode-identifying time τ by which the stability of closed loop holds?

☆ The work was partially supported by National Natural Science Foundation of China (51477146, 51177137, 61134001 and 61322301), National Natural Science Foundation of Heilongjiang (F201417, JC2015015), the Fundamental Research Funds for the Central Universities, China HIT.BRETIII.201211 and HIT.BRETIV.201306, and Air Force Research Laboratory under agreement number FA8750-15-1-0105. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Denis Arzelier under the direction of Editor Richard Middleton.

E-mail addresses: lixianzhang@hit.edu.cn (L. Zhang), xwm1004@163.com (W. Xiang).

http://dx.doi.org/10.1016/j.automatica.2015.11.010
0005-1098/

Then, from the control point of view, designing a set of controllers stabilizing the closed loop in the presence of a mode-identifying time (producing switching delay) τ is of interest. Furthermore, the controller is expected to tolerate the switching delay as large as possible. Therefore, as an opposite to Problem A, the switching-delay tolerant control problem considered in this paper is:

- **Problem B.** How to design a set of stabilizing feedback controllers, which is capable of being tolerant to the switching delay τ as large as possible?

Due to the inevitable and necessary existence of mode-identifying process in actual applications, both of above problems are not only theoretically interesting and challenging, but also very important in practical applications, which motivates the present study in this paper.

Inspired by the discretized Lyapunov function approach widely used in time-delay systems (Gu, Kharitonov, & Chen, 2003), and also inspired by the recent articles for switched system with dwell time constraint (Allerhand & Shaked, 2011, 2013), the elementary time unit (ETU) approach is proposed to solve the problems presented in this paper. Briefly speaking, the ETU is the activation interval being classified by mode-identifying interval and normal-working interval, and both of which consist of a finite number of segments with different resolutions. It is worth mentioning that some previous results such as Mahmoud and Shi (2012), Zhang and Gao (2010), Zhang and Shi (2009) and Xiang et al. (2011), concerned with asynchronously switched system are covered by our ETU approach. The remainder of this paper is organized as follows. In Section 2, the considered systems and problems are formulated. In Section 3, the ETU technique is introduced and the mode-identifying time estimation is studied. The switching-delay tolerant control is studied in Section 4. Conclusions are given in Section 5.

Notations: Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_{\geq 0}$ stand for non-negative real numbers, and $\mathbb{R}^n$ be the n-dimensional real vector space. $\|\cdot\|$ stands for Euclidean norm. The notation $P \succ 0$ ($P \succeq$) means matrix P is real symmetric and positive definite (semi-positive definite). $P^\top$ denotes the transposition of matrix P and $\mathbf{He}\{P\} = P^\top + P$. Function int[$x$] rounds the x to the nearest integer towards zero. $\xi_i(\cdot) : [0, \infty) \to \{0, 1\}$ are indication functions for switching signal $\sigma(t)$, it is defined as $\xi_i(t) = 1$ if $\sigma(t) = i$, otherwise $\xi_i(t) = 0$.

2. System description and problem formulation

In this paper, the switched system is in the form of

$$\dot{x}(t) = A_{\sigma_p(t)}x(t) + B_{\sigma_p(t)}u(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input. Define index set $\mathcal{I} := \{1, 2, \ldots, N\}$ where N is the number of subsystems. $\sigma_p(t) : [0, \infty) \to \mathcal{I}$ denotes the switching signal, which is assumed to be a piecewise constant function continuous from right. $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are known matrices of appropriate dimension. The discontinuities of $\sigma_p(t)$ are called *switches* and the switching sequence is expressed by

$$\mathcal{S}_p := \{(i_0, t_0), \ldots, (i_k, t_k), \ldots | i_k \in \mathcal{I}, k = 0, 1, \ldots\}$$

where t_0 denotes the initial time, t_k is the kth switching instant and $d_k = t_{k+1} - t_k$, $k = 0, 1, \ldots$. It is assumed that there exists a lower bound of d_k, i.e., $d_k \geq d_{\min} > 0$, $\forall k = 0, 1, \ldots$, where $d_{\min}$ is called minimal dwell time.

The mode-dependent controller is considered as

$$u(t) = K_{\sigma_c(t)}x(t) \tag{2}$$

where $K_i \in \mathbb{R}^{m\times n}$, for $\sigma_c(t) = i \in \mathcal{I}$ are constant matrix gains already designed. $\sigma_c(t) : [0, \infty) \to \mathcal{I}$ is the switching signal of controller. In the presence of mode-identifying process, the identification time is denoted as τ which directly leads to $\sigma_c(t) = \sigma_p(t - \tau)$. Trivially, it is assumed $0 < \tau < d_{\min}$. Hence, the switching sequence generated by $\sigma_c(t)$ can be expressed by

$$\mathcal{S}_c := \left\{(i_0, \hat{t}_0), \ldots, (i_k, \hat{t}_k), \ldots | i_k \in \mathcal{I}, k = 0, 1, \ldots\right\}$$

in which $\hat{t}_k = t_k + \tau < t_{k+1}$, $\forall k = 0, 1, \ldots$. Combining the sequences $\mathcal{S}_p$ and $\mathcal{S}_c$ and inspired by the idea in Vale and Miller (2011), each interval $[t_k, t_{k+1})$ can be essentially classified into the mode-identifying period $\mathcal{M}_1 := [t_k, t_k + \tau)$ with $\sigma_p(t) \neq \sigma_c(t)$, and the normal-working period $\mathcal{M}_2 := [t_k + \tau, t_{k+1})$ with $\sigma_p(t) = \sigma_c(t)$.

Let $\bar{A}_{i,j} := A_i + B_iK_j$ and $\mathcal{I}^2 := \mathcal{I} \times \mathcal{I}$, $\mathcal{I}^2_{\mathcal{M}_1}$ be the set of all indices in $\mathcal{I}^2$ such that $i \neq j$, $\forall i, j \in \mathcal{I}$, and $\mathcal{I}^2_{\mathcal{M}_2} := \mathcal{I}^2 \setminus \mathcal{I}^2_{\mathcal{M}_1}$ includes all indices such that $i = j$, $\forall i, j \in \mathcal{I}$. Substituting controller (2) into system (1) and considering the mode estimator, the dynamics of closed loop in interval $[t_k, t_{k+1})$ with $\sigma_p(t) = i$, $t \in [t_k, t_{k+1})$ and $\sigma_c(t) = j$, $t \in [t_k, t_k + \tau)$ can be derived as follows:

$$\dot{x}(t) = \begin{cases} \bar{A}_{i,j}x(t) & t \in [t_k, t_k + \tau),\ (i, j) \in \mathcal{I}^2_{\mathcal{M}_1} \\ \bar{A}_{i,i}x(t) & t \in [t_k + \tau, t_{k+1}),\ (i, i) \in \mathcal{I}^2_{\mathcal{M}_2} \end{cases} \tag{3}$$

where $0 < \tau < d_{\min}$ is the mode-identifying time.

Definition 1. System (3) is said to be exponentially stable with a decay rate $\beta > 0$ if $\|x(t)\| < Ce^{-\beta(t-t_0)}\|x(t_0)\|$ holds for any $x(t_0)$, any $t \geq t_0$ and a constant $C > 0$.

Then, given a mode-identifying time τ, the first problem is restated as follows.

Problem 1. Find sufficient conditions on the controller (2) for system (1) and on the mode-identifying time τ such that the closed loop (3) is exponentially stable.

When τ is uncertain, one could expect that an upper bound on it denoted by τ^*, below which the stability of the closed loop is guaranteed.

Problem 2. Given the controller (2) for system (1), estimate the upper bound of admissible mode-identifying time τ^* such that the exponential stability of closed loop (3) can hold for any $\tau \leq \tau^*$.

As to switching-delay tolerant control problem, the following time-varying controller covering (2) is considered

$$u(t) = \mathcal{K}_{\sigma_c(t)}(t)x(t) \tag{4}$$

where $\mathcal{K}_i(t)$, $i \in \mathcal{I}$ are time-varying gain matrices to be determined. The closed loop system can be expressed by

$$\dot{x}(t) = \begin{cases} \bar{\mathcal{A}}_{i,j}(t)x(t) & t \in [t_k, t_k + \tau),\ (i, j) \in \mathcal{I}^2_{\mathcal{M}_1} \\ \bar{\mathcal{A}}_{i,i}(t)x(t) & t \in [t_k + \tau, t_{k+1}),\ (i, i) \in \mathcal{I}^2_{\mathcal{M}_2} \end{cases} \tag{5}$$

where $\bar{\mathcal{A}}_{i,j}(t) = A_i + B_i\mathcal{K}_j(t)$, $\bar{\mathcal{A}}_{i,i}(t) = A_i + B_i\mathcal{K}_i(t)$.

Problem 3. Consider switched system (1) with a switching delay τ, design a state feedback control scheme (4) guaranteeing the exponential stability of closed loop (5).

At last when τ is uncertain, Problem 3 can be developed accordingly as follows.

Problem 4. Consider switched system (1) with switching delay, design a state feedback control scheme (4) with maximal ability of tolerating switching delay τ^* guaranteeing the exponential stability of closed loop (5).

The above four linked problems are the main concerns to be addressed in the rest of this paper.

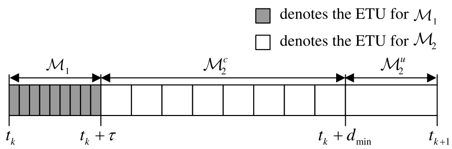

Fig. 1. ETUs for the time interval $[t_k, t_{k+1})$.

3. Mode-identifying time estimation

3.1. Elementary time unit approach

Due to the existence of mode-identifying process, each time interval $[t_k, t_{k+1})$ can be classified into two categories as mode-identifying interval $\mathcal{M}_1 := [t_k, t_k+\tau)$ and normal-working mode interval $\mathcal{M}_2 := [t_k+\tau, t_{k+1})$. For each interval $\mathcal{M}_p, p = 1, 2$, we define the elementary time unit (ETU) which is the basic segment constituting the time intervals. Suppose that the interval $[t_k, t_k+\tau)$ is constituted by L_1 elementary segments, the series of elementary segments are called the ETUs for $\mathcal{M}_1$, which are denoted as $\mathcal{N}_{1,k,n} := [t_k+\theta_{1,n}, t_k+\theta_{1,n+1}), n = 0, 1, \ldots, L_1-1$ of equal length h_1. The value of h_1 is called the resolution of ETU for $\mathcal{M}_1$, and $\theta_{1,n} = nh_1$. The ETUs for $\mathcal{M}_2$ can be defined likewise. However, since the length of interval $[t_k+\tau, t_{k+1})$ is uncertain, the interval $[t_k+\tau, t_{k+1})$ is supposed to be classified into the certain interval $\mathcal{M}_2^c := [t_k+\tau, t_k+d_{\min})$ and uncertain interval $\mathcal{M}_2^u := [t_k+d_{\min}, t_{k+1})$. Obviously, only the certain time interval $\mathcal{M}_2^c$ can be possibly divided into L_2 segments with equal length. The ETUs for $\mathcal{M}_2^c$ are denoted as $\mathcal{N}_{2,k,n} := [t_k+\tau+\theta_{2,n}, t_k+\tau+\theta_{2,n+1}), n = 0, 1, \ldots, L_2-1$ with resolution h_2, and then $\theta_{2,n} = nh_2$. Afterwards the rest uncertain interval $\mathcal{M}_2^u$ is denoted as $\mathcal{N}_{2,k,L_2}$. Finally, the ETUs in $[t_k, t_{k+1}) = \mathcal{M}_1 \cup \mathcal{M}_2$ explicitly have properties that $\bigcup_{n=0}^{L_1-1} \mathcal{N}_{1,k,n} = \mathcal{M}_1, \bigcup_{n=0}^{L_2} \mathcal{N}_{2,k,n} = \mathcal{M}_2$ and $\mathcal{N}_{p,k,n} \cap \mathcal{N}_{p,k,m} = \emptyset, n \neq m, \forall p = 1, 2$. The ETUs in the interval $[t_k, t_{k+1})$ are illustrated by Fig. 1.

Based on ETU, a set of time-scheduled Lyapunov functions $\mathcal{V}_i(t,x) = x^\top(t)\mathcal{P}_i(t)x(t), i \in \mathcal{I}$ is constructed as

$$\mathcal{V}_i(t,x) = \begin{cases} x^\top(t)\mathcal{P}_{1,i,n}(\alpha_{1,n})x(t) & t \in \mathcal{N}_{1,k,n} \\ x^\top(t)\mathcal{P}_{2,i,n}(\alpha_{2,n})x(t) & t \in \mathcal{N}_{2,k,n} \\ x^\top(t)P_{2,i,L_2}x(t) & t \in \mathcal{N}_{2,k,L_2} \end{cases} \tag{6}$$

where $\mathcal{P}_{1,i,n}(\alpha_{1,n}) = (1-\alpha_{1,n})P_{1,i,n} + \alpha_{1,n}P_{1,i,n+1}$ with $\alpha_{1,n} = (t-t_k-\theta_{1,n})/h_1$, $\mathcal{P}_{2,i,n}(\alpha_{2,n}) = (1-\alpha_{2,n})P_{2,i,n} + \alpha_{2,n}P_{2,i,n+1}$ with $\alpha_{2,n} = (t-t_k-\tau-\theta_{2,n})/h_2$, and $P_{p,i,n} \succ 0, n = 0, 1, \ldots, L_p, p = 1, 2, i \in \mathcal{I}$.

3.2. Admissible mode-identifying time estimation

At first, two useful lemmas are given as below.

Lemma 1. *Given a scalar $\eta_1 > 0$ and consider system (3), if there exists a set of matrices $P_{1,i,n} \succ 0, n = 0, 1, \ldots, L_1, i \in \mathcal{I}$ such that $\Omega_{1,i,n}^{(1)} \prec 0, \Omega_{1,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_1-1, \forall i, j \in \mathcal{I}$, where*

$$\Omega_{1,i,n}^{(1)} = \mathbf{He}\{P_{1,i,n}\bar{A}_{i,j}\} + \Psi_{1,i,n} - \eta_1 P_{1,i,n} \tag{7}$$

$$\Omega_{1,i,n}^{(2)} = \mathbf{He}\{P_{1,i,n+1}\bar{A}_{i,j}\} + \Psi_{1,i,n} - \eta_1 P_{1,i,n+1} \tag{8}$$

$$\Psi_{1,i,n} = (P_{1,i,n+1} - P_{1,i,n})/h_1, \tag{9}$$

then during interval $\mathcal{M}_1 := [t_k, t_k+\tau)$, it follows that

$$\mathcal{V}_i(t,x) < e^{\eta_1(t-t_k)}\mathcal{V}_i(t_k,x), \quad \forall t \in \mathcal{M}_1. \tag{10}$$

Proof. See Appendix. □

Lemma 2. *Given a scalar $\eta_2 > 0$ and consider system (3), if there exists a set of matrices $P_{2,i,n} \succ 0, n = 0, 1, \ldots, L_2, i \in \mathcal{I}$ such that $\Omega_{2,i,n}^{(1)} \prec 0, \Omega_{2,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_2-1, \forall i \in \mathcal{I}, \Omega_{2,i,L_2} \prec 0$, where*

$$\Omega_{2,i,n}^{(1)} = \mathbf{He}\{P_{2,i,n}\bar{A}_{i,i}\} + \Psi_{2,i,n} + \eta_2 P_{2,i,n} \tag{11}$$

$$\Omega_{2,i,n}^{(2)} = \mathbf{He}\{P_{2,i,n+1}\bar{A}_{i,i}\} + \Psi_{2,i,n} + \eta_2 P_{2,i,n+1} \tag{12}$$

$$\Omega_{2,i,L_2} = \mathbf{He}\{P_{2,i,L_2}\bar{A}_{i,i}\} + \eta_2 P_{2,i,L_2} \tag{13}$$

$$\Psi_{2,i,n} = (P_{2,i,n+1} - P_{2,i,n})/h_2, \tag{14}$$

then during interval $\mathcal{M}_2 := [t_k+\tau, t_{k+1})$, it follows that

$$\mathcal{V}_i(t,x) < e^{-\eta_2(t-t_k-\tau)}\mathcal{V}_i(t_k+\tau,x), \quad \forall t \in \mathcal{M}_2. \tag{15}$$

Proof. See Appendix. □

Based on the above lemmas, we are ready to present our first result with respect to Problem 1. Since the mode-identifying time τ is given in advance, the resolutions of ETUs h_1 and h_2 for Problem 1 are naturally selected by $h_1 = \tau/L_1$ and $h_2 = (d_{\min}-\tau)/L_2$, where L_1 and L_2 are the numbers of dividing points for intervals $[t_k, t_k+\tau)$ and $[t_k+\tau, t_k+d_{\min})$, respectively.

Theorem 1. *Consider switched system (1) with controller (2) and a mode estimator with mode-identifying time τ, given the division parameters $L_p, p = 1, 2$, if there exist scalars $\mu_1 > 0, \mu_2 > 0, \eta_1 > 0, \eta_2 > 0$, and a set of matrices $P_{p,i,n} \succ 0, n = 0, 1, \ldots, L_p, p = 1, 2, i \in \mathcal{I}$ such that*

(i) $\Omega_{1,i,n}^{(1)} \prec 0, \Omega_{1,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_1-1$;
(ii) $\Omega_{2,i,n}^{(1)} \prec 0, \Omega_{2,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_2-1, \Omega_{2,i,L_2} \prec 0$;
(iii) $P_{2,i,0} \preceq \mu_1 P_{1,i,L_1}, P_{1,i,0} \preceq \mu_2 P_{2,j,L_2}, i \neq j$;
(iv) $\ln\mu_1 + \ln\mu_2 + \eta_1\tau - \eta_2(d_{\min}-\tau) < 0$;

where $\Omega_{1,i,n}^{(1)}, \Omega_{1,i,n}^{(2)}, \Omega_{2,i,n}^{(1)}, \Omega_{2,i,n}^{(2)}, \Omega_{2,i,L_2}$ are defined in (7)–(9), (11)–(14), then the closed loop system (3) is exponentially stable.

Proof. For the closed loop (3), the Lyapunov function is chosen to be $\mathcal{V}(t) = \sum_{i=1}^{N}\xi_i(t)\mathcal{V}_i(t,x)$ where $\xi_i(t), i \in \mathcal{I}$ are the indication functions and $\mathcal{V}_i(t,x), i \in \mathcal{I}$ are defined in (6). We define notations $\mathcal{V}(t^-) = \lim_{t\to t^-}\mathcal{V}(t)$, $\mathcal{V}(t^+) = \lim_{t\to t^+}\mathcal{V}(t)$, and Condition (iii) means that

$$\mathcal{V}(\hat{t}_k^+) \le \mu_1\mathcal{V}(\hat{t}_k^-), \; \hat{t}_k = t_k+\tau \in [t_k, t_{k+1}) \tag{16}$$

$$\mathcal{V}(t_k^+) \le \mu_2\mathcal{V}(t_k^-). \tag{17}$$

Then, from Conditions (i), (ii) and using (10), (15) by Lemmas 1 and 2, it follows that

$$\mathcal{V}(t) < Ce^{-\eta_2(t-t_k)}\mathcal{V}(t_k^+), \quad t \in [t_k, t_{k+1}) \tag{18}$$

where $C = \max\{\mu_1 e^{(\eta_1+\eta_2)\tau}, e^{(\eta_1+\eta_2)\tau}\}$. Likewise, the following inequality holds

$$\mathcal{V}(t_k^+) < \mu_1\mu_2 e^{(\eta_1+\eta_2)\tau-\eta_2(t_k-t_{k-1})}\mathcal{V}(t_{k-1}^+) \tag{19}$$

Combining (18) and (19), we can obtain

$$\begin{aligned}\mathcal{V}(t) &< Ce^{-\eta_2(t-t_k)}\mu_1\mu_2 e^{(\eta_1+\eta_2)\tau-\eta_2(t_k-t_{k-1})}\mathcal{V}(t_{k-1}^+) \\ &< \cdots < Ce^{-\eta_2(t-t_0)}\left(\mu_1\mu_2 e^{(\eta_1+\eta_2)\tau}\right)^k\mathcal{V}(t_0).\end{aligned} \tag{20}$$

If $\mu_1\mu_2 e^{(\eta_1+\eta_2)\tau} < 1$, it is straightforward from (20) that

$$\mathcal{V}(t) < Ce^{-\eta_2(t-t_0)}\mathcal{V}(t_0), \quad t \in [t_k, t_{k+1}). \tag{21}$$

On the other hand, if $\mu_1\mu_2 e^{(\eta_1+\eta_2)\tau} \geq 1$, due to $k \leq (t - t_0)/d_{\min}$, $t \in [t_k, t_{k+1})$, it follows from (20) that

$$\mathcal{V}(t) < Ce^{-\eta_2(t-t_0)} \left(\mu_1\mu_2 e^{(\eta_1+\eta_2)\tau}\right)^{\frac{t-t_0}{d_{\min}}} \mathcal{V}(t_0)$$

which can be rewritten as

$$\mathcal{V}(t) < Ce^{-\rho(t-t_0)}\mathcal{V}(t_0), \quad t \in [t_k, t_{k+1}) \tag{22}$$

where $\rho = \eta_2 - (\ln\mu_1 + \ln\mu_2 + (\eta_1+\eta_2)\tau)/d_{\min} > 0$ according to Condition (iv). Thus the exponential stability can be established by (21) and (22). □

Remark 1. Several previous results can be viewed as particular cases of Theorem 1, such as

(1) Let $P_i = P_{p,i,n}$, $\forall n = 0, 1, \ldots, L_p$, $\forall p = 1, 2, i \in \mathcal{I}$, so it has $\mu_1 = 1$ and $\mu_2 > 1$, which reduces Condition (iii) to $P_i < \mu_2 P_j$, $i \neq j$. Then, Condition (iv) implies the dwell time $d_{\min}$ has to satisfy $d_{\min} > [(\eta_1+\eta_2)\tau + \ln\mu_2]/\eta_2$ to ensure the stability, which is exactly the result for asynchronously switched systems in Zhang and Gao (2010), Zhang and Shi (2009) by MLF approach.
(2) Moreover, if further ignore the impact of mode estimator, that is to say, $\tau = 0$, the dwell time condition reduces to $d_{\min} > \ln\mu/\eta_2$, which is the well-known result in Morse (1996).
(3) Then, reconsidering Theorem 1 with $\tau = 0$, this implies there only exists $\mathcal{M}_2$ so that Condition (i) can be eliminated. Then, if we particularly let $\mu_1 = 1$ and $\mu_2 = 1$, Condition (iv) can be also eliminated since $-\eta_2 d_{\min} < 0$ is always satisfied. Furthermore, by particularly enforcing $\eta_2 = 0$ in Condition (ii), it directly leads to the results in Allerhand and Shaked (2011, 2013), where the case without mode estimator is considered.

As we move forward to Problem 2 where the mode-identifying time is unknown and has to be estimated, Condition (iii) in Theorem 1 has to be modified as follows,

(iii′) $P_{2,i,0} \preceq \mu_1 P_{1,i,n}, \forall n = 0, 1, \ldots, L_1, P_{1,i,0} \preceq \mu_2 P_{2,j,L_2}, \ i \neq j$.

Then, the following result can be obtained.

Proposition 1. *If Theorem 1 with modified Condition (iii′) holds with τ^* and $d^*_{\min}$, then it still holds with any $\tau \in [0, \tau^*]$ and $d_{\min} \in [d^*_{\min}, \infty)$.*

Proof. The modified Condition (iii′) means that $\mathcal{V}(\hat{t}_k^+) \leq \mu_1\mathcal{V}(\hat{t}_k^-)$, $\forall \hat{t}_k \in [t_k, t_k + \tau^*)$, taking place of (16) in the proof of Theorem 1. Then, the rest of the proof can be conducted through similar guidelines in Theorem 1, and it is omitted here. □

Based on Proposition 1, the maximal value of τ^* can be determined. At first, we pre-specified η_2 for some control objectives. Then, we set $\mu_1 = \mu_2 = \mu$ for the sake of simplicity, and from Condition (iv) in Theorem 1, it can be seen that $\ln\mu < [\eta_2 d_{\min} - \tau(\eta_1+\eta_2)]/2 \leq \eta_2 d_{\min}/2$, which implies $\mu \in (0, e^{\eta_2 d_{\min}/2})$. So it is possible to check all the values in $(0, e^{\eta_2 d_{\min}/2})$ with discretized step $\Delta\mu$. Then, with each μ in $(0, e^{\eta_2 d_{\min}/2})$ and an η_1^* can be figured out to satisfy

$$\eta_1^* = [\eta_2(d_{\min} - \tau) - 2\ln\mu]/\tau \tag{23}$$

and η_1 can be selected as $0 < \eta_1 < \eta_1^*$. As a result, the maximal τ^* can be computed by

$$\tau^* = \max_{0<\tau<d_{\min}} \{\tau : \text{(i), (ii), (iii)}' \text{ in Theorem 1 hold}\}. \tag{24}$$

In summary, the computation on the maximal admissible mode-identifying time τ^* is given by Algorithm 1.

Algorithm 1 Computation on τ^* with $d_{\min}$ and η_2

Initialize $\mu = 0$, loop counter $M = 0$, and set a variation $\Delta\mu > 0$;
while $\mu < e^{\eta_2 d_{\min}/2}$ **do**
  Set $M = M + 1$ and $\mu = \mu + \Delta\mu$;
  Solve (24) to obtain τ^*;
  if (24) is feasible and $\eta_1^* > 0$ by (23) **then**
    Record $D(M) = \tau^*$;
  else
    Record $D(M) = 0$;
  end if
end while
if $\exists m = 1, 2, \ldots, M$ such that $D(m) \neq 0$ **then**
  $\tau^* = \max_{m=1,2,\ldots,M}\{D(m)\}$ and exit;
else
  The stability cannot be established and exit;
end if

Table 1
Admissible mode-identifying time τ^* with L_1 and L_2.

τ^*	$L_2 = 1$	$L_2 = 2$	$L_2 = 3$	$L_2 = 4$	$L_2 = 5$	
$L_1 = 1$	0.200	0.217	0.228	0.230	0.243	Small
$L_1 = 2$	0.209	0.224	0.236	0.242	0.252	‖
$L_1 = 3$	0.214	0.230	0.240	0.249	0.256	‖
$L_1 = 4$	0.218	0.235	0.246	0.254	0.259	⇓
$L_1 = 5$	0.222	0.238	0.249	0.257	0.264	Large
Small ================⇒ large						

3.3. Numerical example

Consider the switched system with two subsystems as

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0.5 & -0.3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.2 & 0.3 \\ -1 & 0.4 \end{bmatrix}$$
$$B_1^\top = \begin{bmatrix} -0.4 & 1.8 \end{bmatrix}, \qquad B_2^\top = \begin{bmatrix} -0.1 & 0.5 \end{bmatrix}.$$

The switching signal $\sigma_p(t)$ is assumed to have a minimal dwell time $d_{\min} = 5$. At first, the feedback gains to ensure a decay rate $\eta_2 = 2$ are given as

$$K_1 = [5.8524 \ -0.6080], \qquad K_2 = [-56.5114 \ 1.9070].$$

By the MLF approach (Zhang & Gao, 2010), it is estimated as $\tau^* = 0.143$. From Theorem 1, the τ^* with different L_1 and L_2 are listed in Table 1. It can be observed that less conservative results can be obtained in contrast to MLF approach. Furthermore, the estimated τ^* tends to be larger as L_1 and L_2 are increased, which implies the denser divisions of intervals lead to a less conservative result. Finally, a periodical switching signal $\sigma_p(t)$ satisfying $t_{k+1} - t_k = 5$, $k = 0, 1, 2, \cdots$ is given. Supposing the mode-identifying time $\tau = 0.29$, the exponential stability should hold according to $\tau^* = 0.298$ with $L_1 = L_2 = 10$. Given an initial state as $x^\top(0) = [15\ 30]$, the switching instants and state responses are shown in Fig. 2. It can be seen that state trajectories converge to the origin in the presence of a switching delay $\tau = 0.29$, though some divergence of the state can be observed during the mode identifying interval.

4. Switching-delay tolerant control synthesis

4.1. ETUs for control synthesis problem

The interval $[\hat{t}_k, \hat{t}_{k+1})$ concerned with the switching signal of controller $\sigma_c(t)$ is considered, which can be classified as normal-working interval $\mathcal{M}_2 := [\hat{t}_k, t_{k+1})$ and mode-identifying interval $\mathcal{M}_1 := [t_{k+1}, \hat{t}_{k+1})$. The intervals $\mathcal{M}_2$ and $\mathcal{M}_1$ are constituted by a series of ETUs $\mathcal{N}_{2,k,n} := [\hat{t}_k + \theta_{2,n}, \hat{t}_k + \theta_{2,n+1})$, $n = 0, 1, \ldots, L_2$ and

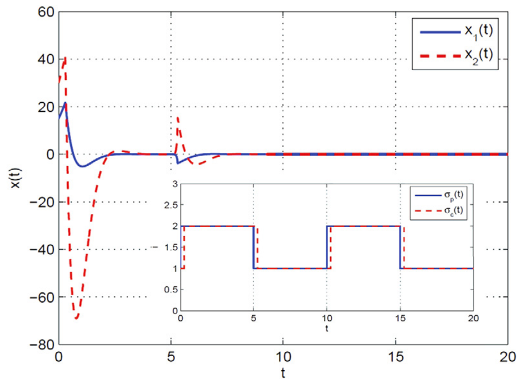

Fig. 2. Switching instants of system and controller.

$\mathcal{N}_{1,k,n} := [t_{k+1}+\theta_{1,n}, t_{k+1}+\theta_{1,n+1})$, $n = 0, 1, \ldots, L_1 - 1$, in which $\theta_2 = nh_2$ and $\theta_1 = nh_1$ with resolutions h_2 and h_1. Moreover, $\mathcal{M}_2$ consists of $\mathcal{M}_2^c := [\hat{t}_k, t_k + d_{\min})$ and $\mathcal{M}_2^u := [t_k + d_{\min}, t_{k+1})$.

Then, a set of continuous matrix functions $\mathcal{Q}_i(t), i \in \mathcal{I}$ is introduced. In the ETUs $\mathcal{N}_{2,k,n}$, $n = 0, 1, \ldots, L_2 - 1$ in $\mathcal{M}_2^c$, $\mathcal{Q}_i(t)$ is defined by

$$\mathcal{Q}_i(t) = \mathcal{Q}_{2,i,n}(\alpha_{2,n}) = (1-\alpha_{2,n})Q_{2,i,n} + \alpha_{2,n}Q_{2,i,n+1} \tag{25}$$

where $\alpha_{2,n} = (t - \hat{t}_k - \theta_{2,n})/h_2$. Also, in $\mathcal{M}_2^u$ which is equivalent to $\mathcal{N}_{2,k,L_2}$, $\mathcal{Q}_i(t)$ is determined as

$$\mathcal{Q}_i(t) = Q_{2,i,L_2}, \quad \forall t \in \mathcal{N}_{2,k,L_2},\ i \in \mathcal{I}. \tag{26}$$

Then, as to $\mathcal{N}_{1,k,n}$ in $\mathcal{M}_1$, $\mathcal{Q}_i(t)$ is defined by

$$\mathcal{Q}_i(t) = \mathcal{Q}_{1,i,n}(\alpha_{1,n}) = (1-\alpha_{1,n})Q_{1,i,n} + \alpha_{1,n}Q_{1,i,n+1} \tag{27}$$

where $\alpha_{1,n} = (t - t_{k+1} - \theta_{1,n})/h_1$.

Letting $Q_{p,i,n} \succ 0, \forall n = 0, 1, \ldots, L_p, \forall p = 1, 2, \forall i \in \mathcal{I}$ which implies $\mathcal{Q}_i(t) \succ 0, \forall i \in \mathcal{I}$ and straightforwardly leads to $\mathcal{Q}_i^{-1}(t) \succ 0, \forall i \in \mathcal{I}$, the time-scheduled Lyapunov function candidates $\mathcal{V}_i(t,x) = x^\top(t)\mathcal{Q}_i^{-1}(t)x(t), i \in \mathcal{I}$ can be constructed as

$$\mathcal{V}_i(t,x) = \begin{cases} x^\top(t)\mathcal{Q}_{2,i,n}^{-1}(\alpha_{2,n})x(t) & t \in \mathcal{N}_{2,k,n} \\ x^\top(t)Q_{2,i,L_2}^{-1}x(t) & t \in \mathcal{N}_{2,k,L_2} \\ x^\top(t)\mathcal{Q}_{1,i,n}^{-1}(\alpha_{1,n})x(t) & t \in \mathcal{N}_{1,k,n}. \end{cases} \tag{28}$$

4.2. Feedback control with detectable switching instants

The following assumption on the detection of switching signal is needed.

Assumption 1. The switching instants $t_k, \forall k = 0, 1, \ldots$ generated by $\sigma_p(t)$ can be detected online.

At first, Problem 3 is taken into account, where the switching delay τ is available.

Theorem 2. *Assuming Assumption 1 holds and there exist scalars $\mu_1 > 0, \mu_2 > 0, \eta_1 > 0, \eta_2 > 0$, and a set of matrices $Q_{p,i,n} \succ 0, X_{p,i,n}, n = 0, 1, \ldots, L_p, p = 1, 2, i \in \mathcal{I}$ such that*

(i) $\Upsilon_{1,i,n}^{(1)} \prec 0, \Upsilon_{1,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_1 - 1$;
(ii) $\Upsilon_{2,i,n}^{(1)} \prec 0, \Upsilon_{2,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_2 - 1, \Upsilon_{2,i,L_2} \prec 0$;
(iii) $Q_{2,i,L_2} \preceq \mu_1 Q_{1,i,0}, Q_{1,j,L_1} \preceq \mu_2 Q_{2,i,0},\ i \neq j$;
(iv) $\ln\mu_1 + \ln\mu_2 + \eta_1\tau - \eta_2(d_{\min} - \tau) < 0$;

where

$$\Upsilon_{1,i,n}^{(1)} = \mathbf{He}\{A_jQ_{1,i,n} + B_jX_{1,i,n}\} - \Psi_{1,i,n} - \eta_1 Q_{1,i,n}$$
$$\Upsilon_{1,i,n}^{(2)} = \mathbf{He}\{A_jQ_{1,i,n+1} + B_jX_{1,i,n+1}\} - \Psi_{1,i,n} - \eta_1 Q_{1,i,n+1}$$
$$\Upsilon_{2,i,n}^{(1)} = \mathbf{He}\{A_iQ_{2,i,n} + B_iX_{2,i,n}\} - \Psi_{2,i,n} + \eta_2 Q_{2,i,n}$$
$$\Upsilon_{2,i,n}^{(2)} = \mathbf{He}\{A_iQ_{2,i,n+1} + B_iX_{2,i,n+1}\} - \Psi_{2,i,n} + \eta_2 Q_{2,i,n+1}$$
$$\Upsilon_{2,i,L_2} = \mathbf{He}\{A_iQ_{2,i,L_2} + B_iX_{2,i,L_2}\} + \eta_2 Q_{2,i,L_2}$$
$$\Psi_{1,i,n} = L_1(Q_{1,i,n+1} - Q_{1,i,n})/\tau$$
$$\Psi_{2,i,n} = L_2(Q_{2,i,n+1} - Q_{2,i,n})/(d_{\min} - \tau)$$

then the closed loop system (5) is exponentially stable, and controller gains are given by $\mathcal{K}_i(t) = \mathcal{X}_i(t)\mathcal{Q}_i^{-1}(t), i \in \mathcal{I}$, where $\mathcal{Q}_i(t)$ is defined by (25)–(27) and $\mathcal{X}_i(t)$ is

$$\mathcal{X}_i(t) = \begin{cases} \mathcal{X}_{2,i,n}(\alpha_{2,n}) & t \in \mathcal{N}_{2,k,n}, n = 0, 1, \ldots L_2 - 1 \\ X_{2,i,L_2} & t \in \mathcal{N}_{2,k,L_2} \\ \mathcal{X}_{1,i,n}(\alpha_{1,n}) & t \in \mathcal{N}_{1,k,n}, n = 0, 1, \ldots L_1 - 1 \end{cases} \tag{29}$$

with $\alpha_{2,n}, \alpha_{1,n}$ defined in (25), (27), and $\mathcal{X}_{2,i,n}(\alpha_{2,n}) = (1-\alpha_{2,n})X_{2,i,n} + \alpha_{2,n}X_{2,i,n+1}$, $\mathcal{X}_{1,i,n}(\alpha_{1,n}) = (1-\alpha_{1,n})X_{1,i,n} + \alpha_{1,n}X_{1,i,n+1}$.

Proof. By constructing Lyapunov function $\mathcal{V}_i(t,x) = x^\top(t)\mathcal{Q}_i^{-1}(t)x(t), i \in \mathcal{I}$ defined by (28), and following the guidelines of Theorem 1, we have to prove that

$$\mathcal{V}_i(t,x) < e^{-\eta_2(t-\hat{t}_k)}\mathcal{V}_i(\hat{t}_k, x),\ \forall t \in \mathcal{M}_2 \tag{30}$$
$$\mathcal{V}_i(t,x) < e^{\eta_1(t-t_{k+1})}\mathcal{V}_i(t_{k+1}, x)\ \forall t \in \mathcal{M}_1 \tag{31}$$

which can be implied by

$$\dot{\mathcal{Q}}_i^{-1}(t) + \mathbf{He}\{\mathcal{Q}_i^{-1}(t)\bar{\mathcal{A}}_{i,i}(t)\} + \eta_2\mathcal{Q}_i^{-1}(t) \prec 0 \tag{32}$$
$$\dot{\mathcal{Q}}_i^{-1}(t) + \mathbf{He}\{\mathcal{Q}_i^{-1}(t)\bar{\mathcal{A}}_{j,i}(t)\} - \eta_1\mathcal{Q}_i^{-1}(t) \prec 0 \tag{33}$$

where $\bar{\mathcal{A}}_{j,i}(t) = A_j + B_j\mathcal{K}_i(t)$ and $\bar{\mathcal{A}}_{i,i}(t) = A_i + B_i\mathcal{K}_i(t)$. Due to fact $\dot{\mathcal{Q}}_i^{-1}(t) = -\mathcal{Q}_i^{-1}(t)\dot{\mathcal{Q}}_i(t)\mathcal{Q}_i^{-1}(t), i \in \mathcal{I}$, multiplying both sides of (32) and (33) by $\mathcal{Q}_i(t)$ and from $\mathcal{X}_i(t) = \mathcal{K}_i(t)\mathcal{Q}_i(t)$, (32) and (33) become

$$-\dot{\mathcal{Q}}_i(t) + \mathbf{He}\{A_i\mathcal{Q}_i(t) + B_i\mathcal{X}_i(t)\} + \eta_2\mathcal{Q}_i(t) \prec 0 \tag{34}$$
$$-\dot{\mathcal{Q}}_i(t) + \mathbf{He}\{A_j\mathcal{Q}_i(t) + B_j\mathcal{X}_i(t)\} - \eta_1\mathcal{Q}_i(t) \prec 0. \tag{35}$$

Due to $h_2 = (d_{\min} - \tau)/L_2$ and $h_1 = \tau/L_1$, it is obtained

$$\dot{\mathcal{Q}}_i(t) = L_2(Q_{2,i,n+1} - Q_{2,i,n})/(d_{\min} - \tau), \quad t \in \mathcal{M}_2$$
$$\dot{\mathcal{Q}}_i(t) = L_1(Q_{1,i,n+1} - Q_{1,i,n})/\tau, \quad t \in \mathcal{M}_1.$$

From Conditions (i), (ii) and by simple manipulations, (34) and (35) can be established, hence (30) and (31) hold.

Then, we construct $\mathcal{V}(t) = \sum_{i=1}^N \xi_i(t)\mathcal{V}_i(t,x)$ where $\xi_i(t), i \in \mathcal{I}$ are the indication functions for controller. If $Q_{2,i,L_2} \preceq \mu_1 Q_{1,i,0}$ in Condition (iii) holds, we have $-Q_{1,i,0} + \mu_1^{-1}Q_{2,i,L_2} \preceq 0$. By Schur complement, it equals $Q_{1,i,0}^{-1} \preceq \mu_1 Q_{2,i,L_2}^{-1}$, leading to $\mathcal{V}(t_k^+) \leq \mu_1\mathcal{V}(t_k^-)$. Similarly, $Q_{1,j,L_1} \preceq \mu_2 Q_{2,i,0}$ in Condition (iii) yields $\mathcal{V}(\hat{t}_k^+) \leq \mu_2\mathcal{V}(\hat{t}_k^-)$. Then, Condition (iv) has $\mu_1\mu_2e^{\eta_1\tau - \eta_2(d_{\min}-\tau)} < 1$. By similar guidelines in Theorem 1, the exponential stability can be proved. □

Two points have to be clarified for controller realization.

(1) The first one is to determine the working ETU at $t, t \in [\hat{t}_k, \hat{t}_{k+1})$. Since $t_k, \hat{t}_k$ are detectable, one sees:
  (a) If $\hat{t}_k \leq t < t_{k+1}$, we have $t \in \mathcal{N}_{2,k,n}$ where $n = \begin{cases} \mathrm{int}[(t-\hat{t}_k)/h_2] & 0 \leq n < L_2 \\ L_2 & n \geq L_2 \end{cases}$.
  (b) If $t_{k+1} \leq t < \hat{t}_{k+1}$, we have $t \in \mathcal{N}_{1,k,n}$ where $n = \mathrm{int}[(t - t_{k+1})/h_1]$.

(2) The second point is to calculate $\alpha_{1,n}$ and $\alpha_{2,n}$ as:
   (a) For $t \in \mathcal{N}_{2,k,n}$, one has $\alpha_{2,n} = (t - \hat{t}_k - \theta_{2,n})/h_2$. Then, by $\theta_{2,n} = nh_2$, we can obtain $\alpha_{2,n} = (t - \hat{t}_k)/h_2 - n$.
   (b) For $t \in \mathcal{N}_{1,k,n}$, we see that $\alpha_{1,n} = (t - t_{k+1} - \theta_{1,n})/h_1$, and due to $\theta_{1,n} = nh_1$, we get $\alpha_{1,n} = (t - t_{k+1})/h_1 - n$.

Furthermore, as to Problem 4, we have to reformulate Condition (iii) in Theorem 2 as

(iii′) $Q_{2,i,L_2} \preceq \mu_1 Q_{1,i,0}, Q_{1,j,n} \preceq \mu_2 Q_{2,i,0}, \forall n = 0, 1, \ldots, L_1, i \neq j$.

Then, we can derive the following proposition.

Proposition 2. *If Theorem 2 with Condition (iii′) holds with a switching delay τ^* and minimal dwell time $d^*_{\min}$, then it still holds for any $\tau \in [0, \tau^*]$ and $d_{\min} \in [d^*_{\min}, \infty)$.*

Proof. It can be demonstrated by similar proof lines in Proposition 1, which is omitted here. □

Remark 2. By the similar discussion for Problem 2, if the Lyapunov function decay rate η_2 for subsystems is prescribed and we let $\mu = \mu_1 = \mu_2$, the feedback controller (4) maximizing the switching-delay tolerant capability τ^* can be obtained by

$$\tau^* = \max_{0<\tau<d_{\min}} \{\tau : \text{(i), (ii), (iii)}' \text{ in Theorem 2 hold}\} \tag{36}$$

in which μ is fixed and η_1 is computed by (23). By checking the values μ in $(0, e^{\eta_2 d_{\min}})$ with a step $\Delta\mu$, the feedback gains maximizing τ^* can be obtained by executing Algorithm 1 with (24) replaced by (36).

4.3. Feedback control with undetectable switching instants

As to the case with undetectable switching, $\mathcal{K}_i(t)$ has to maintain its value in $\mathcal{M}_1$ to wait for the correct identification for the $\sigma_p(t)$. Thus, the controller gains become

$$\mathcal{K}_i(t) = \begin{cases} \mathcal{X}_{2,i,n}(\alpha_{2,n})\mathcal{Q}^{-1}_{2,i,n}(\alpha_{2,n}) & t \in \mathcal{N}_{2,k,n} \\ X_{2,i,L_2} Q^{-1}_{2,i,L_2} & t \in \mathcal{N}_{2,k,L_2} \cup \mathcal{N}_{1,k,n} \end{cases} \tag{37}$$

where $\mathcal{Q}_{2,i,n}(\alpha_{2,n})$, $\mathcal{X}_{2,i,n}(\alpha_{2,n})$ are defined by (25), (29). The time-scheduled Lyapunov function is constructed as

$$\mathcal{V}_i(t) = \begin{cases} x^\top(t)\mathcal{Q}^{-1}_{2,i,n}(\alpha_{2,n})x(t) & t \in \mathcal{N}_{2,k,n} \\ x^\top(t)\mathcal{Q}^{-1}_{2,i,L_2}x(t) & t \in \mathcal{N}_{2,k,L_2} \cup \mathcal{N}_{1,k,n}. \end{cases} \tag{38}$$

Then, the following theorem can be obtained.

Theorem 3. *Assuming Assumption 1 holds and there exist scalars $\mu > 0, \eta_1 > 0, \eta_2 > 0$, and a set of matrices $Q_{i,n} \succ 0, X_{i,n}, n = 0, 1, \cdots, L, i \in \mathcal{I}$ such that*

(i) $\Upsilon_{1,i} \prec 0$;
(ii) $\Upsilon^{(1)}_{2,i,n} \prec 0, \Upsilon^{(2)}_{2,i,n} \prec 0, \forall n = 0, 1, \ldots, L-1, \Upsilon_{2,i,L} \prec 0$;
(iii) $Q_{j,L} \preceq \mu Q_{i,0}, \ i \neq j$;
(iv) $\ln \mu + \eta_1 \tau - \eta_2(d_{\min} - \tau) < 0$;

where

$$\Upsilon_{1,i} = \mathbf{He}\{A_j Q_{i,L} + B_j X_{i,L}\} - \eta_1 Q_{i,L}$$
$$\Upsilon^{(1)}_{2,i,n} = \mathbf{He}\{A_i Q_{i,n} + B_i X_{i,n}\} - \Psi_{i,n} + \eta_2 Q_{i,n}$$
$$\Upsilon^{(2)}_{2,i,n} = \mathbf{He}\{A_i Q_{i,n+1} + B_i X_{i,n+1}\} - \Psi_{i,n} + \eta_2 Q_{i,n+1}$$
$$\Upsilon_{2,i,L} = \mathbf{He}\{A_i Q_{i,L} + B_i X_{i,L}\} + \eta_2 Q_{i,L}$$
$$\Psi_{i,n} = L(Q_{i,n+1} - Q_{i,n})/(d_{\min} - \tau)$$

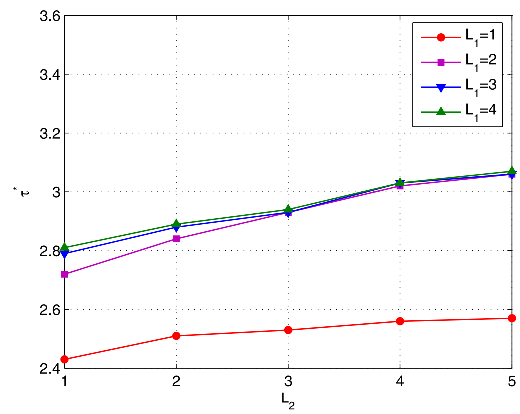

Fig. 3. Capability of tolerating switching delay τ^* with detectable t_k.

then the closed loop system (5) is exponentially stable, and the controller gains $\mathcal{K}_i(t), i \in \mathcal{I}$ are given as

$$\mathcal{K}_i(t) = \begin{cases} \mathcal{X}_{i,n}(\alpha_n)\mathcal{Q}^{-1}_{i,n}(\alpha_n) & t \in \mathcal{N}_{k,n} \\ X_{i,L}Q^{-1}_{i,L} & t \in \mathcal{N}_{2,k,n} \cup \mathcal{N}_{1,k,n} \end{cases} \tag{39}$$

where $\mathcal{Q}_{i,n}(\alpha_n) = (1 - \alpha_n)Q_{i,n} + \alpha_n Q_{i,n+1}$, $\mathcal{X}_{i,n}(\alpha_n) = (1 - \alpha_n)X_{i,n} + \alpha_n X_{i,n+1}$ and $\alpha_n = (t - \hat{t}_k)/h - n$ in which $h = \frac{d_{\min}-\tau}{L}$ and $n = \begin{cases} \text{int}[(t - \hat{t}_k)/h] & 0 \le n < L \\ L & n \ge L. \end{cases}$

Proof. By setting $Q_{1,i,n} = Q_{i,L}, \forall n = 0, 1, \ldots, L_1, \mu_1 = 1, \mu_2 = \mu$ $Q_{2,i,n} = Q_{i,n}$, $h = h_2$ and $L = L_2$, Theorem 3 becomes Theorem 2, which completes the proof. □

The following property of Theorem 3 can be obtained.

Proposition 3. *If Theorem 3 holds with a switching delay τ^* and minimal dwell time $d^*_{\min}$, then it still holds for any $\tau \in [0, \tau^*]$ and $d_{\min} \in [d^*_{\min}, \infty)$.*

Proof. It can be proved by letting $Q_{1,i,n} = Q_{i,L}, \forall n = 0, 1, \ldots, L_1$ and $Q_{2,i,n} = Q_{i,n}$ in Proposition 2. □

Remark 3. Similar to (36), if η_2 is prescribed, the controller with maximal τ^* can be obtained by

$$\tau^* = \max_{0<\tau<d_{\min}} \{\tau : \text{(i), (ii), (iii) in Theorem 3 hold}\} \tag{40}$$

where μ is fixed and $\eta_1 < [\eta_2(d_{\min} - \tau) - \ln \mu]/\tau$.

4.4. Numerical example

Consider a switched linear system with 3 subsystems as

$$A_1 = \begin{bmatrix} -0.6 & -1.0 & 1.3 \\ -1.0 & -0.1 & 0.1 \\ -0.2 & 0.3 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2.3 \\ 1.8 \\ 0.4 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1.6 & 0.6 & 0.2 \\ 0.8 & -0.5 & 0.6 \\ 0.2 & 0.8 & -1.0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0 \\ -0.4 \\ -1.5 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1.5 & 0.4 & 1.3 \\ 0.4 & 1.2 & 1.3 \\ -1.9 & 0.6 & -0.9 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.2 \\ -1.3 \\ -0.8 \end{bmatrix}.$$

Assume $d_{\min} = 5$ and $\eta_2 = 2$, both cases with detectable and undetectable t_k are considered. Given different L_1, L_2 and L, the obtained τ^* are shown in Figs. 3 and 4. The capability of tolerating switching delay τ^* increases monotonically as L_1, L_2 or L increases. This is consistent with the mode-identifying estimation problem, that is, a denser division of intervals leads to a less

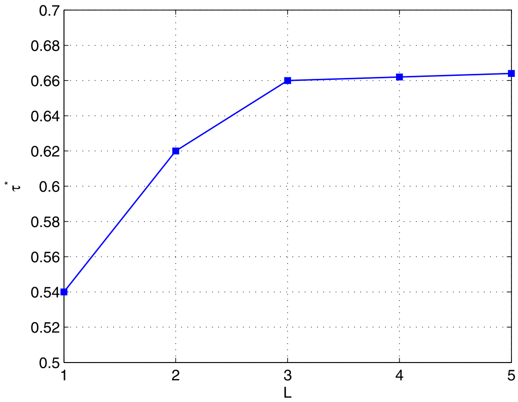

Fig. 4. Capability of tolerating switching delay τ^* with undetectable t_k.

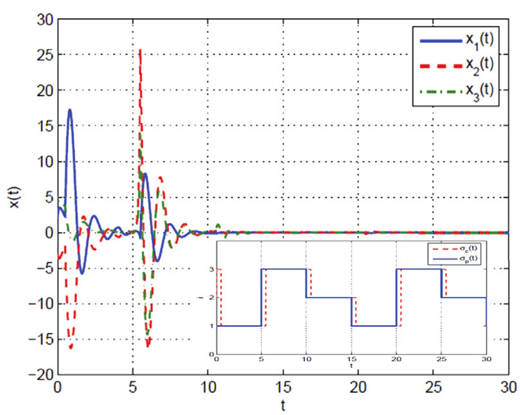

Fig. 5. State response with switching delay $\tau = 5$.

conservative result. In addition, the τ^* obtained with detectable t_k is larger than that with undetectable t_k. This phenomenon is because Theorem 3 is only a particular case of Theorem 2. From the practical point of view, the employment of information t_k can provide less conservativeness, but its cost is an online detection of switching instants of controlled plant.

Finally, assume t_k is undetectable and the maximal switching delay is 10% of the minimal dwell time, i.e. $\tau^* = 0.5$. From Fig. 4, the controller with $L = 1$ ($\tau^* = 0.54$) can tolerate this switching delay. Given a $\sigma_p(t)$ with $t_{k+1} - t_k = 5, k = 0, 1, \ldots,$ the subsystem activating order is cyclic, i.e., $3 \to 2 \to 1 \to 3 \to \cdots$. The switching instants and state responses are shown in Fig. 5, by which the closed loop system is asymptotically stable.

5. Conclusions

In this paper, an elementary time unit method (ETU) is proposed to solve mode-identifying time estimation and switching-delay tolerant control problems. A class of time-scheduled Lyapunov functions involving the information of switching signal are constructed to deal with the problems. By deriving a sufficient condition guaranteeing exponential stability in the presence of mode-identifying time, an algorithm is proposed for the estimation on the maximal admissible mode-identifying time. Then, the switching-delay tolerant control is studied, where the cases with detectable and undetectable switching instants are both considered. Several numerical examples are used to validate our results. Some extension based on ETU can be directly made such as to the case with all modes unstable (Xiang & Xiao, 2014a), asynchronously switching control (Xiang & Xiao, 2012; Zhang & Shi, 2009), and discrete-time switched system (Briat, 2014; Xiang & Xiao, 2014b). As in Allerhand and Shaked (2011, 2013), the proposed ETU approach is particularly suitable to deal with uncertainties. Moreover, this paper only considers the state feedback with all states accessible, the extension to more general case with output feedback control such as in Duan and Wu (2014) and Lu et al. (2006) will be our future study.

Appendix

Proof of Lemma 1. Considering the time-scheduled Lyapunov function $\mathcal{V}_i(t, x), i \in \mathcal{I}$ defined by (6), and in each ETU $\mathcal{N}_{1,k,n}$, one has $\dot{\mathcal{P}}_i(t) = (P_{1,i,n+1} - P_{1,i,n})\dot{\alpha}_{1,n}$. Due to $\alpha_{1,n} = (t - t_k - \theta_{1,n})/h_1$, we have $\dot{\alpha}_{1,n} = 1/h_1$ and further $\dot{\mathcal{P}}_i(t) = \Psi_{1,i,n}, t \in \mathcal{N}_{1,k,n}$. Thus, we obtain

$$\dot{\mathcal{V}}_i(t, x) = x^\top(t)\Xi_{1,i,n}(\alpha_{1,n})x(t)$$

where $\Xi_{1,i,n}(\alpha_{1,n}) = \bar{A}_{i,j}^\top\mathcal{P}_{1,i,n}(\alpha_{1,n}) + \mathcal{P}_{1,i,n}(\alpha_{1,n})\bar{A}_{i,j} + \Psi_{1,i,n}$. By the linear interpolation relationship as shown in Subsection 3.1, we have $\Xi_{1,i,n}(\alpha_{1,n}) = (1 - \alpha_{1,n})\Xi_{1,i,n}^{(1)} + \alpha_{1,n}\Xi_{1,i,n}^{(2)}$, where $\Xi_{1,i,n}^{(1)} = \bar{A}_{i,j}^\top P_{1,i,n} + P_{1,i,n}\bar{A}_{i,j} + \Psi_{1,i,n}$ and $\Xi_{1,i,n}^{(2)} = \bar{A}_{i,j}^\top P_{1,i,n+1} + P_{1,i,n+1}\bar{A}_{i,j} + \Psi_{1,i,n}$. Thus, from $\Omega_{1,i,n}^{(1)} \prec 0, \Omega_{1,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_1 - 1, \forall i \in \mathcal{I}$. it follows that $\dot{\mathcal{V}}_i(t, x) < \eta_1\mathcal{V}_i(t, x), t \in \bigcup_{n=0,1,\ldots,L_1-1}\mathcal{N}_{1,k,n} = [t_k, t_k + \tau)$. Thus the inequality (10) can be established. □

Proof of Lemma 2. As for certain interval $\mathcal{M}_2^c$ constituted by ETUs $\mathcal{N}_{2,k,n}, n = 0, 1, \ldots, L_2 - 1$, from $\Omega_{2,i,n}^{(1)} \prec 0, \Omega_{2,i,n}^{(2)} \prec 0, \forall n = 0, 1, \ldots, L_2 - 1, \forall i \in \mathcal{I}$ and the similar guidelines for interval $\mathcal{M}_1$ in Lemma 1, we obtain that $\dot{\mathcal{V}}_i(t, x) < -\eta_2\mathcal{V}_i(t, x), \forall t \in \mathcal{M}_2^c$. Similarly, as to the uncertain interval $\mathcal{M}_2^u$, $\Omega_{2,i,L_2} \prec 0, \forall i \in \mathcal{I}$, ensures $\dot{\mathcal{V}}_i(t, x) < -\eta_2\mathcal{V}_i(t, x), \forall t \in \mathcal{M}_2^u$. At last, since $\mathcal{M}_2 = \mathcal{M}_2^c \cup \mathcal{M}_2^u$, we can conclude (15) holds. □

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Lixian Zhang received the Ph.D. degree in control science and engineering from Harbin Institute of Technology, China, in 2006. From Jan 2007 to Sep 2008, he worked as a postdoctoral fellow in the Dept. of Mechanical Engineering at Ecole Polytechnique de Montreal, Canada. He was a visiting professor at Process Systems Engineering Laboratory, Massachusetts Institute of Technology (MIT) during Feb 2012 to March 2013. Since Jan 2009, he has been with the Harbin Institute of Technology, China, where he is currently full professor and vice director in the Research Institute of Intelligent Control and Systems.

Dr. Zhang's research interests include nondeterministic and stochastic switched systems, networked control systems, model predictive control and their applications. He serves as Associated Editor for various peer-reviewed journals including IEEE Transactions on Automatic Control, IEEE Transactions on Cybernetics, etc., and was a leading Guest Editor for a Special Section in IEEE Transactions on Industrial Informatics. He is an IEEE Senior Member and Chapter of IEEE SMCS Harbin Section Chapter. He is a Thomson Reuters ISI Highly Cited Researcher in 2014 and 2015.

Weiming Xiang received his B.S. degree from the Department of Electrical Engineering, East China Jiaotong University, Nanchang, China, in 2005, his M.S. degree from the Department of Automation, Nanjing University of Science and Technology, Nanjing, China, in 2007, and Ph.D degree in Transportation Planning and Management at Southwest Jiaotong University, Chengdu, China in 2014. He has been the Associate Professor of School of Transportation and Logistics, Southwest Jiaotong University, Chengdu, China since 2015. During May 2015 to October 2015, he held a position of Research Associate in the Department of Mechanical Engineering at the University of Hong Kong. He is currently a Postdoctoral Research Associate in the Department of Computer Science and Engineering, University of Texas at Arlington, USA.

Dr. Xiang's research interests are in the area of switched systems and control, robust control and filtering, nonlinear systems and control, fuzzy systems and transportation systems. He has authored or co-authored more than 40 papers. He is an IEEE member and on the Editorial Board of Neurocomputing (2014–present).