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Some inequalities for the atom-bond connectivity index of graph operations

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ABSTRACT

The atom-bond connectivity index is a useful topological index in studying the stability of alkanes and the strain energy of cycloalkanes. In this paper some inequalities for the atom-bond connectivity index of a series of graph operations are presented. We also prove our bounds are tight. As an application, the *ABC* indices of C_4 nanotubes and nanotori are computed.

1. Introduction

In this section we recall some definitions that will be used in the paper. Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex v is denoted by $\deg_G(v)$ ($\deg(v)$ for short). Suppose *Graph* is the collection of all graphs. A mapping $Top : Graph \longrightarrow \mathbb{R}$ is called a topological index, if $G \cong H$ implies that $Top(G) = Top(H)$. The atom-bond connectivity index is a valuable predictive index in the study of the heat of formation in alkanes [5,6]. It is defined as $ABC(G) = \sum_{e=uv} \sqrt{\frac{\deg(u)+\deg(v)-2}{\deg(u)\deg(v)}}$. The mathematical properties of this index and its new version were reported in [2,3,8,10,24].

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2), see [12, p. 185].

The Cartesian product $G\square H$ of graphs G and H has the vertex set $V(G\square H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G\square H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$.

The strong product of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1\nabla G_2$ having $V_1 \times V_2$ as a vertex set and two vertices (x_1, x_2), (y_1, y_2) of $G_1\nabla G_2$ are adjacent if either (i) $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or (ii) $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or (iii) $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. It can be easily checked that $G_1\nabla G_2$ is a connected graph if and only if both G_1 and G_2 are connected. It is also clear that $G_1\nabla G_2$ is a complete graph if and only if both factors are complete.

The corona product GoH of two graphs G and H is defined to be the graph Γ obtained by taking one copy of G (which has p_1 vertices) and p_1 copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. If G is a (p_1, q_1) graph and H is a (p_2, q_2) graph, then it follows from the definition of the corona that GoH has $p_1(1+p_2)$ vertices and $q_1+p_1q_2+p_1p_2$ edges. It is clear that if G is connected, then GoH is connected, and in general GoH is not isomorphic to HoG.

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The Wiener index of the Cartesian product of graphs was studied in [9,25]. In [19], Klavžar et al. computed the Szeged index of Cartesian product graphs and in [20] the PI index of the Cartesian product of graphs is computed. We refer the interested readers to [21] for a polynomial approach about the Wiener index of graph operations. In some recent papers [1,7,11,13–18,22,26], the PI, vertex PI, hyper-Wiener, edge Wiener, Szeged, edge Szeged and Zagreb group indices of some graph operations are considered. Here, we continue this progress to investigate the *ABC* index of some graph operations.

In the rest of this paper, P_n, C_n, S_n, and K_n stand for the path, the cycle, the star, and the complete graph of n vertices. Our other notations are standard and taken mainly from [4,23].

2. Main results

In this section some bounds for the join, Cartesian product, composition, strong product and corona of graphs without isolated vertices are calculated. It is also proved that all bounds presented here are tight. We begin with an example.

Example 1. It is clear that $ABC(P_2) = 0$. For $n \geq 3$, the ABC index of the path P_n, cycle C_n and complete graph K_n are computed as follows:

$$ABC(P_n) = (n-1)\frac{\sqrt{2}}{2}; \qquad ABC(C_n) = n\frac{\sqrt{2}}{2}; \qquad ABC(K_n) = \frac{\sqrt{2}}{2}n\sqrt{n-2}.$$

Suppose G and H are graphs with a disjoint vertex set. The union $G\cup H$ is defined as a graph with $V(G\cup H) = V(G)\cup V(H)$ and $E(G\cup H) = E(G)\cup E(H)$. The union of m graphs, each of them isomorphic to a graph G is denoted by mG.

Theorem 1. *Suppose G and H are arbitrary graphs. Then*

$$\begin{aligned} ABC(G+H) \leq\ & \frac{\Delta_G}{\delta_G+|H|}ABC(G) + |E(G)|\frac{\sqrt{2|H|}}{\delta_G+|H|} + \frac{\Delta_H}{\delta_H+|G|}ABC(H) + |E(H)|\frac{\sqrt{2|G|}}{\delta_H+|G|} \\ & + |G|.|H|\sqrt{\frac{\Delta_G+\Delta_H+|G|+|H|-2}{(\delta_G+|H|)(\delta_H+|G|)}} \end{aligned}$$

with equality if and only if for two positive integers r and s, $G \cong rK_2$ *and* $H \cong sK_2$.

Proof. By definition of the ABC index,

$$\begin{aligned} ABC(G+H) &= \sum_{e=uv}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \\ &= \sum_{\substack{e=uv\\ u,v\in V(G)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} + \sum_{\substack{e=uv\\ u,v\in V(H)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} + \sum_{\substack{e=uv\\ u\in V(G),v\in V(H)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}. \end{aligned}$$

Clearly, if $u \in V(G)$ then $d(u) = d_G(u) + |H|$ and so

$$\sum_{\substack{e=uv\\ u,v\in V(G)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} = \sum_{\substack{e=uv\\ u,v\in V(G)}}\sqrt{\frac{d_G(u)+d_G(v)+2|H|-2}{(d_G(u)+|H|)(d_G(v)+|H|)}}.$$

On the other hand,

$$\frac{d_G(u)+d_G(v)+2|H|-2}{(d_G(u)+|H|)(d_G(v)+|H|)} = \frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}\frac{d_G(u)d_G(v)}{(d_G(u)+|H|)(d_G(v)+|H|)} + \frac{2|H|}{(d_G(u)+|H|)(d_G(v)+|H|)}.$$

Since $\frac{d_G(u)d_G(v)}{(d_G(u)+|H|)(d_G(v)+|H|)} \leq \frac{\Delta_G^2}{(\delta_G+|H|)^2}$ and $\frac{1}{(d_G(u)+|H|)(d_G(v)+|H|)} \leq \frac{1}{(\delta_G+|H|)^2}$, we have

$$\sqrt{\frac{d_G(u)+d_G(v)+2|H|-2}{(d_G(u)+|H|)(d_G(v)+|H|)}} \leq \frac{\Delta_G}{\delta_G+|H|}\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}} + \frac{\sqrt{2|H|}}{\delta_G+|H|}. \tag{1}$$

Equality holds if and only if $d_G(u) = d_G(v) = 1$. In a similar way,

$$\sqrt{\frac{d_H(u)+d_H(v)+2|G|-2}{(d_H(u)+|G|)(d_H(v)+|G|)}} \leq \frac{\Delta_H}{\delta_H+|G|}\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}} + \frac{\sqrt{2|G|}}{\delta_H+|G|} \tag{2}$$

with equality if and only if $d_H(u) = d_H(v) = 1$. But

$$d(u) + d(v) - 2 = d_G(u) + |H| + d_H(v) + |G| - 2 \leq \Delta_G + \Delta_H + |G| + |H| - 2,$$
$$d(u)d(v) = (d_G(u) + |H|)(d_H(v) + |G|) \geq (\delta_G + |H|)(\delta_H + |G|),$$

and so $\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \leq \sqrt{\frac{\Delta_G+\Delta_H+|G|+|H|-2}{(\delta_G+|H|)(\delta_H+|G|)}}$. By the above calculations, we have:

$$\begin{aligned}
ABC(G+H) &= \sum_{e=uv} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \\
&\leq \sum_{\substack{e=uv\\ u,v\in V(G)}} \left(\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}}\frac{\Delta_G}{(\delta_G+|H|)} + \frac{\sqrt{2|H|}}{\delta_G+|H|}\right) \\
&\quad + \sum_{\substack{e=uv\\ u,v\in V(H)}} \left(\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}}\frac{\Delta_H}{(\delta_H+|G|)} + \frac{\sqrt{2|G|}}{\delta_H+|G|}\right) \\
&\quad + \sum_{\substack{e=uv\\ u\in V(G), v\in V(H)}} \sqrt{\frac{\Delta_G+\Delta_H+|G|+|H|-2}{(\delta_G+|H|)(\delta_H+|G|)}} \\
&= \frac{\Delta_G}{\delta_G+|H|}ABC(G) + |E(G)|\frac{\sqrt{2|H|}}{\delta_G+|H|} + \frac{\Delta_H}{\delta_H+|G|}ABC(H) \\
&\quad + |E(H)|\frac{\sqrt{2|G|}}{\delta_H+|G|} + |G||H|\sqrt{\frac{\Delta_G+\Delta_H+|G|+|H|-2}{(\delta_G+|H|)(\delta_H+|G|)}},
\end{aligned}$$

which proves the first part of theorem. To prove the second part, it is enough to apply inequalities (1) and (2). □

Theorem 2. *Suppose G and H are arbitrary graphs. Then*

$$\begin{aligned}
ABC(G+H) &\geq \left|\frac{\delta_G}{\Delta_G+|H|}ABC(G) - |E(G)|\frac{\sqrt{2|H|}}{\Delta_G+|H|}\right| + \left|\frac{\delta_H}{\Delta_H+|G|}ABC(H) - |E(H)|\frac{\sqrt{2|G|}}{\Delta_H+|G|}\right| \\
&\quad + |G||H|\sqrt{\frac{\delta_G+\delta_H+|G|+|H|-2}{(\Delta_G+|H|)(\Delta_H+|G|)}}
\end{aligned}$$

with equality if and only if for two positive integers r and s, $G \cong rK_2$ and $H \cong sK_2$.

Proof. Suppose $u, v \in V(G)$. Then

$$\begin{aligned}
&\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}\frac{d_G(u)d_G(v)}{(d_G(u)+|H|)(d_G(v)+|H|)} + \frac{2|H|}{(d_G(u)+|H|)(d_G(v)+|H|)} \\
&\quad \geq \frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}\frac{\delta_G^2}{(\Delta_G+|H|)^2} + \frac{2|H|}{(\Delta_G+|H|)^2}.
\end{aligned}$$

Clearly $\sqrt{a+b} \geq |\sqrt{a} - \sqrt{b}|$ with equality if and only if $a = 0$ or $b = 0$. Thus,

$$\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \geq \left|\frac{\delta_G}{\Delta_G+|H|}\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}} - \frac{\sqrt{2|H|}}{\delta_G+|H|}\right|. \tag{3}$$

Equality holds if and only if $d_G(u) = d_G(v) = 1$. By choosing $u, v \in V(H)$ and a similar argument as those are given above, we have:

$$\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \geq \left|\frac{\delta_H}{\Delta_H+|G|}\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}} - \frac{\sqrt{2|G|}}{\delta_H+|G|}\right|. \tag{4}$$

Suppose $u \in V(G)$ and $v \in V(H)$. Then $d(u) + d(v) - 2 \geq \delta_G + \delta_H + |G| + |H| - 2$ and $d(u)d(v) \leq (\Delta_G + |H|)(\Delta_H + |G|)$. Thus $\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \geq \sqrt{\frac{\delta_G+\delta_H+|G|+|H|-2}{(\Delta_G+|H|)(\Delta_H+|G|)}}$ and we have:

$$ABC(G+H) = \sum_{e=uv} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$$

$$\geq \sum_{\substack{e=uv\\u,v\in V(G)}} \left| \frac{\delta_G}{\Delta_G+|H|}\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}} - \frac{\sqrt{2|H|}}{\Delta_G+|H|}\right|$$

$$+ \sum_{\substack{e=uv\\u,v\in V(H)}} \left| \frac{\delta_H}{\Delta_H+|G|}\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}} - \frac{\sqrt{2|G|}}{\Delta_H+|G|}\right| + \sum_{\substack{e=u,v\\u\in V(G),v\in V(H)}} \sqrt{\frac{\delta_G+\delta_H+|G|+|H|-2}{(\Delta_G+|H|)(\Delta_H+|G|)}}$$

$$\geq \left| \sum_{\substack{e=uv\\u,v\in V(G)}} \left(\frac{\delta_G}{\Delta_G+|H|}\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}} - \frac{\sqrt{2|H|}}{\Delta_G+|H|}\right)\right|$$

$$+ \left| \sum_{\substack{e=uv\\u,v\in V(H)}} \left(\frac{\delta_H}{\Delta_H+|G|}\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}} - \frac{\sqrt{2|G|}}{\Delta_H+|G|}\right)\right| + |G||H|\sqrt{\frac{\delta_G+\delta_H+|G|+|H|-2}{(\Delta_G+|H|)(\Delta_H+|G|)}}$$

$$= \left| \frac{\delta_G}{\Delta_G+|H|}ABC(G) - |E(G)|\frac{\sqrt{2|H|}}{\Delta_G+|H|}\right| + \left| \frac{\delta_H}{\Delta_H+|G|}ABC(H) - |E(H)|\frac{\sqrt{2|G|}}{\Delta_H+|G|}\right|$$

$$+ |G||H|\sqrt{\frac{\delta_G+\delta_H+|G|+|H|-2}{(\Delta_G+|H|)(\Delta_H+|G|)}}.$$

This completes the first part of our theorem. To prove the second part, it is enough to apply the inequalities (3) and (4). □

Theorem 3. *Suppose G and H are arbitrary graphs. Then*

$$ABC(G\square H) \leq \frac{\Delta_G}{\delta_G+\delta_H}|H|ABC(G) + \frac{\Delta_H}{\delta_G+\delta_H}|G|ABC(H) + (|H||E(G)| + |G||E(H)|)\frac{\sqrt{2}}{\delta_G+\delta_H}.$$

Equality holds if and only if there are two positive integers r and s such that $G \cong rK_2$ and $H \cong sK_2$.

Proof. Suppose $u = (a, b)$ and $v = (c, d)$ are vertices of $G \times H$. Then we have:

$$\begin{aligned}
\frac{d(u)+d(v)-2}{d(u)d(v)} &= \frac{(d_G(a)+d_H(b)) + (d_G(c)+d_H(d)) - 2}{(d_G(a)+d_H(b))(d_G(c)+d_H(d))}\\
&= \frac{(d_G(a)+d_G(c)-2) + (d_H(b)+d_H(d)-2) + 2}{(d_G(a)+d_H(b))(d_G(c)+d_H(d))}\\
&= \frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}\,\frac{d_G(a)d_G(c)}{(d_G(a)+d_H(b))(d_G(c)+d_H(d))}\\
&\quad + \frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}\,\frac{d_H(b)d_H(d)}{(d_G(a)+d_H(b))(d_G(c)+d_H(d))}\\
&\quad + \frac{2}{(d_G(a)+d_H(b))(d_G(c)+d_H(d))}\,\frac{d(u)+d(v)-2}{d(u)d(v)}\\
&\leq \frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}\,\frac{\Delta_G^2}{(\delta_G+\delta_H)^2} + \frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}\,\frac{\Delta_H^2}{(\delta_G+\delta_H)^2} + \frac{2}{(\delta_G+\delta_H)^2}.
\end{aligned}$$

Hence,

$$\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \leq \frac{\Delta_G}{\delta_G+\delta_H}\sqrt{\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}} + \frac{\Delta_H}{\delta_G+\delta_H}\sqrt{\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}} + \frac{\sqrt{2}}{\delta_G+\delta_H} \quad (5)$$

with equality if and only if $d_G(a) = d_G(c) = 1$ and $d_G(b) = d_G(d) = 1$. So,

$$\sum_{\substack{e=uv\\u=(a,b),v=(c,d)}} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \leq \frac{\Delta_G}{\delta_G+\delta_H}\sum_{e=(a,b)(c,d)} \sqrt{\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}}$$

$$+ \frac{\Delta_H}{\delta_G+\delta_H}\sum_{e=(a,b)(c,d)} \sqrt{\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}} + \sum_{e=(a,b)(c,d)} \frac{\sqrt{2}}{\delta_G+\delta_H}.$$

Therefore,

$$\begin{aligned}
ABC(G\square H) &= \sum_{\substack{e=uv\\ u=(a,b),v=(c,d)}} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}\\
&\le \frac{\Delta_G}{\delta_G+\delta_H}|H|\sum_{e=ac}\sqrt{\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}}+\frac{\Delta_H}{\delta_G+\delta_H}|G|\sum_{e=bd}\sqrt{\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}}\\
&\quad+(|H||E(G)|+|G||E(H)|)\frac{\sqrt{2}}{\delta_G+\delta_H}\\
&= \frac{\Delta_G}{\delta_G+\delta_H}|H|ABC(G)+\frac{\Delta_H}{\delta_G+\delta_H}|G|ABC(H)+(|H||E(G)|+|G||E(H)|)\frac{\sqrt{2}}{\delta_G+\delta_H}.
\end{aligned}$$

This proves the first part and inequality (5) implies the second part of theorem. □

Example 2. Suppose R and S denote a C_4 nanotube and nanotorus, respectively. Then $R \cong P_n\square C_m$, $S \cong C_n\square C_m$ and $T = P_n\square P_m$, where $m, n \ge 3$. Then we have:

$$\begin{aligned}
ABC(R) &= ABC(P_n\square C_m)\\
&= m\sqrt{\frac{5}{3}}+\frac{4}{3}m+\frac{2mn-5m}{4}\sqrt{6},\\
ABC(S) &= ABC(C_n\square C_m)\\
&= \frac{mn}{2}\sqrt{6},\\
ABC(T) &= ABC(P_n\square P_m)\\
&= (m+n-4)\sqrt{\frac{5}{3}}+\frac{4(m+n-6)}{3}+\frac{2mn-5(m+n)+12}{4}\sqrt{6}+4\sqrt{2}.
\end{aligned}$$

By Theorem 3, it is easy to see that $\lim_{n\to\infty}\frac{ABC(P_n\square C_n)}{Our\ bound}=\frac{3\sqrt{3}}{8}$.

Theorem 4. *Suppose G and H are arbitrary graphs. Then*

$$\begin{aligned}
ABC(G[H]) \le \frac{1}{|H|\delta_G+\delta_H}&\left(\Delta_G\sqrt{|H|}\left(ABC(G)+|G||E(H)|\frac{\sqrt{2\Delta_G-2}}{\delta_G}\right)\right.\\
&\left.+\Delta_H\left(|E(G)|\frac{\sqrt{2\Delta_H-2}}{\delta_H}+|G|ABC(H)\right)+|E(G[H])|\sqrt{2|H|}\right).
\end{aligned}$$

Equality holds if and only if there are two positive integers r and s such that $G \cong rK_2$ and $H \cong sK_2$.

Proof. Let $u = (a, b)$ and $v = (c, d)$ then $d_{G[H]}(u) = |H|d_G(a) + d_H(b)$. Now

$$\begin{aligned}
\frac{d(u)+d(v)-2}{d(u)d(v)} &= \frac{|H|d_G(a)+d_H(b)+|H|d_G(c)+d_H(d)-2}{(|H|d_G(a)+d_H(b))(|H|d_G(c)+d_H(d))}\\
&= |H|\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}\frac{d_G(a)d_G(c)}{(|H|d_G(a)+d_H(b))(|H|d_G(c)+d_H(d))}\\
&\quad+\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}\frac{d_H(b)d_H(d)}{(|H|d_G(a)+d_H(b))(|H|d_G(c)+d_H(d))}\\
&\quad+\frac{2|H|}{(|H|d_G(a)+d_H(b))(|H|d_G(c)+d_H(d))}\\
&\le |H|\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}\frac{\Delta_G^2}{(|H|\delta_G+\delta_H)^2}\\
&\quad+\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}\frac{\Delta_H^2}{(|H|\delta_G+\delta_H)^2}+\frac{2|H|}{(|H|\delta_G+\delta_H)^2}.
\end{aligned}$$

Thus

$$\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \le \sqrt{|H|}\frac{\Delta_G}{|H|\delta_G+\delta_H}\sqrt{\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}}$$

$$+\frac{\Delta_H}{|H|\delta_G+\delta_H}\sqrt{\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}}+\frac{\sqrt{2|H|}}{|H|\delta_G+\delta_H} \tag{6}$$

with equality if and only if $d_G(a)=d_G(c)=1$ and $d_H(b)=d_H(d)=1$. Therefore

$$\begin{aligned}
ABC(G[H]) &= \sum_{e=uv}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}\\
&\le \frac{1}{|H|\delta_G+\delta_H}\Bigg(\Delta_G\sqrt{|H|}\sum_{\substack{ac\in E(G)\text{ or}\\ a=c,bd\in E(H)}}\sqrt{\frac{d_G(a)+d_G(c)-2}{d_G(a)d_G(c)}}\\
&\quad+\ \Delta_H\sum_{\substack{ac\in E(G)\text{ or}\\ a=c,bd\in E(H)}}\sqrt{\frac{d_H(b)+d_H(d)-2}{d_H(b)d_H(d)}}+|E(G[H])|\sqrt{2|H|}\Bigg)\\
&\le \frac{1}{|H|\delta_G+\delta_H}\left(\Delta_G\sqrt{|H|}\left(ABC(G)+|G||E(H)|\frac{\sqrt{2\Delta_G-2}}{\delta_G}\right)\right.\\
&\quad\left.+\Delta_H\left(|E(G)|\frac{\sqrt{2\Delta_H-2}}{\delta_H}+|G|ABC(H)\right)+|E(G[H])|\sqrt{2|H|}\right).
\end{aligned}$$

Thus one can see that

$$\begin{aligned}
ABC(G[H]) &\le \frac{1}{|H|\delta_G+\delta_H}\left(\Delta_G\sqrt{|H|}\left(ABC(G)+|G||E(H)|\frac{\sqrt{2\Delta_G-2}}{\delta_G}\right)\right.\\
&\quad\left.+\Delta_H\left(|E(G)|\frac{\sqrt{2\Delta_H-2}}{\delta_H}+|G|ABC(H)\right)+|E(G[H])|\sqrt{2|H|}\right)
\end{aligned}$$

which proves the first part of theorem. The equality is implied by (6). □

Theorem 5. *The ABC index of the corona product is computed as follows:*

$$\begin{aligned}
ABC(GoH) &\le \frac{\Delta_G}{\delta_G+|H|}ABC(G)+\frac{\sqrt{2|H|}}{\delta_G+|H|}|E(G)|+|G|\left(\frac{\Delta_H}{\delta_H+1}ABC(H)+\frac{\sqrt{2}}{\delta_H+1}|E(H)|\right)\\
&\quad+|G||H|\sqrt{\frac{\Delta_G+\Delta_H+|H|-1}{(\delta_G+|H|)(\delta_H+1)}}.
\end{aligned}$$

Equality holds if and only if there are two positive integers r and s such that $G\cong rK_2$ and $H\cong sK_2$.

Proof. By definition of the *ABC* index, we have:

$$ABC(GoH)=\sum_{\substack{e=uv\\ u,v\in V(G)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}+|G|\sum_{\substack{e=uv\\ u,v\in V(H)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}+\sum_{\substack{e=uv\\ u\in V(G),v\in V(H)}}\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}.$$

We first assume that $u, v\in V(G)$ are arbitrary. Then

$$\begin{aligned}
\frac{d(u)+d(v)-2}{d(u)d(v)} &= \frac{d_G(u)+d_G(v)-2+2|H|}{(d_G(u)+|H|)(d_G(v)+|H|)}\\
&= \frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}\frac{d_G(u)d_G(v)}{(d_G(u)+|H|)(d_G(v)+|H|)}+\frac{2|H|}{(d_G(u)+|H|)(d_G(v)+|H|)}\\
&\le \frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}\frac{\Delta_G^2}{(\delta_G+|H|)^2}+\frac{2|H|}{(\delta_G+|H|)^2}.
\end{aligned}$$

If $u, v\in V(H)$ then,

$$\begin{aligned}
\frac{d(u)+d(v)-2}{d(u)d(v)} &= \frac{d_H(u)+1+d_H(v)+1-2}{(d_H(u)+1)(d_H(v)+1)}\\
&= \frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}\frac{d_H(u)d_H(v)}{(d_H(u)+1)(d_H(v)+1)}+\frac{2}{(d_H(u)+1)(d_H(v)+1)}
\end{aligned}$$

$$\leq \frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}\frac{\Delta_H^2}{(\delta_H+1)^2}+\frac{2}{(\delta_H+1)^2}.$$

Finally, if $u \in V(G)$ and $v \in V(H)$ then,

$$\frac{d(u)+d(v)-2}{d(u)d(v)} = \frac{d_G(u)+|H|+d_H(v)+1-2}{(d_G(u)+|H|)(d_H(v)+1)} \leq \frac{\Delta_G+\Delta_H+|H|-1}{(\delta_G+|H|)(\delta_G+1)}.$$

Therefore,

$$\begin{aligned} ABC(GoH) &\leq \sum_{uv\in E(G)}\left(\sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}}\frac{\Delta_G}{\delta_G+|H|}+\frac{\sqrt{2|H|}}{\delta_G+|H|}\right) \\ &\quad +|G|\left(\sum_{uv\in E(H)}\sqrt{\frac{d_H(u)+d_H(v)-2}{d_H(u)d_H(v)}}\frac{\Delta_H}{\delta_H+1}+\frac{\sqrt{2}}{\delta_H+1}\right)+|G||H|\sqrt{\frac{\Delta_G+\Delta_H+|H|-1}{(\delta_G+|H|)(\delta_H+1)}} \\ &= \frac{\Delta_G}{\delta_G+|H|}ABC(G)+\frac{\sqrt{2|H|}}{\delta_G+|H|}|E(G)|+|G|\left(\frac{\Delta_H}{\delta_H+1}ABC(H)+\frac{\sqrt{2}}{\delta_H+1}|E(H)|\right) \\ &\quad +|G||H|\sqrt{\frac{\Delta_G+\Delta_H+|H|-1}{(\delta_G+|H|)(\delta_H+1)}}, \end{aligned}$$

proving the result. The equality is obtained in the same way as Theorems 1–5. □

Example 3. Suppose T is an n-vertex tree and H is a trivial graph with one vertex then

$$ABC(ToH) \leq \frac{\Delta_T}{2}ABC(T)+\frac{\sqrt{2}}{2}(n-1)+n\sqrt{\frac{\Delta_T}{2}}.$$

Moreover if T is a chemical tree with $\Delta_T \leq 3$ then

$$ABC(ToH) \leq \frac{3}{2}ABC(T)+\frac{\sqrt{2}}{2}(n-1)+n\sqrt{\frac{3}{2}}.$$

Theorem 6. *Suppose G and H are arbitrary graphs. Then*

$$\begin{aligned} ABC(G\nabla H) &\leq \frac{1}{\delta_G+\delta_H+\delta_G\delta_H}\left(\Delta_G\left(|E(H)||G|\frac{\sqrt{2\Delta_G-2}}{\delta_G}+|H|ABC(G)+|E(H)|ABC(G)\right)\right. \\ &\quad \left.+\Delta_H\left(|G|ABC(H)+|E(G)||H|\frac{\sqrt{2\Delta_H-2}}{\delta_H}+|E(G)|ABC(H)\right)+|E(G\nabla H)|\sqrt{2\Delta_G\Delta_H+2}\right). \end{aligned}$$

Equality holds if and only if there are two positive integers r and s such that $G \cong rK_2$ *and* $H \cong sK_2$.

Proof. The proof is similar to those given in Theorems 1–5. □

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