Finance Research Letters 11 (2014) 183–193

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

Computing present values: Capital budgeting done correctly ☆

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ARTICLE INFO

Article history:
Received 2 February 2014
Accepted 3 May 2014
Available online 15 May 2014

JEL classification codes:
G31
G12
G10
G13

Keywords:
Present value
No arbitrage
Capital budgeting

ABSTRACT

This paper shows that the standard textbook formula for computing the present value of a future random cash flow – the discounted expected value – is formally incorrect and can generate significant errors when used to compute present values. The correct present value method is provided as well as a simple adjustment to the textbook formula which can be used to obtain an approximation to the correct value.

1. The introduction

Perhaps the most fundamental concept in corporate finance and capital budgeting is the notion of a present value. Computing a present value is one of the first tasks that a student of finance is expected to master. Therefore, it may be surprising to learn that the standard textbook formula for computing the present value of a future random cash flow – taking the discounted expected value of the cash flow (see Ross et al. (2008), chap. 12; Brealey and Myers (2003), chap. 9; Brealey et al. (2011), chap. 10; Berk and DeMarzo (2014), chaps. 18 and 19) – is formally incorrect and often generates significantly biased values.

☆ Helpful comments from Arkadev Chatterjea and Andrew Karolyi are greatly appreciated.

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http://dx.doi.org/10.1016/j.frl.2014.05.001
1544-6123/

The purpose of this paper is to prove this claim by: (1) showing the correct method, (2) documenting the size of the error when using the standard textbook formula, and (3) providing an adjustment to the textbook formula which, as an approximation, removes this error. The error in the textbook formula is due to the fact that it ignores the correlations between the cash flows and the discount rates when computing present values.

The goal, of course, is to improve decision making in standard capital budgeting procedures. Although many of these results are provable from existing equilibrium asset pricing models, I could find no source presenting these results. A minor contribution here is proving these results using only the absence of arbitrage.

An outline of the paper is as follows. Section 2 formalizes the argument for a discrete time model. Section 3 repeats the analysis for a continuous time model. This section contains the documentation of the size of the errors in using the textbook formula as well as the adjustment to obtain an approximation to the correct value. Section 4 concludes. All proofs are contained in the Appendix A.

2. The discrete time model

The discrete time model is presented first because the mathematics is simpler than in the continuous time case, and consequently the intuition behind the correct present valuation formula is clear.

We consider a finite horizon model where time increments in units, $t = 0, 1, \ldots, T$. The economy's randomness is characterized by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1,\ldots,T]}, \mathbf{P})$ satisfying the usual conditions, see Protter (2005), with $\mathbf{P}$ the statistical probability measure.

Let C_T be a random cash flow at time T assumed to be $\mathcal{F}_T$- measurable that we want to compute its present value. We assume that the cash flow is non-zero and strictly positive with positive probability, i.e. $\mathbf{P}(C_T \geqslant 0) = 1$ and $\mathbf{P}(C_T > 0) > 0$.

Let V_t be the present value of the cash flow at time t. Note that by definition $V_T = C_T$.

Assumption (No Arbitrage) $E(|V_t|) < \infty$ and $V_t > 0$ for all t.

This is a *no arbitrage* assumption because if V_t traded and for some t either $V_t = \infty$ or $V_t = 0$, then an arbitrage opportunity would exist. Indeed, in the first case (∞) buying V_t and holding the position until time T generates the arbitrage opportunity, and in the second case (0) shorting V_t and holding the position until time T generates the arbitrage opportunity.

Next, define μ_{T-1} to be the time $T-1$ conditional expected return on the cash flow's present value, i.e.

$$E_{T-1}\left(\frac{V_T}{V_{T-1}}\right) = 1 + \mu_{T-1}. \tag{2.1}$$

Note that when viewed at a time $t < T-1$, μ_{T-1} is a random variable. We call μ_{T-1} the *discount rate*. The no arbitrage assumption guarantees[1] both that μ_{T-1} exists and that $\mu_{T-1} > -1$.

Expression (2.1) implies that $V_{T-1} = E_{T-1}\left(\frac{V_T}{1+\mu_{T-1}}\right)$. Next, using the same logic at time $T-2$, we obtain $V_{T-2} = E_{T-2}\left(\frac{V_{T-1}}{1+\mu_{T-2}}\right)$.

Substitution of V_{T-1} into this last expression and using the law of iterated expectations gives $V_{T-2} = E_{T-2}\left(\frac{E_{T-1}\left(\frac{V_T}{1+\mu_{T-1}}\right)}{1+\mu_{T-2}}\right) = E_{T-2}\left(\frac{V_T}{[1+\mu_{T-2}][1+\mu_{T-1}]}\right)$.

Continuing, we get the present value formula.[2]

$$V_t = E_t\left(\frac{V_T}{[1+\mu_t][1+\mu_{t+1}]\cdots[1+\mu_{T-1}]}\right). \tag{2.2}$$

[1] The proof of this statement is simple. $V_T > 0$ and $V_{T-1} > 0$ imply that $\frac{V_T}{V_{T-1}} > 0$ so that $\frac{V_T}{V_{T-1}} - 1 > -1$. Taking the conditional expectation proves the claim.

[2] This is really a version of the Doob decomposition theorem, see Follmer and Schied (2004), p. 277.

This expression is consistent with no arbitrage. To see this, let us first simplify the notation and define $B_t = [1+\mu_0][1+\mu_1]\cdots[1+\mu_{t-1}]$. Rewriting expression (2.2) using B_t then gives

$$\frac{V_t}{B_t} = E_t\left(\frac{V_T}{B_T}\right). \tag{2.3}$$

This expression shows that under $\mathbf{P}, \frac{V_t}{B_t}$ is a martingale. This is a no arbitrage condition. Indeed, to prove this claim, suppose that investing in V_t represents an arbitrage. We seek a contradiction of this supposition. If an arbitrage, then $\mathbf{P}(V_T \geqslant 0) = 1, \mathbf{P}(V_T > 0) = 1$, and $V_t = 0$. Noting that $B_t > 0$ because $\mu_t > -1$ for all t, this implies $E_t\left(\frac{V_T}{B_T}\right) > 0$. But, $E_t\left(\frac{V_T}{B_T}\right) = \frac{V_t}{B_t} > 0$ implies $V_t > 0$, which is the contradiction.

For future use, we can rewrite the present valuation formula one last time as

$$V_t = E_t(V_T)E_t\left(\frac{1}{B_T}\right) + cov_t\left(V_T, \frac{1}{B_T}\right). \tag{2.4}$$

We note that the derivation of this present value formula is valid even if the asset V_T does not trade. In this case, however, the assumption given above loses its interpretation as a no-arbitrage condition. When the asset does not trade, this assumption can be interpreted as a necessary condition for the existence of equilibrium prices.

2.1. Identification of the discount rates

The discount rates used to compute the present value can be obtained from estimating a standard equilibrium asset pricing model applied to a traded asset which proxies for (has the same risk as) the cash flow. Alternatively, one can use the arbitrage based asset pricing model contained in Jarrow and Protter (2013). Jarrow and Protter show that if V_t represents the value of a traded asset, then in a frictionless and competitive market with no arbitrage opportunities, the discount rate satisfies

$$\mu_t = r_0(t) + \sum_{j=1}^{N_\mu} \beta_j(t)[E_t(r_j(t)) - r_0(t)], \tag{2.5}$$

where $r_0(t)$ is the riskless rate over $[t, t+1], r_j(t)$ corresponds to the return on the jth risk factor for $j = 1, \ldots, N_\mu$ where the number and particular risk factors included from the set of all risk factors[3] depends on μ_t, and $\beta_j(t)$ is the asset's beta with respect to the jth risk factor. The advantage of a no arbitrage pricing methodology versus an equilibrium based pricing model is that equilibrium requires more structure on traders' preferences, beliefs, endowments and the market clearing mechanism than does arbitrage free pricing.

2.2. The error

This section characterizes the error between the textbook present value formula and the correct one. Given the cash flow $C_T = V_T$, recall that the textbook formula is:

$$\textit{Textbook Formula} = \frac{E(V_T)}{[1+\mu_0][1+E(\mu_1)]\cdots[1+E(\mu_{T-1})]}. \tag{2.6}$$

To facilitate the economic intuition, we introduce some new notation. Define the discount factor

$$D(t,T) = E_t\left(\frac{1}{[1+\mu_t][1+\mu_{t+1}]\cdots[1+\mu_{T-1}]}\right). \tag{2.7}$$

This represents the present value of a dollar received at time T that has the same risk as the cash flow C_T. This is an abstract construct that does not trade. Given these discount factors, define the implied forward rates analogous to those in a default free term structure setting, i.e. $1 + F(t,T) = \frac{D(t,T)}{D(t,T+1)}$. This is the time t forward rate for the future time period $[T, T+1]$. A direct calculation shows that

[3] In the most general version of this model there can be an uncountably infinite number of risk factors trading in the economy.

$$D(t,T) = \frac{1}{[1+F(t,t)][1+F(t,t+1)]\cdots[1+F(t,T-1)]}. \quad (2.8)$$

Using the new notation, the error in computing the present value can be written as:

$$\begin{aligned} &Error = Correct\ Present\ Value - Textbook\ Formula \\ &= \left[E\left(\frac{V_T}{B_T}\right) - E(V_T)D(0,T)\right] + \left[E(V_T)D(0,T) - \frac{E(V_T)}{E(B_T)}\right] \\ &+ \left[\frac{E(V_T)}{E(B_T)} - \frac{E(V_T)}{E(1+\mu_0)E(1+\mu_1)\cdots E(1+\mu_{T-1})}\right] \end{aligned}. \quad (2.9)$$

We analyze each bias in turn. Note that the biases are numbered in reverse order, with the extreme right side of the previous expression representing bias 1.

2.2.1. Bias 1

The first bias is

$$\begin{aligned} Bias\ 1 &= \frac{E(V_T)}{E(B_T)} - \frac{E(V_T)}{E(1+\mu_0)E(1+\mu_1)\cdots E(1+\mu_{T-1})} \\ &= \frac{E(V_T)}{E(B_T)}\left[1 - \frac{E(B_T)}{E(B_T) - \sum_{t=1}^{T-2} cov((1+\mu_t),(1+\mu_{t+1})\cdots(1+\mu_{T-1}))}\right]. \end{aligned} \quad (2.10)$$

The first bias is characterized by the term in brackets. This bias is small if the autocorrelation between discount rates is small. Note that if the correlation between consecutive discount rates is positive (they exhibit mean reversion), then this component generates a negative bias.

2.2.2. Bias 2

The second bias is

$$\begin{aligned} Bias\ 2 &= E(V_T)\left[D(0,T) - \frac{1}{E(B_T)}\right] \\ &= E(V_T)\left[var(B_T) - \sum_{i=3}^{\infty}(-1)^i\left\{E([B_T-1]^i) - [E(B_T-1)]^i\right\}\right] > 0. \end{aligned} \quad (2.11)$$

As shown, the second bias is strictly positive and approximately equal in magnitude to the variance of the discount factor, i.e. $var(B_T)$.

The first and second biases combined yield

$$Biases\ 1\ \text{and}\ 2 = E(V_T)D(0,T) - \frac{E(V_T)}{E(1+\mu_0)E(1+\mu_1)\cdots E(1+\mu_{T-1})}. \quad (2.12)$$

This combined bias is the result of using the discount rates μ_t instead of the "correct" forward rates $F(0,t)$. If the "risk premium" embedded in the forward rates is positive, i.e. $F(0,t) > \mu_t$ for all t, then the sum of these two biases will be negative.

2.2.3. Bias 3

Last, we can rewrite expression (2.2) as:

$$V_t = E_t(V_T)D(t,T) + corr_t\left(V_T, \frac{1}{B(T)}\right)sd_t(V_T)sd_t(B(T)),$$

where the third bias is given by

$$Bias\ 3 = corr_t\left(V_T, \frac{1}{B(T)}\right)sd_t(V_T)sd_t(B(T)). \quad (2.13)$$

The third bias depends on the correlation between the cash flow's value and the (reciprocal of) the discount rates. The bias is positive if and only if this correlation is positive.

Interestingly, the third bias's mathematics is analogous to the mathematics characterizing the difference between forward and futures prices for commodities (see Jarrow (2009)). The differences between forward and futures prices for various commodities are known to be economically significant, and hence one would expect the same to be true for bias 3 in this context as well. To see this identification, let V_t be the spot price of a commodity. Then $\frac{V_t}{D(t,T)}$ is the forward price, and $E_t(C_T)$ corresponds to the futures price when $E(\cdot)$ is interpreted as the risk-neutral probability measure. The difference between forward and futures prices is well known to be equal to $\frac{1}{D(t,T)} cov_t\left(V_T, \frac{1}{B(T)}\right)$.

3. The continuous time model

Consider a finite horizon continuous time economy with the randomness characterized by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ satisfying the usual conditions, see Protter (2005), with $\mathbf{P}$ the statistical probability measure. The continuous time model is useful for generating analytic representations of the error from using the textbook present valuation formula that are conducive to calibration and numerical calculations.

Let C_T be a cash flow at time T for computing its time 0 present value assumed to be $\mathcal{F}_T$- measurable. As in the discrete time model, we assume that the cash flow is non-zero and strictly positive with positive probability, i.e. $\mathbf{P}(C_T \geqslant 0) = 1$ and $\mathbf{P}(C_T > 0) > 0$.

Analogous to the discrete time model, we want to define the present value's conditional expected return over an infinitesimal time interval $[t, t+dt]$.

Assumption (No Arbitrage). *A solution $V_t > 0$ for all t exists to the following backward stochastic differential equation (bsde):*

$$\begin{array}{l} dV_t = V_t\mu_t dt + V_t dM_t \\ \text{subject to } V_T = C_T \end{array}, \tag{3.1}$$

where μ_t is the present value's time t conditional expected return per unit time assumed $\mathcal{F}_t$ - measurable with $\int_0^T |\mu_s V_t| ds < \infty$, and M_t is a continuous martingale[4] adapted to the filtration $\mathcal{F}_t$.

It is shown in the Appendix A that the solution to this bsde yields the present value formula:

$$V_t = E_t\left(C_T e^{-\int_t^T \mu_s ds}\right). \tag{3.2}$$

As shown, the present value corresponds to the expected discounted cash flow.

Defining $B_t = e^{-\int_t^T \mu_s ds}$, analogous to the discrete time case, this implies

$$\frac{V_t}{B_t} = E_t\left(\frac{V_T}{B_T}\right). \tag{3.3}$$

For later usage, the textbook formula is

$$\text{Textbook Formula} = E(C_T)e^{-\int_0^T E(\mu_s)ds}. \tag{3.4}$$

3.1. The error

This section characterizes the error between the textbook present value formula and the correct one. To facilitate the analytic expressions, define the discount factor

$$D(t,T) = E_t\left(e^{-\int_t^T \mu_s ds}\right), \tag{3.5}$$

[4] This is related to a Doob–Meyer Decomposition theorem, see Protter (2005), p. 111, with the additional assumption that the predictable process in the decomposition is absolutely continuous. This assumption could be generalized to allow the present value process to be discontinuous.

and the forward rates

$$F(t,T) = -\frac{\partial log D(t,T)}{\partial T}, \tag{3.6}$$

assuming, of course, that they exist.

Using this new notation, the correct present value formula is:

$$V_0 = E(C_T)D(0,T) + cov\left(C_T, \frac{1}{B_T}\right). \tag{3.7}$$

The error from using the textbook formula is

$$\begin{aligned} Error &= Correct\ Present\ Value - Textbook\ Formula \\ &= E(C_T)\left[D(0,T) - e^{-\int_0^T E(\mu_s)ds}\right] + cov\left(C_T, \frac{1}{B_T}\right). \end{aligned} \tag{3.8}$$

In continuous time we see that the first bias disappears:

$$Bias\ 1 = E(C_T)\left[e^{-E\left(\int_0^T \mu_s ds\right)} - e^{-\int_0^T E(\mu_s)ds}\right] = 0.$$

This occurs because when using exponentials, the denominator in the discrete time setting corresponds to an exponent in the continuous time setting, and the expectation of an integral is the integral of the expectation.

The second bias is strictly positive:

$$Bias\ 2 = E(C_T)\left[D(0,T) - e^{-\int_0^T E(\mu_s)ds}\right] = E(C_T)\left[e^{-\int_0^T F(0,s)ds} - e^{-\int_0^T E(\mu_s)ds}\right] > 0. \tag{3.9}$$

As in the discrete time case, this bias is generated because the textbook formula uses the discount rates $E(\mu_s)$ instead of the forward rates $F(0,s)$. This bias is strictly positive because e^{-x} is a strictly convex function, and Jensen's inequality applies.

The third bias is given by the covariance between the cash flow and the discount factor:

$$Bias\ 3 = cov\left(C_T, \frac{1}{B_T}\right). \tag{3.10}$$

This bias is strictly positive if and only if the correlation between the cash flow and the discount factor is strictly positive.

3.2. A calibration

The purpose of the section is to estimate the size of the error generating from using the textbook formula instead of the correct one. Surprisingly, the error is shown to be quite large. The example used is where both the random cash flow and the discount factor are jointly lognormally distributed.

3.2.1. The cash flow

We assume that the cash flow is lognormally distributed, i.e.

$$C_T = E(C_T)e^{-\frac{1}{2}\eta^2 + \eta Y}, \tag{3.11}$$

where Y is normal $(0,1)$. Note that the parameters do not depend on T.

3.2.2. The forward rates

Using an analogy to the term structure of interest rate mathematics, we impose our assumption directly on the implied forward rates from the discount factors. In particular, we assume

$$dF(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \tag{3.12}$$

where $\sigma(t,T)$ is a deterministic function of time and W_t is a standard Brownian motion adapted to the filtration $\mathcal{F}_t$. This implies that forward rates are normally distributed and the discount factors are log-normally distributed.

Define $a(t,T) = -\int_t^T \sigma(t,s)ds$. Then, (see Heath et al. (1992))

$$B_T = \frac{1}{D(0,T)} e^{\frac{1}{2}\int_0^T a(s,T)^2 ds - \int_0^T a(s,T)dW_s}, \tag{3.13}$$

where $\int_0^T a(s,T)dW_s$ is normal $\left(0, \int_0^T a(s,T)^2 ds\right)$.[5]

It is shown in the Appendix A that

$$e^{-\int_0^T E(\mu_s)ds} = D(0,T)e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds}, \quad and \tag{3.14}$$

$$cov\left(C_T, \frac{1}{B_T}\right) = E(C_T)D(0,T)\left[e^{cov(\log C_T, \log\frac{1}{B_T})} - 1\right]. \tag{3.15}$$

3.3. The biases

Using the formulas just derived, the biases satisfy the following expressions:

$$Bias\ 2 = E(C_T)e^{-\int_0^T E(\mu_s)ds}\left[e^{\frac{1}{2}\int_0^T a(s,T)^2 ds} - 1\right] > 0. \tag{3.16}$$

$$Bias\ 3 = E(C_T)e^{-\int_0^T E(\mu_s)ds}e^{\frac{1}{2}\int_0^T a(s,T)^2 ds}\left[e^{cov(\log C_T, \log\frac{1}{B_T})} - 1\right]. \tag{3.17}$$

Combined, the total error from using the textbook formula is:

$$Error = E(C_T)e^{-\int_0^T E(\mu_s)ds}\left[e^{cov(\log C_T, \log\frac{1}{B_T}) + \frac{1}{2}\int_0^T a(s,T)^2 ds} - 1\right]. \tag{3.18}$$

3.4. The parameters

To calibrate the model, we define the following parameters: $sd(\log C_T) = \eta$, and $\rho = corr\left(\log C_T, \log\frac{1}{B_T}\right)$. Assuming $\sigma(t,s) = \sigma$ we also get: $a(s,T) = -\sigma(T-s)$ and $\int_0^T a(s,T)^2 ds = \frac{\sigma^2 T^3}{3}$, so that $d\left(\log\frac{1}{B_T}\right) = \sqrt{\int_0^T a(s,T)^2 ds} = \sigma\sqrt{\frac{T^3}{3}}$.

Using this parameterization, the error from using the textbook formula is:

$$Error = E(C_T)e^{-\int_0^T E(\mu_s)ds}\left[e^{\rho\eta\sigma\sqrt{\frac{T^3}{3}} + \frac{1}{2}\frac{\sigma^2 T^3}{3}} - 1\right]. \tag{3.19}$$

We set $\sigma = 0.01$, which implies that the forward rate's per year standard deviation is 1%; and we set $\eta = .3$, which implies that standard deviation of the percentage change in the time T cash flow is 30%. We normalize the textbook's formula's cash flow present value to be \$100, i.e $E(C_T)e^{-\int_0^T E(\mu_s)ds} = 100$. Given these values, Table 1 contains the percentage errors for different correlations between the cash flow and discount factors, and different discounting periods.

In Table 1, the zero correlation column gives the errors due to only bias 2. As shown, these are strictly positive and increasing as the discounting period increases. This implies, of course, that the textbook formula underestimates the true present value. For discounting periods 7 years or longer, the percentage errors exceed 2%. At 15 years the percentage error is over 25%.

Next, moving to the left or right of the zero correlation column, the differences in errors are due to the inclusion of bias 3 – the correlation between the cash flows and discount factors. When the correlation is negative, biases 2 and 3 offset each other. For shorter discounting periods, bias 3 dominates and causes

[5] We are assuming, of course, that all given integrals exist (for sufficient conditions see Heath et al. (1992)).

Table 1
Percentage errors from using the textbook formula for different time periods $T = 1, \ldots, 30$ measured in years and different correlations $\rho = corr\left(\log C_T, \log \frac{1}{B_T}\right)$.

T years	Correlation ρ						
	−0.7	−0.5	−0.2	0	0.2	0.5	0.7
1	−0.24	−0.17	−0.06	0.01	0.08	0.18	0.25
2	−0.63	−0.44	−0.14	0.05	0.25	0.54	0.74
3	−1.07	−0.72	−0.18	0.18	0.54	1.09	1.45
4	−1.50	−0.95	−0.13	0.43	0.99	1.83	2.39
5	−1.86	−1.10	0.06	0.84	1.62	2.81	3.61
6	-2.10	−1.10	0.42	1.45	2.49	4.07	5.13
7	-2.18	−0.92	1.01	2.31	3.63	5.65	7.01
10	−1.00	1.20	4.58	6.89	9.26	12.91	15.41
15	-8.78	13.24	20.29	25.23	30.38	38.49	44.18
20	37.22	46.00	60.22	70.46	81.36	99.02	111.75
25	109.29	128.22	159.88	183.39	209.03	251.90	283.73
30	306.15	355.12	439.87	504.96	577.91	704.14	801.10

the error to be negative, i.e. the textbook formula exceeds the correct value. For discounting periods less than or equal to 5 years, the errors are less than 2%. As the discounting period increases, however, bias 2 dominates bias 3, and the errors eventually become positive. In contrast, when the correlation is positive, biases 2 and 3 complement each other, and the two biases sum. The errors are all positive, and depending upon the correlation, they exceed 2% when the discounting period exceeds 3 years.

As seen in Table 1, the percentage errors from using the textbook present value formula are small when computing present values for cash flows received 3 years or less into the future. For 4 years or more, depending on the correlation between the cash flows and discount factor, the errors become significant. For example, for 7 years and a correlation equal to +0.5, the error is 5.65%. For 15 years or longer, the errors are quite large. These errors are sufficiently large that they can generate incorrect decisions when making investment decisions. The implication, of course, is that the textbook formula should not be used for capital budgeting decisions.

Instead, a simple approximation can be employed to adjust the textbook formula and reduce the error in the resulting present value estimate. This approximation is obtained by using a joint lognormal distribution for both the cash flow and discount factor (as done in the calibration example). The adjustment factor is:

$$Adjustment\ Factor = e^{\rho\eta\sigma\sqrt{\frac{T^3}{3}+\frac{1}{2}\frac{\sigma^2 T^3}{3}}}. \tag{3.20}$$

Multiplying the textbook formula by this adjustment factor will generate the following approximation for the correct present value, i.e.

$$V_0 \cong E(C_T)e^{-\int_0^T E(\mu_s)ds} e^{\rho\eta\sigma\sqrt{\frac{T^3}{3}+\frac{1}{2}\frac{\sigma^2 T^3}{3}}}, \tag{3.21}$$

where $sd(\log C_T) = \eta$, $\rho = corr\left(\log C_T, \log \frac{1}{B_T}\right)$, and $sd(dF(t,s)) = \sigma$.

To use this adjustment factor, one needs to estimate the parameters (η, ρ, σ). Since the discount factor $D(t,T)$ or the forward rates $F(t,T)$ are not observable, these parameters cannot be estimated directly. Instead, given a traded asset with a market price which proxies for the cash flow, one can use calibration to obtain these parameter estimates. First, one needs to estimate the present value of the cash flow using the textbook formula, expression (3.4). Then, using the market prices for the proxy asset and expression (3.21), one can find those parameters that equate the market price (left side of the expression) to the theoretical price (right side of the expression). A sum of squared errors minimization procedure can be employed.

4. Conclusion

For such a simple concept, it is perhaps surprising to learn that the standard textbook formula for computing the present value of a random and future cash flow – the discounted expected value – is formally incorrect. This paper proves this claim, documents the magnitude of the error in using the textbook formula, and provides a simple adjustment to the standard textbook formula that can be used to approximate the correct value.

Appendix A. Proofs

A.1. Expression (2.10)

The denominator can be written (using $E(xy) - E(x)E(y) = cov(x, y)$) as:

$$E[B_T] = E((1+\mu_0)(1+\mu_1)\cdots(1+\mu_{T-1}))$$
$$= E(1+\mu_0)E(1+\mu_1)\cdots E(1+\mu_{T-1}) + \sum_{t=1}^{T-2} cov((1+\mu_t), (1+\mu_{t+1})\cdots(1+\mu_{T-1})).$$

Substitution and algebra gives the result.

A.2. Expression (2.11)

Using a Taylor series expansion with $x \neq -1$ gives.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots. \text{ When } 0 < x < 1, \text{ then } x^i > x^{i+1}.$$
$$E\left(\frac{1}{1+x}\right) = 1 - E(x) + E(x^2) - E(x^3) \ldots.$$
$$\frac{1}{E(1+x)} = \frac{1}{1+E(x)} = 1 - E(x) + E(x)^2 - E(x)^3 \ldots.$$

Then,

$$E\left(\frac{1}{1+x}\right) - \frac{1}{E(1+x)} = E(x^2) - E(x)^2 - [E(x^3) - E(x)^3] + \cdots.$$
$$E\left(\frac{1}{1+x}\right) - \frac{1}{E(1+x)} = var(x) - \sum_{i=3}^{\infty}(-1)^i[E(x^i) - E(x)^i]$$

Set $x = B_T - 1$, since $\mu_t > -1$ a.s. implies $B_T > 0$, the resulting ratio exists.
By Jensen's inequality since $\frac{1}{1+x}$ is a strictly convex function, $E\left(\frac{1}{1+x}\right) - \frac{1}{E(1+x)} > 0$.
This completes the proof.

A.3. Expression (3.2)

By Ito's lemma. $d\log V_t = \frac{dV_t}{V_t} - \frac{1}{2}\frac{(dV_t)^2}{V_t^2} = \frac{dV_t}{V_t} - \frac{1}{2}d[M,M]_t$ where $[\cdot,\cdot]_t$ is the quadratic variation.

$$= \mu_t dt - \frac{1}{2}d[M,M]_t + dM_t.$$

The solution is: $\log V_T = \log V_t + \int_t^T \mu_s ds - \frac{1}{2}\int_t^T d[M,M]_s + M_T - M_t$

$$V_T = V_t e^{\int_t^T \mu_s ds - \frac{1}{2}\int_t^T d[M,M]_s + M_T - M_t}.$$

Next,

$$\frac{V_T}{e^{\int_t^T \mu_s ds}} = V_t e^{-\frac{1}{2}\int_t^T d[M,M]_s + M_T - M_t}.$$

Taking expectations gives:

$$E_t\left(\frac{V_T}{e^{\int_t^T \mu_s ds}}\right) = E_t\left(V_t e^{-\frac{1}{2}\int_t^T d[M,M]_s + M_T - M_t}\right) = V_t,$$

since

$$E_t\left(e^{-\frac{1}{2}\int_t^T d[M,M]_s + M_T - M_t}\right) = 1,$$

see Protter (2005), p. 85.

This implies that $V_t = E_t\left(V_T e^{-\int_t^T \mu_s ds}\right)$. Using the condition that $V_T = C_T$ gives the final result.

A.4. Expressions (3.14) and (3.15)

$$E(B_T) = E\left(\frac{1}{D(0,T)} e^{\frac{1}{2}\int_0^T a(s,T)^2 ds - \int_0^T a(s,T)dW_s}\right) = \frac{1}{D(0,T)} e^{\int_0^T a(s,T)^2 ds} E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds - \int_0^T a(s,T)dW_s}\right).$$

But,

$$E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds - \int_0^T a(s,T)dW_s}\right) = 1.$$

So,

$$E(B_T) = \frac{1}{D(0,T)} e^{\int_0^T a(s,T)^2 ds}.$$

Next,

$$\log\frac{1}{B_T} = \log D(0,T) - \frac{1}{2}\int_0^T a(s,T)^2 ds + \int_0^T a(s,T)dW_s,$$

which implies

$$E\left(\frac{1}{B(T)}\right) = E\left(D(0,T)e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds + \int_0^T a(s,T)dW_s}\right) = D(0,T)E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds + \int_0^T a(s,T)dW_s}\right).$$

But,

$$E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds + \int_0^T a(s,T)dW_s}\right) = 1.$$

So,

$$E\left(\frac{1}{B(T)}\right) = D(0,T).$$

This gives

$$e^{-\int_0^T E(\mu_s)ds} = D(0,T)e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds + E\left(\int_0^T a(s,T)dW_s\right)} = D(0,T)e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds}.$$

Thus,

$$Bias\ 2 = E(C_T)\left[D(0,T) - e^{-\int_0^T E(\mu_s)ds}\right] = E(C_T)e^{-\int_0^T E(\mu_s)ds}\left[e^{\frac{1}{2}\int_0^T a(s,T)^2 ds} - 1\right].$$

Next, note that

$$cov\left(C_T, \frac{1}{B_T}\right) = cov\left(E(C_T)e^{-\frac{1}{2}\eta^2+\eta Y}, D(0,T)e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds+\int_0^T a(s,T)dW_s}\right)$$
$$= E(C_T)D(0,T)cov\left(e^{-\frac{1}{2}\eta^2+\eta Y}, e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds+\int_0^T a(s,T)dW_s}\right)$$
$$= E(C_T)D(0,T)\left[E\left(e^{-\frac{1}{2}\eta^2+\eta Y-\frac{1}{2}\int_0^T a(s,T)^2 ds+\int_0^T a(s,T)dW_s}\right) - E\left(e^{-\frac{1}{2}\eta^2+\eta Y}\right)E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds+\int_0^T a(s,T)dW_s}\right)\right].$$

but

$$E\left(e^{-\frac{1}{2}\eta^2+\eta Y}\right) = 1,$$

and

$$E\left(e^{-\frac{1}{2}\int_0^T a(s,T)^2 ds+\int_0^T a(s,T)dW_s}\right) = 1,$$

by the characteristic functions of normal distributions. Substitution gives

$$cov\left(C_T, \frac{1}{B_T}\right) = E(C_T)D(0,T)\left[E\left(e^{-\frac{1}{2}\eta^2-\frac{1}{2}\int_0^T a(s,T)^2 ds+\eta Y+\int_0^T a(s,T)dW_s}\right) - 1\right].$$

Using the characteristic function for a sum of normals:

$$E\left(e^{-\frac{1}{2}\eta^2-\frac{1}{2}\int_0^T a(s,T)^2 ds+\eta Y+\int_0^T a(s,T)dW_s}\right) = e^{cov\left(\eta Y, \int_0^T a(s,T)dW_s\right)} = e^{cov(\log C_T, \log\frac{1}{B_T})}.$$

So,

$$cov\left(C_T, \frac{1}{B_T}\right) = E(C_T)D(0,T)\left[e^{cov(\log C_T, \log\frac{1}{B_T})} - 1\right] = E(C_T)e^{-\int_0^T E(\mu_s)ds}e^{\frac{1}{2}\int_0^T a(s,T)^2 ds}\left[e^{cov(\log C_T, \log\frac{1}{B_T})} - 1\right].$$

This completes the proof.

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