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Filter theory of BL algebras

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Abstract In this paper we consider fundamental properties of some types of filters (Boolean, positive implicative, implicative and fantastic filters) of BL algebras defined in Haveshki et al. (Soft Comput 10:657–664, 2006) and Turunen (Arch Math Logic 40:467–473, 2001). It is proved in Haveshki et al. (2006) that if F is a maximal and (positive) implicative filter then it is a Boolean filter. In that paper there is an open problem

> Under what condition are Boolean filters positive implicative filters?

One of our results gives an answer to the problem, that is, we need no more conditions. Moreover, we give simple characterizations of those filters by an identity form $\forall x, y(t(x, y) \in F)$, where $t(x, y)$ is a term containing x, y.

Keywords (Positive) implicative filter · Boolean filter · Fantastic filter · BL algebra

1 Introduction

BL-algebras were invented by Hájek (1998) in order to prove the completeness theorem of basic fuzzy logic, BL-logic in short. Soon after Cignoli et al. (2000) proved that Hájek's logic really is the logic of continuous t-norms as conjectured by Hájek. At the same time started a systematic study of BL-algebras, too. Indeed, (Turunen 1999) published where BL-algebras were studied by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called filters, too. In Turunen (2001), Boolean deductive systems and implicative deductive systems were introduced. Moreover, it was proved that these deductive systems coincide. In Haveshki et al. (2006) continued an algebraic analysis of BL-algebras and they introduced e.g. implicative filters of BL-algebras. Notice that implicative deductive systems and implicative filters are not, in general, the same subsets. However, in this paper we show that positive implicative filters introduced in Haveshki et al. (2006) and the original implicative deductive systems introduced in Turunen (2001) coincide.

Moreover, they proved in Haveshki et al. (2006) that if F is a maximal and (positive) implicative filter then it is a Boolean filter. But they could not prove the converse. It is left as an open problem

> Under what condition are Boolean filters positive implicative filters?

That is,

> For a Boolean filter F, under what suitable condition does it become a maximal and (positive) implicative filter?

In this paper we give an answer to the problem by showing stronger result that

> The class **PIF** of positive implicative filters coincides with the class **BF** of Boolean filters.

Thus, we can show that every positive implicative filter is a Boolean filter without maximality. This is a complete characterization of Boolean filters.

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We show that the class of positive implicative filters (or Boolean filters) is the same as the class of implicative and fantastic filters and they all coincide with the original implicative deductive systems defined in Turunen (1999).

In Haveshki et al. (2006), those filters are defined by quasi-identity forms such as "if $t(x, y) \in F$ then $t'(x, y) \in F$" for terms $t(x, y)$ and $t'(x, y)$. But, in considering the properties of quotient algebras by those filters, since the class of BL algebras is a variety, if we define those filters by identity-forms then we can see that the quotient algebras have the same properties of original algebras by the general theory of universal algebras. Thus we give simple characterizations of those filters by an identity form $t(x, y) \in F$, where $t(x, y)$ is a term containing x, y.

2 Preliminaries

In the following we define BL algebras and some types of filters according to Haveshki et al. (2006) and Turunen (2001) to express our statement exactly. For an algebraic structure $\mathcal{A} = (A, \wedge, \vee, *, \rightarrow, 0, 1)$ of type (2, 2, 2, 2, 0, 0) is called a *BL algebra* if it satisfies the following conditions

(BL1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice
(BL2) $(A, *, 1)$ is a commutative monoid
(BL3) $*$ is a left adjoint of $\rightarrow$, that is, $x * z \leq y$ if and only if $z \leq x \rightarrow y$
(BL4) $x \wedge y = x*(x \rightarrow y)$, thus $x*(x \rightarrow y) = y*(y \rightarrow x)$
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$

A BL algebra A is called a *Gödel algebra* if $x^2 = x*x = x$ for every $x \in A$. An element x^- is defined by $x \rightarrow 0$. Then a BL algebra A is also called an *MV algebra* if $(x^-)^- = x$ or equivalently $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ for all $x, y \in A$.

Hájek (1998) proved the following

Proposition 1 *Let A be a BL algebra. For all elements $x, y, z \in A$,*

(1) $x * (x \rightarrow y) \leq y$
(2) $x \leq y \rightarrow x * y$
(3) $x \leq y \iff x \rightarrow y = 1$
(4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
(5) $x \leq y \implies y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y$
(6) $y \leq (y \rightarrow x) \rightarrow x$
(7) $y \rightarrow x \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$
(8) $x \rightarrow y \leq (t \rightarrow z) \rightarrow (x \rightarrow z)$
(9) $x \vee y = [(x \rightarrow y) \rightarrow y)] \wedge [(y \rightarrow x) \rightarrow x]$

A *filter* F of a BL algebra A is a non-empty subset of A such that for all $x, y \in A$,

(F1) $x, y \in F$ implies $x * y \in F$;
(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

Since $x \leq 1$ for every $x \in F$, non-emptiness of a filter means (a) $1 \in F$. It is proved in Turunen (2001) that if F is a filter then it satisfies

(a) $1 \in F$
(DS2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

A subset D of a BL algebra A is called a *deductive system* in Turunen (2001) if it satisfies the above two conditions. It is obvious that for a non-empty subset D, D is a deductive system if and only if it is a filter.

For any filter F of a BL algebra A, we can define a relation $\equiv_F$ on A from F as follows : for all $x, y \in A$,

$$x \equiv_F y \iff x \rightarrow y, y \rightarrow x \in F.$$

It is easy to show that the relation $\equiv_F$ is a congruence on A and, since the class of BL algebras is a variety, a quotient structure $A/F = \{[x] \mid x \in A\}$ by $\equiv_F$ is also a BL algebra by the following definition : for all $[x], [y] \in A/F$,

$$[x] \wedge [y] = [x \wedge y]$$
$$[x] \vee [y] = [x \vee y]$$
$$[x] \rightarrow [y] = [x \rightarrow y]$$
$$[x] * [y] = [x * y].$$

Let A be a BL algebra and F a proper filter of A, that is, $F \neq A$. F is called *maximal* if it is not properly contained in any other proper filter of A. In Turunen (2001), implicative deductive systems were defined and it was proved [theorem 1 in Turunen (2001)] that Boolean deductive systems and implicative deductive systems coincide in BL-algebras. A proper filter F is called *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for all $x, y \in A$. A proper filter F is said to be *Boolean* if $x \vee x^- \in F$ for every $x \in A$. By **BF** (**MF**) we mean the class of all (maximal) Boolean filters of A. In Turunen (2001) it is proved that F is a Boolean filter if and only if the quotient algebra A/F is a Boolean algebra. Moreover it is proved in Turunen (2001) that

Proposition 2 *Let A be a BL algebra and F a proper filter of A. Then the following conditions are equivalent:*

(1) *F is a maximal and Boolean filter.*
(2) *F is a prime and Boolean filter.*
(3) *F is a proper filter such that $x \in F$ or $x^- \in F$ for every $x \in A$.*

It is easy to show that F is a maximal and Boolean filter if and only if $|A/F| = 2$.

A non-empty subset $I \subseteq A$ is called an *implicative filter* of A in Haveshki et al. (2006) if it satisfies

(a) $1 \in I$
(I2) $x \to (y \to z) \in I$ and $x \to y \in I$ imply $x \to z \in I$ for all $x, y, z, \in A$.

Also a non-empty subset $F \subseteq A$ is called *positive implicative filter* if

(a) $1 \in F$
(PI2) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z, \in A$.

A non-empty subset F of A is said to be *fantastic* if

(a) $1 \in F$
(FF2) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z, \in A$.

By **IF** (**PIF**, **FF**) we mean the class of all implicative (positive implicative, fantastic) filters of A, respectively.

It is proved in Haveshki et al. (2006) that

Proposition 3 (1) $\mathbf{MF} \cap \mathbf{IF}, \mathbf{MF} \cap \mathbf{PIF} \subseteq \mathbf{BF}$
(2) $\mathbf{PIF} \subset \mathbf{IF}$, *that is, any positive implicative filter is an implicative filter, but the converse is not true.*
(3) *For a filter F of a BL algebra A, $F \in$ **IF** if and only if A/F is a Gödel algebra.*
(4) *For a filter F of a BL algebra A, $F \in$ **FF** if and only if A/F is an MV algebra.*

3 Characterization of filters

In this section we give simple characterizations of filters defined in the previous section. First of all we treat implicative filters of BL algebras. In Haveshki et al. (2006), it is proved that

Proposition 4 [Theorem 3.5 Haveshki et al. 2006]. *The following are equivalent:*

.
(a) *F is an implicative filter.*
(b) *F is a filter and $y \to (y \to x) \in F$ implies $y \to x \in F$ for all $x, y \in A$.*
(c) *F is a filter and $z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$ for all $x, y, z \in A$.*
(d) *$1 \in F$ and $z \to (y \to (y \to x)) \in F$ implies $y \to x \in F$ for all $x, y, z \in A$.*

We note that the characterization of implicative filter above has a quasi-identity form, that is, it has a form of "If $t(x, y) \in F$ then $t'(x, y) \in F$" for terms $t(x, y)$ and $t'(x, y)$. But considering properties of a quotient algebra, for example, to prove the quotient algebra is a BL algebra, it is more useful to use identity forms such as "$t(x, y) \in F$" than to use the quasi-identity forms "if $t(x, y) \in F$ then $t'(x, y) \in F$". In the following we give an identity form of characterization of implicative filters.

Proposition 5 *For any filter F of a BL algebra A, the following conditions are equivalent.*

(1) *F is an implicative filter,*
(2) *$x \to x^2 \in F$ for every $x \in A$.*

Proof Suppose that F is an implicative filter. Since $x \to (x \to x^2) = x^2 \to x^2 = 1 \in F$ and $x \to x = 1 \in F$, we have $x \to x^2 \in F$ by definition.

Conversely, we suppose that F satisfies the condition (2) and that $x \to (y \to z), x \to y \in F$. Since $(x \to (y \to z)) * (x \to y) * x * x \leq (y \to z) * y \leq z$, we have $(x \to (y \to z)) * (x \to y) \leq x^2 \to z$ and thus $x^2 \to z \in F$ by $x \to (y \to z), x \to y \in F$. It follows from assumption $x \to x^2 \in F$ that $x \to z \in F$. This means that F is the implicative filter. □

From the above we can show the next result without difficulty.

Theorem 1 [Theorem 3.7, 3.8 Haveshki et al. (2006)]. *For any filter F, F is an implicative filter if and only if A/F is the Gödel algebra, that is, BL algebra with meeting the condition $\alpha = \alpha^2$ for every $\alpha \in A/F$.*

A non-empty subset $F \subseteq A$ is called a *positive implicative filter* in Haveshki et al. (2006) if

(1) $1 \in F$
(PI2) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$.

The definition is rather artificial, since the role of z is not clear. The following is obvious from the original definition of positive implicative filters.

Proposition 6 [Theorem 3.14 Haveshki et al. (2006)]. *Let F be a filter of A. F is a positive implicative filter if and only if $(x \to y) \to x \in F$ implies $x \in F$ for all $x, y \in A$ if and only if $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$ for all $x, y \in A$.*

Remark It is easy to show that, for a filter F, it is a positive implicative filter if and only if $(a \to 0) \to a \in F$ implies $a \in F$ for every $a \in A$.

We also have an identity form of characterization of positive implicative filters.

Proposition 7 *Let F be a filter of a BL algebra A. F is a positive implicative filter if and only if $(x^- \to x) \to x \in F$ for all $x \in A$.*

Proof Suppose that F is a positive implicative filter. Let $\alpha = (x^- \to x) \to x$. Then we have the following sequence of formulas :

$$\begin{aligned}(\alpha \to 0) \to \alpha &= (((x^- \to x) \to x) \to 0) \to ((x^- \to x) \to x)\\ &= (x^- \to x) \to ((((x^- \to x) \to x) \to 0) \to x)\\ &\geq (((x^- \to x) \to x) \to 0) \to x^-\\ &= (((x^- \to x) \to x) \to 0) \to (x \to 0)\\ &\geq x \to ((x^- \to x) \to x) = 1 \in F.\end{aligned}$$

It follows from the remark that $\alpha = (x^- \to x) \to x \in F$ for every $x \in A$.

Conversely, we assume that $(x \to y) \to x \in F$ for all $x, y \in F$. By Theorem 3.13 in Haveshki et al. (2006), it is sufficient to show that $x \in F$ by the proposition above. Since $0 \leq y$, we have $x \to 0 \leq x \to y$ and thus $(x \to y) \to x \leq (x \to 0) \to x = x^- \to x$. This means that $x^- \to x \in F$. It follows from the assumption $(x^- \to x) \to x \in F$ that $x \in F$. Hence F is the positive implicative filter. □

Moreover, we can prove from the result above that Boolean filters and positive implicative filters are the same.

Theorem 2 *Let F be a filter of A. Then we have F is a Boolean filter if and only if it is a positive implicative filter.*

Proof Suppose that F is a Boolean filter, that is, $x \vee x^- = ((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x) \in F$ for every $x \in A$. In particular, this implies $(x^- \to x) \to x \in F$. Thus F is a positive implicative filter by the proposition above.

Conversely, assume that F is a positive implicative filter. Of course, it is also an implicative filter, thus we have $x \to x^2 \in F$. Since

$$\begin{aligned}(x \to x^-) \to x^- &= (x \to (x \to 0)) \to (x \to 0)\\ &= (x^2 \to 0) \to (x \to 0)\\ &\geq x \to x^2 \in F,\end{aligned}$$

we get that $(x \to x^-) \to x^- \in F$. This implies that $(x^- \to x) \to x \in F$ by assumption. Thus we obtain that

$$x \vee x^- = ((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x) \in F.$$

Hence F is a Boolean filter. □

From the above we have

Theorem 3 *Let F be a filter. The following conditions are equivalent:*

(a) *F is a Boolean filter,*
(b) *F is a positive implicative filter,*
(c) *A/F is a Boolean algebra,*
(d) *F is an implicative deductive system.*

It is proved in Haveshki et al. (2006) as Cor.3.21 that A/F is a Boolean algebra if F is a maximal and (positive) implicative filter. From the result above we can show a stronger result than Haveshki et al. (2006).

Corollary 1 *For a filter F of a BL algebra A, F is a maximal and (positive) implicative filter if and only if $A/F \cong \{0, 1\}$, that is, A/F is isomorphic to the simplest Boolean algebra $\{0, 1\}$.*

Theorem 2 solves the problem presented in Haveshki et al. (2006).

4 Fantastic filters

In Haveshki et al. (2006), a subset F of BL-algebra L was called a *fantastic filter* if

(1) $1 \in F$
(FF2) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$.

It is obvious from definition that

Proposition 8 [Theorem 4.3 Haveshki et al. (2006)]. *F is a fantastic filter if and only if*

(1) $1 \in F$
(FF2)' $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$.

Here we can also give an identity form of characterization of fantastic filters.

Lemma 1 *Let F be a filter. The following conditions are equivalent:*

(1) *F is a fantastic filter,*
(2) *$((x \to 0) \to 0) \to x \in F$ for all $x \in A$,*
(3) *$x \to u \in F$ and $y \to u \in F$ imply $((x \to y) \to y) \to u \in F$.*

Proof • (1) $\Longrightarrow$ (2). Suppose that F is a fantastic filter, hence it satisfies the condition (FF2)' : $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$. If we take $y = 0$ in this formula, since $0 \to x = 1 \in F$, then we have $((x \to 0) \to 0) \to x \in F$.

• (2) $\Longrightarrow$ (3). If $x \to u, y \to u \in F$, since

$$x \to u \leq (u \to 0) \to (x \to 0)$$
$$y \to u \leq (u \to 0) \to (y \to 0),$$

then we have $(u \to 0) \to (x \to 0), (u \to 0) \to (y \to 0) \in F$. Thus

$$((u \to 0) \to (x \to 0)) \wedge ((u \to 0) \to (y \to 0)) \in F.$$

Since

$$((u \to 0) \to (x \to 0)) \wedge ((u \to 0) \to (y \to 0))$$
$$= (u \to 0) \to ((x \to 0) \wedge (y \to 0))$$
$$= (u \to 0) \to ((y \to 0) * ((y \to 0) \to (x \to 0)))$$
$$= (u \to 0) \to (y \to 0) * (x \to ((y \to 0) \to 0)) \in F,$$

it follows from

$$\{(u \to 0) \to (y \to 0) * (x \to ((y \to 0) \to 0))\}$$
$$\to \{(u \to 0) \to (y \to 0) * (x \to y)\}$$
$$\geq (y \to 0) * (x \to ((y \to 0) \to 0))$$
$$\to (y \to 0) * (x \to y)$$
$$\geq (x \to ((y \to 0) \to 0)) \to (x \to y)$$
$$\geq ((y \to 0) \to 0) \to y \in F$$

that

$$(u \to 0) \to (y \to 0) * (x \to y) \in F.$$

Moreover, from

$$(u \to 0) \to (y \to 0) * (x \to y)$$
$$\leq ((y \to 0) * (x \to y) \to 0) \to ((u \to 0) \to 0)$$
$$= \{(x \to y) \to ((y \to 0) \to 0)\} \to ((u \to 0) \to 0),$$

we have

$$\{(x \to y) \to ((y \to 0) \to 0)\} \to ((u \to 0) \to 0) \in F.$$

Since $\{(x \to y) \to ((y \to 0) \to 0)\} \to ((u \to 0) \to 0) \in F$ and

$$[\{(x \to y) \to ((y \to 0) \to 0)\} \to \{(u \to 0) \to 0\}]$$
$$\to [\{(x \to y) \to y\} \to \{(u \to 0) \to 0\}]$$
$$\geq \{(x \to y) \to y\} \to \{(x \to y) \to ((y \to 0) \to 0)\}$$
$$\geq y \to ((y \to 0) \to 0) = 1 \in F,$$

we get that

$$((x \to y) \to y) \to ((u \to 0) \to 0) \in F.$$

It follows from

$$[\{(x \to y) \to y\} \to \{(u \to 0) \to 0\}]$$
$$\to [\{(x \to y) \to y\} \to u]$$
$$\geq ((u \to 0) \to 0) \to u \in F$$

that $((x \to y) \to y) \to u \in F$.

- (3) $\Longrightarrow$ (1). if we take $u = x$ in the condition (3) then we can show that F is the fantastic filter. □

The above means that for a filter F it is a fantastic filter if and only if

$$\sup\{x/F, y/F\} = ((x \to y) \to y)/F$$

in the quotient algebra A/F. Thus we have

Theorem 4 **(Haveshki et al. 2006)** *Let F be a filter of a BL-algebra A. Then we have F is a fantastic filter iff the quotient algebra A/F is an MV-algebra.*

Proof Here we give another proof of the theorem by use of our simple characterization of fantastic filters. Let F be a filter. The result comes from the following:

F is a fantastic filter

$$\Longleftrightarrow ((x \to 0) \to 0) \to x \in F \ (\forall x \in A)$$
$$\Longleftrightarrow (x/F \to 0/F) \to 0/F = x/F \text{ in } A/F$$
$$\Longleftrightarrow (x/F)^{--} = x/F \text{ in } A/F$$
$$\Longleftrightarrow A/F \text{ is an MV algebra.}$$

□

Moreover we can characterize Boolean filters by implicative filters and fantastic filters as follows.

Theorem 5 *Let F be a filter of A. Then*
F is a Boolean filter if and only if it is an implicative and fantastic filter.

Proof Let F be a filter of A. By Theorems 1,3 and 4, we have that F is a Boolean filter iff A/F is a Boolean algebra iff A/F is a BL algebra with $\alpha^{--} = \alpha$ and $\alpha^2 = \alpha$ for every $\alpha \in A/F$ iff F is a fantastic and implicative filter. □

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