Insurance: Mathematics and Economics 65 (2015) 46–54

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On minimizing drawdown risks of lifetime investments

Xinfu Chen [a], David Landriault [b], Bin Li [b], Dongchen Li [b,*]

[a] Mathematics Department, University of Pittsburgh, PA 15260, USA

[b] Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

ARTICLE INFO

Article history:
Received July 2015
Received in revised form
August 2015
Accepted 20 August 2015
Available online 8 September 2015

Keywords:
Drawdown risk
Portfolio optimization
Lifetime investment
Minimum variance strategy
HJB equation

ABSTRACT

Drawdown measures the decline of portfolio value from its historic high-water mark. In this paper, we study a lifetime investment problem aiming at minimizing the risk of drawdown occurrences. Under the Black–Scholes framework, we examine two financial market models: a market with two risky assets, and a market with a risk-free asset and a risky asset. Closed-form optimal trading strategies are derived under both models by utilizing a decomposition technique on the associated Hamilton–Jacobi–Bellman (HJB) equation. We show that it is optimal to minimize the portfolio variance when the fund value is at its historic high-water mark. Moreover, when the fund value drops, the proportion of wealth invested in the asset with a higher instantaneous rate of return should be increased. We find that the instantaneous return rate of the minimum lifetime drawdown probability (MLDP) portfolio is never less than the return rate of the minimum variance (MV) portfolio. This supports the practical use of drawdown-based performance measures in which the role of volatility is replaced by drawdown.

1. Introduction

Drawdown, measuring the decline of portfolio value from its historic high-water mark, is a frequently quoted risk metric to evaluate the performance of portfolio managers in the fund management industry (see, e.g., Burghardt et al., 2003). Drawdown focuses primarily on extreme downward risks (as opposed to other standard risk measures such as volatility and Beta), making it particularly relevant for risk management purposes. Also, drawdown can easily be measured and interpreted by both portfolio managers and clients. A significant drawdown not only leads to large portfolio losses but may also trigger a long-term recession. Bailey and Lopez de Prado (2015) recently provided some justification to the so-called "triple penance rule", where the recovery period was shown to be on average three times as long as the time to produce a drawdown. Also, drawdown is considered a key determinant of sustainable investments as investors tend to overestimate their tolerance to risk. For instance, a sharp drop in portfolio's value is often accompanied by investors exercising their fund redemption options. Moreover, investors tend to assess their investment success by comparing their current portfolio value to the historical maximum value. This resulted in much hardship during the global financial crisis of 2008 when substantial drops in portfolio value were experienced across the board. Therefore, portfolio managers have strong incentives to adopt strategies with low drawdown risks (and more stable growth rate).

Portfolio optimization problems related to drawdown risks have long focused on maximizing the long-term (asymptotic) growth rate of a portfolio subject to a strict drawdown constraint. Grossman and Zhou (1993) pioneered this research topic by considering a market model with a risky asset and a risk-free asset in the Black–Scholes framework. This problem has been extended to a multi-asset framework and a general semimartingale framework by Cvitanic and Karatzas (1995) and Cherny and Obloj (2013), respectively. Klass and Nowicki (2005) later showed that the strategy proposed by Grossman and Zhou (1993) is not always optimal in a discrete-time setting. Moreover, the objective to maximize the long-term growth rate has been criticized because any strategy which coincides with the optimal strategy of Grossman and Zhou (1993) after any fixed time is optimal. Roche (2006) studied the infinite-horizon optimal consumption–investment problem for a power utility subject to the same drawdown constraint. Elie and Touzi (2008) later extended Roche (2006) to a general class of utility functions. Portfolio optimization problems with drawdown constraints are also considered in discrete-time settings (see, e.g., Chekhlov et al., 2005 and Alexander and Baptista, 2006).

In this paper, we consider the optimization problem of minimizing the probability that a significant drawdown occurs over a lifetime investment. Mathematically speaking, our problem

* Corresponding author.
E-mail addresses: xinfu@pit.edu (X. Chen), david.landriault@uwaterloo.ca (D. Landriault), bin.li@uwaterloo.ca (B. Li), d65li@uwaterloo.ca (D. Li).

http://dx.doi.org/10.1016/j.insmatheco.2015.08.007
0167-6687/

is formulated as follows. On a filtered complete probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, we consider a $\boldsymbol{F}$-progressively measurable trading strategy $\pi = \{\pi_t\}_{t\geq 0}$. The associated fund value process is denoted by $W^\pi = \{W_t^\pi\}_{t\geq 0}$ with initial value $W_0 = w > 0$. We define the (floored) running maximum of the fund value at time t by

$$M_t^\pi = \max\Big\{\sup_{0\leq s\leq t} W_s^\pi, m\Big\},$$

with $m \geq w$. Note that the initial values w and m are fixed positive constants, and hence are independent of the trading strategy π. The ratios $(M_t^\pi - W_t^\pi)/M_t^\pi$ and W_t^π/M_t^π are respectively called the *relative drawdown level* and the *relative fund level* at time t. To quantify and measure the drawdown risk, for a fixed significance level $\alpha \in (0, 1)$, we define

$$\tau_\alpha^\pi = \inf\left\{t \geq 0 : M_t^\pi - W_t^\pi > \alpha M_t^\pi\right\},$$

to be the first time the relative drawdown of the fund value W^π exceeds the significance level $100\alpha\%$. Equivalently, the event $(\tau_\alpha^\pi > t)$ for some fixed $t > 0$ implies that the relative drawdown of the fund value in time period $[0, t]$ never exceeds α.

Our main objective is to solve for the optimal trading strategy $\pi^* = \{\pi_t^*\}_{t\geq 0}$ that minimizes the probability that a relative drawdown of size over α occurs before e_λ, the random time of death of a client with constant force of mortality $\lambda > 0$, i.e.,

$$\min_{\pi\in\Pi} \mathbb{P}\left\{\tau_\alpha^\pi < e_\lambda | W_0 = w, M_0 = m\right\}, \tag{1.1}$$

where Π is the set of admissible trading strategies defined as

$$\Pi = \left\{\pi : \pi \text{ is } \boldsymbol{F}\text{-progressively measurable and } \int_0^t \pi_s^2 \mathrm{d}s < \infty \text{ for any } t \geq 0\right\}. \tag{1.2}$$

Thus, e_λ is an $\mathcal{F}$-measurable exponentially distributed random variable with mean $1/\lambda > 0$, independent of the fund value process by assumption. For ease of notation, we denote the objective function in (1.1) as

$$\psi(w, m) = \min_{\pi\in\Pi} \mathbb{P}^{w,m}\left\{\tau_\alpha^\pi < e_\lambda\right\} = \min_{\pi\in\Pi} \mathbb{E}^{w,m}[e^{-\lambda\tau_\alpha^\pi}], \tag{1.3}$$

where the last equation is due to the independence of τ_α^π and e_λ. Here and henceforth, we write $\mathbb{E}^{w,m}[\,\cdot\,] = \mathbb{E}[\,\cdot\,|W_0 = w, M_0 = m]$.

The present work proposes to minimize the lifetime drawdown probability rather than impose a strict drawdown constraint, as is commonly done in the literature. This is because a strict drawdown constraint may not be attainable in some contexts (such as those discussed in Sections 2 and 3). As for other similar optimization problems (e.g., the minimum lifetime ruin probability (MLRP) of Young (2004), Bayraktar and Young (2007), Bayraktar and Zhang (2015) and references therein), we consider the drawdown probability over the lifetime of a client with a constant force of mortality. For the treatment of non-constant forces of mortality, one may adopt the approximative scheme of Moore and Young (2006). Finally, the solution of our resulting Hamilton–Jacobi–Bellman (HJB) equation does not possess a simple form, which makes its solution form difficult to guess. Instead, we decompose the HJB equation into two nonlinear equations of first order which are solved consecutively.

We point out that a recent paper by Angoshtari et al. (2015b) also studied the minimum drawdown probability problem but over an infinite-time horizon. By utilizing the results of Bäuerle and Bayraktar (2014), the authors found that the minimum infinite-time drawdown probability (MIDP) strategy coincides with the minimum infinite-time ruin probability (MIRP) strategy which consists in maximizing the ratio of the drift of the value process to its volatility squared. However, we point out that such a relationship does not hold for a random (or finite) maturity setting such as in (1.3) as the time-change arguments in Bäuerle and Bayraktar (2014) do not apply.

We will study the MLDP problem (1.3) by examining two different market models: a market with two risky assets and a market with a risk-free asset and a risky asset. We point out that several conclusions and implications of market model I are determinant to the subsequent analysis of market model II. Also, the following financial implications hold for both market models: (1) it is optimal to minimize the portfolio's variance when the fund value is at its historic high-water mark; (2) when the fund value drops, it is optimal to increase the proportion invested in the asset with a higher instantaneous rate of return (even though its volatility may also be higher). It follows that the instantaneous return rate of the MLDP strategy is never less than the return rate of the minimum variance (MV) strategy, which supports the practical use of drawdown-based performance measures.

The rest of the paper is organized as follows. The parallel Sections 2 and 3 are respectively devoted to the market models I and II. For each model, we provide a verification theorem, obtain closed-form expressions for the MLDP and its corresponding optimal trading strategy, as well as prove some properties of the optimal trading strategy. At the end of each section, we complement the analysis with some numerical examples.

2. Market model I

In this section, we study problem (1.3) under the market model consisting of two risky assets. We assume that the ith risky asset $(i = 1, 2)$ is governed by a geometric Brownian motion with dynamics

$$\mathrm{d}S_t^{(i)} = \mu_i S_t^{(i)}\mathrm{d}t + \sigma_i S_t^{(i)}\mathrm{d}B_t^{(i)}, \quad S_0^{(i)} > 0,$$

where $\mu_i \in \mathbb{R}, \sigma_i > 0$, and $\{B_t^{(i)}\}_{t\geq 0}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$. In addition, $\{B_t^{(1)}\}_{t\geq 0}$ and $\{B_t^{(2)}\}_{t\geq 0}$ are assumed to be dependent with

$$\mathrm{d}B_t^{(1)}\mathrm{d}B_t^{(2)} = \rho\mathrm{d}t,$$

where $\rho \in (-1, 1)$ is the correlation coefficient. To avoid triviality, we exclude cases where the two assets are either perfectly positively or negatively correlated. Given a trading strategy $\pi \in \Pi$ defined in (1.2), where π_t represents *the fraction of wealth invested in Asset* 1 at time t, the evolution of the fund value process W^π is governed by

$$\begin{aligned}\mathrm{d}W_t^\pi &= \pi_t W_t^\pi \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} + (1-\pi_t)W_t^\pi \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} \\ &= (\pi_t\mu_1 + (1-\pi_t)\,\mu_2)\,W_t^\pi\mathrm{d}t + \pi_t W_t^\pi\sigma_1\mathrm{d}B_t^{(1)} \\ &\quad + (1-\pi_t)\,W_t^\pi\sigma_2\mathrm{d}B_t^{(2)}\end{aligned} \tag{2.1}$$

with initial value $W_0 = w > 0$.

2.1. Verification theorem

We first prove a verification theorem for the MLDP. By a dimension reduction, the MLDP problem (1.3) will later be reduced to a one-dimensional stochastic control problem.

Let

$$D = \left\{(w, m) \in \mathbb{R}^2 : m(1-\alpha) \leq w \leq m \text{ and } m > 0\right\},$$

and define a differential operator $\mathcal{L}^\beta$ $(\beta \in \mathbb{R})$ as

$$\mathcal{L}^\beta f = (\beta\mu_1 + (1-\beta)\,\mu_2)\,xf_x + \frac{1}{2}\left(\beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2 + 2\rho\beta\,(1-\beta)\,\sigma_1\sigma_2\right)x^2 f_{xx} - \lambda f,$$

where f is a twice-differentiable function in x with $f_x := \frac{\partial f}{\partial x}$ and $f_{xx} := \frac{\partial^2 f}{\partial x^2}$.

Theorem 2.1. *Suppose that $f : D \to (0, 1]$ satisfies the following conditions:*

(1) *For any fixed $m > 0$, $f(\cdot, m) \in C^2([m(1-\alpha), m])$ is strictly decreasing and strictly convex;*
(2) *For any fixed $w > 0$, $f(w, \cdot) \in C^1([w, w/(1-\alpha)])$ is strictly increasing;*
(3) *For any fixed $m > 0$ and $\beta \in \mathbb{R}$, $\mathcal{L}^\beta f(\cdot, m) \geq 0$ for $w \in [m(1-\alpha), m]$;*
(4) *For any fixed $m > 0$, there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that $\mathcal{L}^{\pi^*} f(\cdot, m) = 0$ for $w \in [m(1-\alpha), m]$;*
(5) *For any $m > 0$, $f(m(1-\alpha), m) = 1$;*
(6) *For any $m > 0$, $f_m(m, m) = 0$.*

Then $f(w, m) = \psi(w, m)$ on D, where $\psi(w, m)$ is the MLDP defined in (1.3), and π^* is the corresponding optimal trading strategy.

Proof. For an admissible trading strategy π satisfying (1.2), we define a sequence of stopping time $\{\gamma_n^\pi\}_{n\in\mathbb{N}}$ with

$$\gamma_n^\pi = \inf\left\{t \geq 0 : \int_0^t \pi_s^2 ds \geq n\right\}.$$

By applying Itô's formula to the process $e^{-\lambda t} f(W_t^\pi, M_t^\pi)$ for $t \in [0, \tau_{\alpha,n}^\pi]$, where $\tau_{\alpha,n}^\pi := \tau_\alpha^\pi \wedge \gamma_n^\pi$, and then using (2.1), we arrive at

$$\begin{aligned}
&e^{-\lambda \tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi) - f(w, m) \\
&= -\lambda \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f(W_t^\pi, M_t^\pi) dt \\
&\quad + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi) dW_t^\pi \\
&\quad + \frac{1}{2} \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_{ww}(W_t^\pi, M_t^\pi) (dW_t^\pi)^2 \\
&\quad + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_m(W_t^\pi, M_t^\pi) dM_t^\pi \\
&= \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} \mathcal{L}^\pi f(W_t^\pi, M_t^\pi) dt \\
&\quad + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi) \pi_t W_t^\pi \sigma_1 dB_t^{(1)} \\
&\quad + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi) (1-\pi_t) W_t^\pi \sigma_2 dB_t^{(2)}.
\end{aligned} \tag{2.2}$$

Note that the operator $\mathcal{L}^\pi f(\cdot, \cdot)$ is applied on the argument w of f in (2.2). Also, the passage from the first to the second equality in (2.2) was made possible given that $f_m(W_t^\pi, M_t^\pi) dM_t^\pi = 0$ a.s. This is because either $dM_t^\pi = 0$ when $W_t^\pi < M_t^\pi$ or $f_m(W_t^\pi, M_t^\pi) = 0$ when $W_t^\pi = M_t^\pi$ by condition (6). Taking the conditional expectation $\mathbb{E}^{w,m}[\cdot]$ on both sides of (2.2) and invoking condition (3), we obtain

$$\mathbb{E}^{w,m}\left[e^{-\lambda \tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi)\right] \geq f(w, m), \tag{2.3}$$

for all $\pi \in \Pi$. Since f is assumed to be bounded, by the dominated convergence theorem and condition (5), it follows that

$$\mathbb{E}^{w,m}\left[e^{-\lambda \tau_\alpha^\pi}\right] \geq f(w, m), \tag{2.4}$$

for all $\pi \in \Pi$. Further, by condition (4), there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that the equality holds in (2.4). In other words, we deduce that

$$f(w, m) = \psi(w, m) = \inf_{\pi \in \Pi} \mathbb{E}^{w,m}\left[e^{-\lambda \tau_\alpha^\pi}\right] = \mathbb{E}^{w,m}[e^{-\lambda \tau_\alpha^{\pi^*}}],$$

which completes the proof. ■

Let f be the function satisfying all the conditions of Theorem 2.1. It is not difficult to see that $f(cw, cm) = f(w, m)$ for any constant $c > 0$. This scaling relation implies that we can reduce the dimension of f by considering

$$f(w, m) = f\left(\frac{w}{m}, 1\right) := g\left(\frac{w}{m}\right), \quad 1-\alpha \leq \frac{w}{m} \leq 1, \tag{2.5}$$

where the ratio w/m is the relative fund level. Using the change of variable formulas $f_w = \frac{1}{m} g'$, $f_{ww} = \frac{1}{m^2} g''$, and $f_m = -\frac{w}{m^2} g'$, we immediately obtain the following corollary from Theorem 2.1.

Corollary 2.1. *Suppose that $g : [1-\alpha, 1] \to (0, 1]$ satisfies the following conditions:*

(1) *$g(\cdot) \in C^2([1-\alpha, 1])$ is strictly decreasing and strictly convex;*
(2) *$\mathcal{L}^\beta g(z) \geq 0$ for any $\beta \in \mathbb{R}$ and $z \in [1-\alpha, 1]$;*
(3) *There exists an admissible strategy $\pi^* : [1-\alpha, 1] \to \mathbb{R}$ such that $\mathcal{L}^{\pi^*} g(z) = 0$ for $z \in [1-\alpha, 1]$;*
(4) *$g(1-\alpha) = 1$;*
(5) *$g'(1) = 0$.*

Then $g(z) = \phi(z) := \inf_{\pi \in \Pi} \mathbb{E}^{w,m}\left[e^{-\lambda \tau_\alpha^\pi}\right]$ for $z = \frac{w}{m} \in [1-\alpha, 1]$, and π^* is the corresponding optimal trading strategy.

2.2. MLDP and optimal trading strategy

In this section, we aim to solve for the MLDP $\phi(\cdot)$ and the corresponding optimal trading strategy π^*. By conditions (2) and (3) of Corollary 2.1, we have

$$\inf_{\beta \in \mathbb{R}} \left\{\mathcal{L}^\beta g(z)\right\} = 0, \quad z \in [1-\alpha, 1]. \tag{2.6}$$

By the first-order condition of Eq. (2.6), the minimizer is given in the feedback form

$$\begin{aligned}
\pi^*(z) = &\frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\
&- \frac{(\mu_1 - \mu_2) g'(z)}{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) z g''(z)}, \quad z \in [1-\alpha, 1].
\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) followed by algebraic manipulations, we obtain the nonlinear equation

$$\frac{A}{2} z^2 g'' - \frac{B}{2} \frac{(g')^2}{g''} - Czg' - \lambda g = 0, \quad z \in [1-\alpha, 1], \tag{2.8}$$

where $A := \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} > 0$, $B := \frac{(\mu_2-\mu_1)^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} \geq 0$, and $C := \frac{(\mu_2-\mu_1)(\sigma_2^2-\rho\sigma_1\sigma_2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} - \mu_2$.

Theorem 2.2. *Under market model I, the MLDP and its corresponding optimal trading strategy are respectively given by*

$$\phi(z) = \exp\left(-A \int_{h^{-1}(z)}^{h^{-1}(1-\alpha)} \frac{x}{k(x)} dx\right), \tag{2.9}$$

and

$$\begin{aligned}
\pi^*(z) = &\frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} - \frac{\mu_1 - \mu_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\
&\times \frac{Ah^{-1}(z)}{A(h^{-1}(z))^2 + k(h^{-1}(z)) - Ah^{-1}(z)},
\end{aligned} \tag{2.10}$$

for $z \in [1-\alpha, 1]$, where

$$k(x) := \lambda + (A+C)x - Ax^2 + \sqrt{(\lambda + Cx)^2 + ABx^2},$$

and $h(v) := \exp\left(-\int_v^0 \frac{A}{k(x)} dx\right)$ for $v \in (v_*, 0]$ with $v_* := \sup\{x < 0 : k(x) = 0\}$. Furthermore, $\phi(\cdot), \pi^*(\cdot) \in C^\infty([1-\alpha, 1])$.

Proof. In light of Eq. (2.8) and Corollary 2.1, we consider the following non-linear equation

$$\begin{cases} \frac{A}{2}z^2G'' - \frac{B}{2}\frac{(G')^2}{G''} - CzG' - \lambda G = 0, \quad z \in (0,1], \\ G(1) = 1, \\ G'(1) = 0, \\ G''(z) > 0, \quad z \in (0,1]. \end{cases} \tag{2.11}$$

Next, we show that (2.11) admits a unique solution G and furthermore, $G \in C^\infty((0,1])$. The advantage to consider the function G is that it is independent of α.

Define two auxiliary functions

$$u(z) := \frac{zG'(z)}{z^2G''(z)} \quad \text{and} \quad v(z) := \frac{zG'(z)}{G(z)}, \quad z \in (0,1]. \tag{2.12}$$

Since $G'(1) = 0$ and $G''(z) > 0$ for $z \in (0,1)$, we have $G'(z) < 0$ for $z \in (0,1)$, which further implies that both $u(z)$ and $v(z)$ are strictly negative functions on $(0,1)$. Dividing both sides of the first equation of (2.11) by $zG'(z)$, we obtain

$$\frac{A}{2u} - \frac{B}{2}u - C - \frac{\lambda}{v} = 0, \quad z \in (0,1). \tag{2.13}$$

Solving the algebraic equation (2.13) with $u(z) < 0$ and $v(z) < 0$, we have

$$\frac{1}{u} = \frac{\lambda + Cv + \sqrt{(\lambda + Cv)^2 + ABv^2}}{Av}. \tag{2.14}$$

Differentiating v in z from the second relation of (2.12) and subsequently using (2.14), it follows that

$$zv' = z\frac{(zG'' + G')G - z(G')^2}{G^2} = \frac{v}{u} + v - v^2 = \frac{1}{A}k(v), \tag{2.15}$$

where $k(x) := \lambda + (A+C)x - Ax^2 + \sqrt{(\lambda + Cx)^2 + ABx^2}$ for $x \in \mathbb{R}$. Since $k(\cdot) \in C^\infty(\mathbb{R})$, $k(0) = 2\lambda > 0$ and $\lim_{v\downarrow -\infty} k(v) = -\infty$, there exists some point v_* such that

$$v_* := \sup\{x < 0 : k(x) = 0\} > -\infty.$$

Furthermore, by $v'(z)z'(v) = 1$, Eq. (2.15) becomes

$$z'(v) = \frac{A}{k(v)}z(v),$$

which admits a unique solution

$$z(v) = h(v) := \exp\left(-\int_v^0 \frac{A}{k(x)}\mathrm{d}x\right), \quad v \in (v_*, 0], \tag{2.16}$$

under the boundary condition $z(0) = 1$. Moreover, by $v = h^{-1}(z)$, it can be shown that $v'(z) > 0$ for $z \in (0,1]$, $v(1) = 0$, and $\lim_{z\downarrow 0} v(z) = v_*$. Now, letting $H(v) := G(h(v)) = G(z)$, it follows that

$$\begin{cases} \frac{\mathrm{d}H}{\mathrm{d}v} = \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{vG(z)}{z}\frac{Az}{k(v)} = \frac{Av}{k(v)}H(v), \quad v \in (v_*, 0], \\ H(0) = 1. \end{cases} \tag{2.17}$$

The solution to (2.17) is given by

$$H(v) = \exp\left(-A\int_v^0 \frac{x}{k(x)}\mathrm{d}x\right), \quad v \in (v_*, 0],$$

or equivalently, we have shown that (2.11) admits a unique solution

$$G(z) = \exp\left(-A\int_{h^{-1}(z)}^0 \frac{x}{k(x)}\mathrm{d}x\right) \in C^\infty((0,1]).$$

Letting

$$g(z) := \frac{G(z)}{G(1-\alpha)} = \exp\left(-A\int_{h^{-1}(z)}^{h^{-1}(1-\alpha)} \frac{x}{k(x)}\mathrm{d}x\right), \tag{2.18}$$

for $z \in [1-\alpha, 1]$, it is straightforward to verify that $g(\cdot)$ satisfies all the conditions of Corollary 2.1. Hence, we conclude that $g(z) = \phi(z)$ for $z \in [1-\alpha, 1]$ which proves (2.9). Finally, differentiating (2.18) yields

$$\begin{aligned} g'(z) &= g(z)\frac{h^{-1}(z)}{z} \\ g''(z) &= g(z)\frac{A(h^{-1}(z))^2 + k(h^{-1}(z)) - Ah^{-1}(z)}{Az^2}. \end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.7) leads to the optimal strategy $\pi^*(\cdot) \in C^\infty([1-\alpha, 1])$ given in (2.10). This completes the proof. ■

We have some interesting observations to make of the MLDP strategy (2.10), which relate to the classical MV strategy.

1. Suppose that $\mu_1 = \mu_2$, the optimal strategy (2.10) reduces to a constant proportional strategy

$$\hat{\pi} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{2.20}$$

It is easy to see from (2.1) that

$$\begin{aligned} &\min_{\pi\in\Pi} \mathrm{Var}\left[\log W_t^\pi\right] \\ &= \min_{\pi\in\Pi} \int_0^t \left(\pi_s^2\sigma_1^2 + (1-\pi_s)^2\sigma_2^2 + 2\pi_s(1-\pi_s)\sigma_1\sigma_2\rho\right)\mathrm{d}s \\ &= \mathrm{Var}\left[\log W_t^{\hat{\pi}}\right]. \end{aligned}$$

Hence, when $\mu_1 = \mu_2$, the MLDP strategy (2.10) coincides with the MV strategy (2.20).

2. Even if $\mu_1 \neq \mu_2$, we can see from (2.7) and condition (5) of Corollary 2.1 that

$$\pi^*(1) = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \hat{\pi}. \tag{2.21}$$

Relation (2.21) implies that, when the fund value is at its running maximum, the MLDP strategy is identical to the MV strategy.

3. By (2.1), we denote by $\mu^\pi := \mu_1\pi_t + \mu_2(1-\pi_t)$ the instantaneous return rate of the portfolio at time t under strategy π. By (2.7) and the fact that the MLDP ϕ is decreasing and convex, we have

$$\begin{aligned} \mu^{\pi^*} - \mu^{\hat{\pi}} &= (\mu_2 - \mu_1)(\hat{\pi} - \pi^*(z)) \\ &= \frac{-(\mu_2 - \mu_1)^2\phi'(z)}{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)z\phi''(z)} \geq 0, \end{aligned} \tag{2.22}$$

for all $z \in [1-\alpha, 1]$. In other words, the instantaneous return rate of the MLDP portfolio is never less than the return rate of the MV portfolio. This result supports the practical use of drawdown-based performance measures in which the role of volatility is replaced by drawdown. Intuitively speaking, this conclusion is consistent with the fact that volatility-based measures penalize for both upside and downside movements of the fund process while drawdown-based measures only penalize for downside movements.

This leads to a natural question: *How does the MLDP strategy behave when the fund value is away from a historic high-water mark?* We find that, as shown in the next proposition, it is optimal to increase the proportion invested in the asset with a higher instantaneous rate of return as the portfolio's relative drawdown level increases (even though this may increase the portfolio's variance).

Proposition 2.1. *Suppose that $\mu_1 \neq \mu_2$. We have*

$$(\mu_1-\mu_2)\frac{d\pi^*}{dz}<0, \quad z\in[1-\alpha,1].$$

Proof. By (2.7) and the definitions of $u(\cdot)$ and $v(\cdot)$ in (2.12), it follows that the optimal strategy can be rewritten as

$$\pi^*(z)=\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}+\frac{\mu_2-\mu_1}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}u(z),$$

which implies that

$$(\mu_1-\mu_2)\frac{d\pi^*}{dz}=-\frac{(\mu_2-\mu_1)^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\frac{du}{dz}.$$

By (2.15), we have

$$\frac{dv}{dz}=\frac{1}{Az}k(v)>0. \tag{2.23}$$

On the other hand, solving v from (2.13), we obtain $v=\frac{2\lambda u}{A-Bu^2-2Cu}$ which yields

$$\frac{dv}{du}=\frac{2\lambda A+2\lambda Bu^2}{(A-Bu^2-2Cu)^2}>0. \tag{2.24}$$

Using (2.23), (2.24), and $\frac{dv}{dz}=\frac{dv}{du}\frac{du}{dz}$, we conclude that $\frac{du}{dz}>0$. This ends the proof. ■

Remark 2.1. As for the market model II of Section 3, a proportional management fee of the fund with rate $\eta \in (0,1)$ can easily be incorporated into the above analysis. Then the dynamics of the fund value process (2.1) becomes

$$dW_t^\pi=(\pi_t\mu_1+(1-\pi_t)\mu_2-\eta)W_t^\pi dt+\pi_tW_t^\pi\sigma_1dB_t^{(1)} + (1-\pi_t)W_t^\pi\sigma_2dB_t^{(2)}.$$

It is clear that the formulas of the MLDP (2.9) and the optimal trading strategy (2.10) still hold by simply replacing μ_1 and μ_2 by $\mu_1-\eta$ and $\mu_2-\eta$, respectively.

2.3. Numerical examples

In this section, we provide some numerical examples to illustrate the main results of Section 2. We consider a relative drawdown level of $\alpha = 0.2$ and an investor's expected future lifetime of 20 years (i.e. $\lambda = 0.05$).

In Fig. 1, we set $\mu_1=0.1$, $\mu_2=0.15$, $\sigma_1=0.125$, $\sigma_2=0.15$ and $\rho=0.2$. We first examine the diversification benefit by comparing in Fig. 1 (left plot) the MLDP to the drawdown probability for investment in Asset 1 or 2 only. The drawdown probabilities for geometric Brownian motions were first derived by Taylor (1975) and can also be found more recently in, e.g., Theorem 1 of Avram et al. (2004). We recall this result here. For $S := \{S_t\}_{t\geq 0}$ a geometric Brownian motion with dynamics

$$dS_t=\mu S_tdt+\sigma S_tdB_t, \quad S_0:=w>0,$$

where $\mu\in\mathbb{R}$, $\sigma>0$, and $\{B_t\}_{t\geq0}$ is a standard Brownian motion, we define the first time the relative drawdown of S exceeds level α as

$$\tau_\alpha:=\inf\{t\geq 0: M_t-S_t>\alpha M_t\},$$

where $M_t := \max\{\sup_{0\leq u\leq t}S_u, m\}$ and $m\geq w$. Then,

$$\mathbb{P}^z\{\tau_\alpha<e_\lambda\}:=\mathbb{P}^{w,m}\{\tau_\alpha<e_\lambda\}=\frac{\beta^+z^{\beta^-}-\beta^-z^{\beta^+}}{\beta^+(1-\alpha)^{\beta^-}-\beta^-(1-\alpha)^{\beta^+}},$$

where $z:=\frac{w}{m}\in[1-\alpha,1]$ and $\beta^\pm=\frac{-\mu+\sigma^2/2\pm\sqrt{(\mu-\sigma^2/2)^2+2\lambda\sigma^2}}{\sigma^2}$.

We observe that the drawdown probabilities are considerably lower under the MLDP strategy (than investing in either Asset 1 or 2). In Fig. 1 (right plot), we provide the curve of the corresponding MLDP strategy as a function of the relative fund level $z = w/m$. Notice that π is increasing in z, which is consistent with Proposition 2.1 as $\mu_1=0.1<0.15=\mu_2$.

Next, we are interested in studying the impact of the correlation coefficient ρ of the two risky assets on the MLDP and the corresponding optimal trading strategy. We set $\mu_1=0.05$, $\mu_2=0.3$, $\sigma_1=0.2$ and $\sigma_2=0.36$ to produce the numerical values of Fig. 2. We find that neither of these two quantities is necessarily monotone in ρ. In the left plot, we observe that the MLDPs are first increasing and then decreasing in ρ for any $z\in[1-\alpha,1]$. This shows that a selection of highly correlated assets (ρ close to -1 or 1 in this example) in a portfolio can help reduce the MLDP of the portfolio. In the right plot, we can see that the impact of ρ on the optimal strategy $\pi^*(z)$ is even more complex. However, when $z=1$, we find that $\pi^*(1)$ is increasing in ρ. This observation can easily be verified from (2.21) as

$$(\sigma_2-\sigma_1)\frac{\partial\pi^*(1)}{\partial\rho}=\frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)^2(\sigma_2+\sigma_1)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)^2}\geq 0.$$

Note that we choose $\sigma_2=0.36>0.2=\sigma_1$.

3. Market model II

In this section, we examine the second market model consisting of a risk-free asset with constant interest rate $r>0$ and a risky asset governed by a geometric Brownian motion with dynamics

$$dS_t=\mu S_tdt+\sigma S_tdB_t, \quad S_0>0,$$

where $\mu\in\mathbb{R}$, $\sigma>0$, and $\{B_t\}_{t\geq0}$ is a standard Brownian motion defined on $(\Omega,\mathcal{F},\boldsymbol{F},\mathbb{P})$. To avoid triviality, a proportional management fee with rate $r<\eta<1$ is continuously deducted from the fund. Therefore, for an admissible strategy $\pi\in\Pi$ representing *the fraction of wealth invested in the risky asset*, the dynamics of the fund value process W^π is then given by

$$\begin{aligned}dW_t^\pi&=\pi_tW_t^\pi\frac{dS_t}{S_t}+(1-\pi_t)rW_t^\pi dt-\eta W_t^\pi dt\\&=(\pi_t(\mu-r)+r-\eta)W_t^\pi dt+\pi_tW_t^\pi\sigma dB_t,\end{aligned} \tag{3.1}$$

with initial value $W_0=w>0$.

At first glance, one may view market model II as a limiting case of market model I by letting $\sigma_2\to0$ and $\mu_2=r$. However, as will be shown, the treatment of these two models and the associated HJB equations are structurally different. First, it is not obvious to find the limit of the MLDP (2.9) and the optimal strategy (2.10) by letting $\sigma_2\to0$ given that the form of h^{-1} is not fully explicit. Also, even if an explicit limit exists, the continuity of the MLDP and the optimal strategy w.r.t. σ_2 at 0+ needs to be justified. Second, a major difference in the analysis of market model II is that we shall first narrow down the candidate pool of the optimal trading strategy. Interestingly, this intuition is based on some observations we made under market model I.

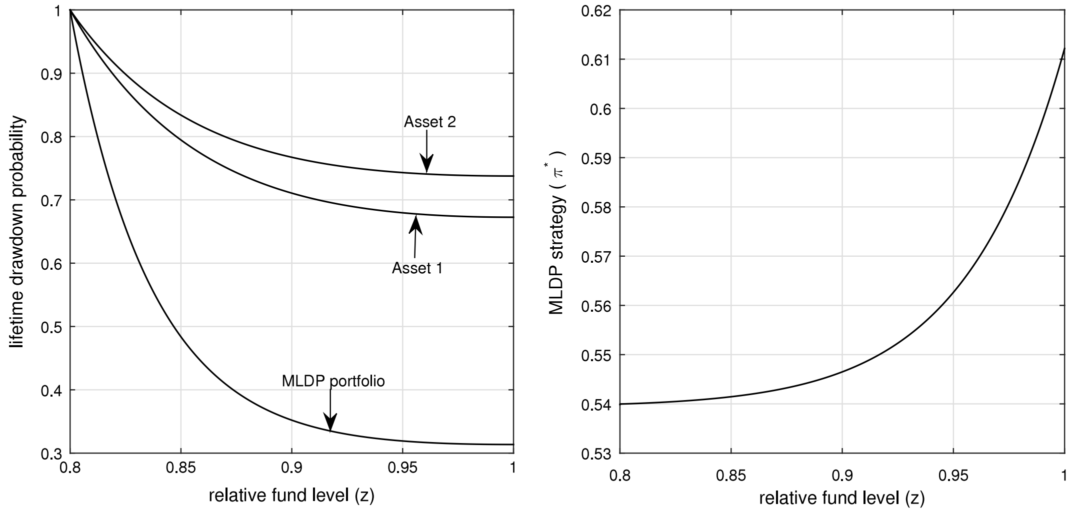

Fig. 1. Lifetime drawdown probabilities (left) and the MLDP trading strategy (right).

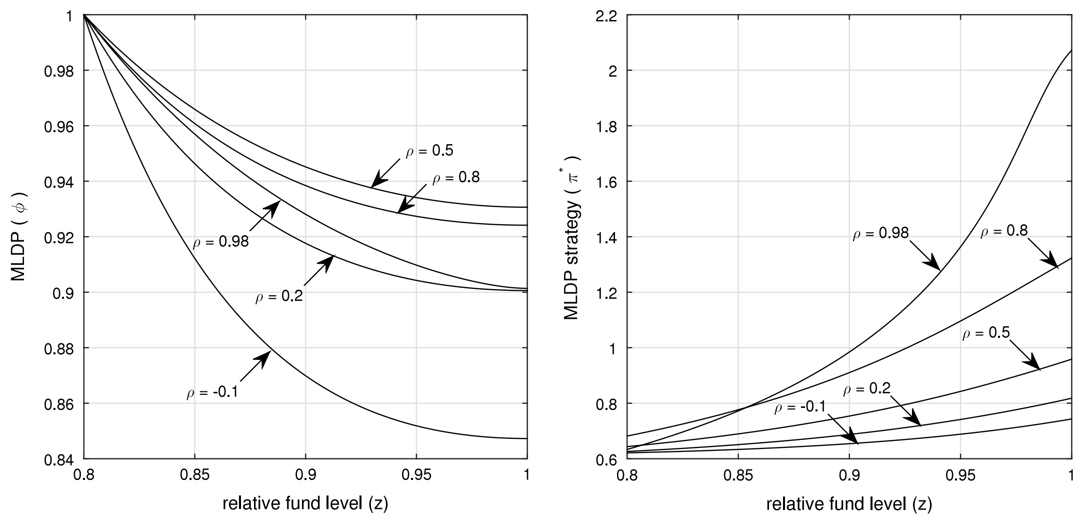

Fig. 2. Impact of ρ on the MLDP (left) and the MLDP trading strategy (right).

3.1. Verification theorem

We define a differential operator $\tilde{\mathcal{L}}^\beta$ $(\beta \in \mathbb{R})$ as

$$\tilde{\mathcal{L}}^\beta f = (\beta(\mu - r) + r - \eta)\, x f_x + \frac{1}{2}\beta^2\sigma^2 x^2 f_{xx} - \lambda f,$$

where f is a twice-differentiable function in x. Then we decompose the admissible set of trading strategies Π as

$$\Pi = \Pi_0 \cup \Pi_1,$$

where $\Pi_0 = \left\{\pi \in \Pi : \pi_t = 0 \text{ a.s. on } (M_t^\pi = W_t^\pi)\right\}$ and $\Pi_1 = \Pi \setminus \Pi_0$. Therefore, Π_0 is the set of admissible strategies which has no risky investment whenever the associated fund value is at its running maximum. For any $\pi \in \Pi_0$, due to the absence of diffusion component when the fund value process reaches its running maximum and the negative drift $r - \eta$ of the value process at that moment, a new running maximum of the associated value process W^π will never occur, i.e.,

$$dM_t^\pi = 0 \quad \text{a.s. for any } \pi \in \Pi_0 \text{ and } t > 0. \tag{3.2}$$

A verification theorem for the MLDP and the optimal trading strategy of market model II is given below.

Theorem 3.1. *Suppose that $f : D \to (0, 1]$ satisfies the following conditions:*

(1) *For any fixed $m > 0$, $f(\cdot, m) \in C^2([m(1-\alpha), m])$ is strictly decreasing and strictly convex;*
(2) *For any fixed $w > 0$, $f(w, \cdot) \in C^1([w, w/(1-\alpha)])$ is strictly increasing;*
(3) *For any fixed $m > 0$ and $\beta \in \mathbb{R}$, $\mathcal{L}^\beta f(\cdot, m) \geq 0$ for $w \in [m(1-\alpha), m]$;*
(4) *For any fixed $m > 0$, there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that $\pi^* \in \Pi_0$ and $\mathcal{L}^{\pi^*} f(\cdot, m) = 0$ for $w \in [m(1-\alpha), m]$;*
(5) *For any $m > 0$, $f(m(1-\alpha), m) = 1$.*

Then $f(w, m) = \psi(w, m)$ on D, where $\psi(w, m)$ is the MLDP defined in (1.3), and π^* is the corresponding optimal trading strategy.

Proof. Suppose that $f : D \to (0, 1]$ satisfies conditions (1)–(5) of Theorem 3.1 and $\pi^* \in \Pi_0$ is an admissible strategy satisfying condition (4). By condition (2) and the fact that M_t^π is a non-decreasing process, we know that $f_m(W_t^\pi, M_t^\pi) dM_t^\pi \geq 0$ a.s. Along the same lines as in the proof of Theorem 2.1, one can see that (2.3) still holds for all $\pi \in \Pi$. Moreover, by $\pi^* \in \Pi_0$ and (3.2), the equality holds in (2.3) for π^*. Using the same arguments as the rest of the proof of Theorem 2.1, we complete the proof of Theorem 3.1. ■

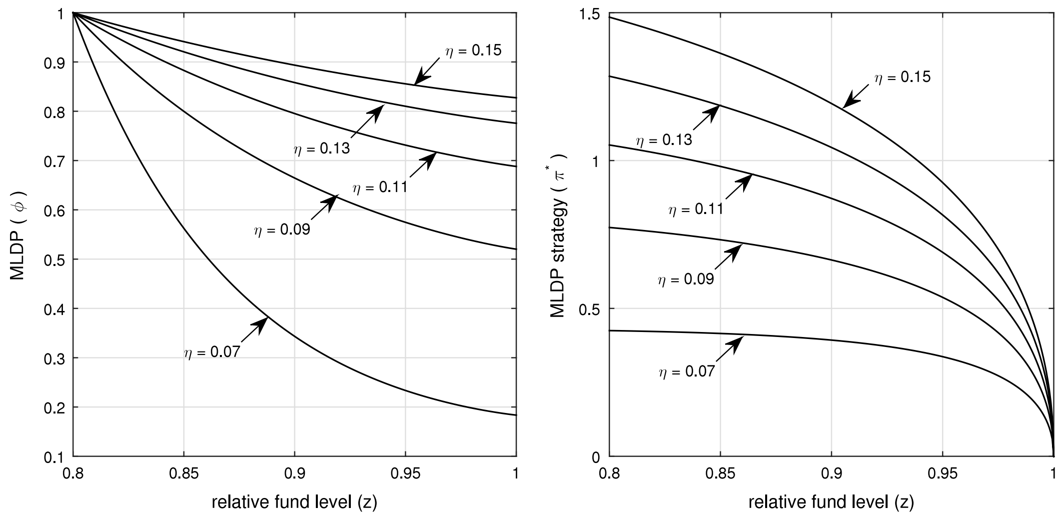

Fig. 3. Impact of η on the MLDP (left) and the MLDP trading strategy (right).

Similar as (2.5), the dimension of f in Theorem 3.1 can be reduced by considering

$$f(w,m) = f\left(\frac{w}{m},1\right) := g\left(\frac{w}{m}\right), \quad 1-\alpha \le \frac{w}{m} \le 1,$$

which immediately yields the following corollary.

Corollary 3.1. *Suppose that* $g : [1-\alpha, 1] \to (0, 1]$ *satisfies the following conditions:*

(1) $g(\cdot) \in C^2([1-\alpha, 1])$ *is strictly decreasing and strictly convex;*
(2) $\mathcal{L}^{\beta} g(z) \ge 0$ *for any* $\beta \in \mathbb{R}$ *and* $z \in [1-\alpha, 1]$*;*
(3) *There exists an admissible strategy* $\pi^* : [1-\alpha, 1] \to \mathbb{R}$ *such that* $\pi^* \in \Pi_0$ *and* $\tilde{\mathcal{L}}^{\pi^*} g(z) = 0$ *for* $z \in [1-\alpha, 1]$*;*
(4) $g(1-\alpha) = 1$*;*
(5) $\lim_{z\uparrow 1} g''(1) = \infty$ *if* $\mu \ne r$*.*

Then $g(z) = \phi(z) := \inf_{\pi\in\Pi} \mathbb{E}^{w,m}\left[e^{-\lambda\tau_\alpha^\pi}\right]$ *for* $z = \frac{w}{m} \in [1-\alpha, 1]$*, and* π^* *is the corresponding optimal trading strategy.*

In comparison to Corollary 2.1, the presence of the two new conditions $\pi^* \in \Pi_0$ and $\lim_{z\uparrow 1} g''(z) = \infty$ if $\mu \ne r$ may appear abrupt. However, both conditions are in agreement with conclusions reached under market model I. First, the condition $\pi^* \in \Pi_0$ is consistent with the conclusion that the MLDP strategy is identical to the MV strategy when the portfolio value is at its running maximum. On the other hand, one can argue $\pi^* \notin \Pi_1$. Otherwise, by (3.2), we should have $\mathbb{P}\{M_t^{\pi^*} > m \text{ for some } t > 0\} > 0$, which further implies that $g'(1) = 0$ from the proof of Theorem 2.2. Moreover, by the first-order condition, we have

$$\pi^*(z) = \begin{cases} -\dfrac{\mu-r}{\sigma^2}\dfrac{g'(z)}{zg''(z)}, & \text{if } \mu \ne r, \\ 0, & \text{if } \mu = r. \end{cases} \tag{3.3}$$

Substituting (3.3) into the equation $\tilde{\mathcal{L}}^{\pi^*} g(z) = 0$, we obtain the nonlinear equation

$$\frac{(\mu-r)^2}{2\sigma^2}\frac{(g')^2}{g''} + (\eta - r)zg' + \lambda g = 0, \quad z \in [1-\alpha, 1]. \tag{3.4}$$

However, by the conditions of Corollary 3.1, we have

$$\frac{(\mu-r)^2}{2\sigma^2}\frac{(g'(1))^2}{g''(1)} + (\eta - r)zg'(1) + \lambda g(1) \ge \lambda g(1) > 0,$$

which contradicts (3.4). Therefore, we deduce $\pi^* \in \Pi_0$ and $g'(1) \ne 0$, which further implies that $\lim_{z\uparrow 1} g''(z) = \infty$ if $\mu \ne r$ by (3.3).

Remark 3.1. Under market model II, we have $\pi^* \in \Pi_0$, which implies that the fund value process will never reach a new running maximum by (3.2). Intuitively speaking, this conclusion is consistent with the fact that the objective function of the MLDP problem (1.3) only penalizes downside risk and does not offer incentives to reach a new running maximum. As shown in Proposition 3.1 and Fig. 3 later, the MLDP strategy becomes more conservative as the fund value increases. As such, since $\eta > r$, when the fund value recovers its running maximum, it is preferable to invest all in the risk-free asset (even if the instantaneous return rate of the portfolio is negative) rather than "gamble" by investing a nonzero proportion of the portfolio in the risky asset and increase the exposure to substantial drawdowns.

3.2. MLDP and optimal trading strategy

By (3.4) and Corollary 3.1, we only need to find a positive, strictly decreasing, strictly convex, and $C^2([1-\alpha, 1])$ solution to the following nonlinear equation

$$\begin{cases} \dfrac{(\mu-r)^2}{2\sigma^2}\dfrac{(g')^2}{g''} + (\eta - r)zg' + \lambda g = 0, \quad z \in [1-\alpha, 1], \\ g(1-\alpha) = 1, \\ \lim\limits_{z\uparrow 1} g''(z) = \infty, \text{ if } \mu \ne r. \end{cases} \tag{3.5}$$

Theorem 3.2. *Under market model* II*, the MLDP and its corresponding optimal trading strategy are respectively given by*

$$\phi(z) = \begin{cases} \exp\left(-\displaystyle\int_{\tilde{h}^{-1}(z)}^{\tilde{h}^{-1}(1-\alpha)} \frac{x}{\tilde{k}(x)} \mathrm{d}x\right), & \text{if } \mu \ne r, \\ \left(\dfrac{1-\alpha}{z}\right)^{\lambda/(\eta - r)}, & \text{if } \mu = r, \end{cases} \tag{3.6}$$

and

$$\pi^*(z) = \begin{cases} \dfrac{2}{\mu - r}\left(\eta - r + \dfrac{\lambda}{\tilde{h}^{-1}(z)}\right), & \text{if } \mu \ne r, \\ 0, & \text{if } \mu = r, \end{cases} \tag{3.7}$$

for $z \in [1-\alpha, 1]$, *where* $\tilde{k}(x) := -\frac{(\mu-r)^2}{2\sigma^2}\frac{x^2}{(\eta-r)x+\lambda} + x - x^2$ *and* $\tilde{h}(v) := \exp\left(-\int_v^{-\lambda/(\eta-r)} \frac{1}{\tilde{k}(x)}\mathrm{d}x\right)$ *for* $v \in (\tilde{v}_*, -\lambda/(\eta-r)]$ *with*

$$\tilde{v}_* = \frac{\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2} - \sqrt{\left(\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2}\right)^2 + 4\lambda(\eta - r)}}{2(\eta - r)}.$$

Furthermore, $\phi(\cdot), \pi^*(\cdot) \in C^\infty([1-\alpha, 1])$.

Proof. For the simple case $\mu = r$, the solution to (3.5) is easily found to be $g(z) = \left(\frac{1-\alpha}{z}\right)^{\lambda/(\eta-r)}$ for $z \in [1-\alpha, 1]$. By Corollary 3.1, one concludes that $g(\cdot) = \phi(\cdot)$.

For the case $\mu \neq r$, similar to the proof of Theorem 2.2, we consider the following equation:

$$\begin{cases} \frac{(\mu-r)^2}{2\sigma^2}\frac{(G')^2}{G''} + (\eta - r)zG' + \lambda G = 0, \quad z \in (0, 1], \\ G(1) = 1, \\ \lim_{z\uparrow 1} G''(z) = \infty, \\ G'(z) < 0, \quad z \in (0, 1], \\ G''(z) > 0, \quad z \in (0, 1]. \end{cases} \tag{3.8}$$

We show that (3.8) admits a unique solution with $G \in C^\infty((0, 1])$. First, substituting the auxiliary functions $u(\cdot)$ and $v(\cdot)$ defined in (2.12) into the first equation of (3.8) yields $\frac{(\mu-r)^2}{2\sigma^2}u = -\frac{\lambda}{v} - (\eta - r)$. This together with (2.15) leads to

$$\begin{aligned} zv' &= \frac{v}{u} + v - v^2 \\ &= -\frac{(\mu-r)^2}{2\sigma^2}\frac{v^2}{(\eta-r)v+\lambda} + v - v^2 := \tilde{k}(v). \end{aligned} \tag{3.9}$$

Note that $\tilde{k}(v) \in C^\infty((-\infty, -\lambda/(\eta-r)))$ with $\lim_{v\uparrow -\lambda/(\eta-r)} \tilde{k}(v) = \infty$ and $\lim_{v\downarrow -\infty} \tilde{k}(v) = -\infty$. Hence, we denote by

$$\begin{aligned} \tilde{v}_* &:= \sup\left\{x < -\lambda/(\eta - r) : \tilde{k}(x) = 0\right\} \\ &= \frac{\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2} - \sqrt{\left(\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2}\right)^2 + 4\lambda(\eta - r)}}{2(\eta - r)}. \end{aligned}$$

By (3.8), it is easy to see that $v(1) = G'(1) = -\lambda/(\eta - r)$. Moreover, by (3.9) and using the relation $z'(v)v'(z) = 1$, we obtain

$$z(v) = \tilde{h}(v) := \exp\left(-\int_v^{-\lambda/(\eta-r)} \frac{1}{\tilde{k}(x)}\mathrm{d}x\right), \tag{3.10}$$

for $v \in (\tilde{v}_*, -\lambda/(\eta - r)]$. Now, by (3.10), let $H(v) := G(\tilde{h}(v)) = G(z)$. It follows from the second relation of (2.12) and (3.9) that $H(v)$ is the solution to the following equation

$$\begin{cases} \frac{\mathrm{d}H}{\mathrm{d}v} = \frac{vG(z)}{z}\frac{z}{\tilde{k}(v)} = \frac{v}{\tilde{k}(v)}H(v), \quad v \in \left(v_*, -\frac{\lambda}{\eta - r}\right], \\ H\left(-\frac{\lambda}{\eta - r}\right) = G(1) = 1. \end{cases}$$

Solving the above initial value problem, we have

$$\begin{aligned} G(z) &= \exp\left(-\int_{v(z)}^{-\lambda/(\eta-r)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right) \\ &= \exp\left(-\int_{\tilde{h}^{-1}(z)}^{-\lambda/(\eta-r)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right) \in C^\infty((0, 1]). \end{aligned}$$

Finally, letting

$$g(z) := \frac{G(z)}{G(1-\alpha)} = \exp\left(-\int_{\tilde{h}^{-1}(z)}^{\tilde{h}^{-1}(1-\alpha)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right), \tag{3.11}$$

for $z \in [1-\alpha, 1]$, it is straightforward to verify that $g(\cdot)$ satisfies all the conditions of Corollary 3.1, which ends the proof of (3.6). By differentiating (3.11) and further using (3.3), we obtain the optimal strategy π^* given in (3.7). ■

The proof of the following proposition is similar to Proposition 2.1, and hence is omitted.

Proposition 3.1. *Under market model* II, *for* $\mu \neq r$, *we have*

$$(\mu - r)\frac{\mathrm{d}\pi^*}{\mathrm{d}z} < 0, \quad z \in [1-\alpha, 1].$$

By Theorem 3.2 and Proposition 3.1, the following implications of market model I also hold under market model II.

1. At high-water mark (i.e. $\pi^* \in \Pi_0$ or equivalently $\pi^*(1) = 0$), the MLDP strategy (3.7) is consistent with the MV strategy.
2. When the drawdown level increases, the MLDP strategy tends to increase the proportion invested in the asset with a higher return rate.
3. Similarly as in (2.22), it is easy to verify that the instantaneous return rate of the MLDP portfolio is never less than the return rate of the MV portfolio.

3.3. Numerical examples

We numerically implement the main results of Section 3 by first conducting a sensitivity analysis on the management fee rate η. For this purpose, we let $\alpha = 0.2$, $\lambda = 0.05$, $\mu = 0.12$, $\sigma = 0.12$ and $r = 0.05$. The numerical values of the MLDPs and the corresponding optimal trading strategies can be found in Fig. 3 for various η values.

For a fixed η, one can see that the MLDP satisfies all the conditions of Corollary 3.1. In particular, we see that $\phi'(1) < 0$, which is different from market model I (condition 5 of Corollary 2.1). For the optimal trading strategy, as $\mu > r$, we find π^* is decreasing in z which is consistent with Proposition 3.1. Moreover, we see that $\pi^*(1) = 0$ which satisfies condition (3) of Corollary 3.1. As for the impact of η, not surprisingly, we find that both the MLDP and the optimal trading strategy are increasing in η, i.e., a high management fee will incur a higher drawdown probability and result in a more aggressive investment strategy.

In Fig. 4, we are interested in comparing the MLDP strategy with the MLRP strategy $\breve{\pi}$ of Young (2004). We recall that the MLRP strategy is a constant proportional strategy given by

$$\breve{\pi} = \frac{\mu - r}{\sigma^2(1 - \tilde{v}_*)}.$$

In Fig. 4, we use the same parameter setting as in Fig. 3 except we choose $\eta = 0.07$, the floored maximum $m = 100$, and the ruin level $w_r = 80$. We see that the MLDP strategy is always more conservative than the MLRP strategy. In fact, with some calculations, one can verify from (3.7) and Proposition 3.1 that

$$\pi^*(z) < \lim_{z\downarrow 0} \pi^*(z) = \frac{2}{\mu - r}\left(\eta - r + \frac{\lambda}{\tilde{v}_*}\right) = \breve{\pi}, \quad 0 < z \leq 1.$$

This relation is also proved in Theorem 3.2 of Angoshtari et al. (2015a). Intuitively, this is because, for any admissible strategy, the first drawdown time of the associated wealth process always occurs before (or equal to) the ruin time. To prevent the occurrence of an earlier stopping time, an investor tends to adopt a more conservative strategy.

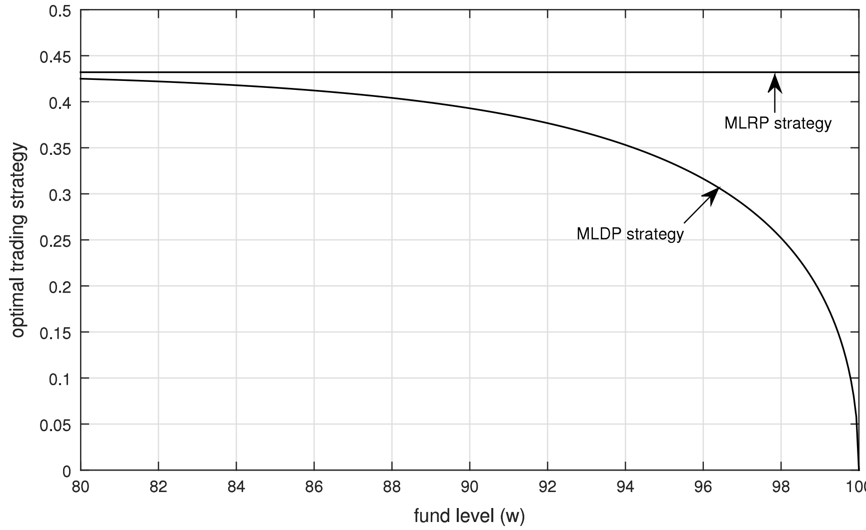

Fig. 4. The MLDP and MLRP trading strategies.

Acknowledgments

The authors would like to thank an anonymous referee for his/her helpful remarks. Support from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by David Landriault and Bin Li (grant numbers 341316 and 05828, respectively). Bin Li also acknowledges the support from a start-up grant (203378) from the University of Waterloo, as is the support from the Canada Research Chair program by David Landriault.

References

Alexander, G.J., Baptista, A.M., 2006. Portfolio selection with a drawdown constraint. J. Banking Finance 30 (11), 3171–3189.

Angoshtari, B., Bayraktar, E., Young, V.R., 2015a. Minimizing the expected lifetime spent in drawdown under proportional consumption. Finance Res. Lett. (forthcoming).

Angoshtari, B., Bayraktar, E., Young, V.R., 2015b. Optimal investment to minimize the probability of drawdown. Preprint. Available at: http://arxiv.org/abs/1506.00166.

Avram, F., Kyprianou, A.E., Pistorius, M.R., 2004. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. Ann. Appl. Probab. 14 (1), 215–238.

Bailey, D.H., Lopez de Prado, M., 2015. Stop-outs under serial correlation and 'the triple penance rule'. J. Risk (forthcoming). Available at SSRN: http://ssrn.com/abstract=2201302.

Bäuerle, N., Bayraktar, E., 2014. A note on applications of stochastic ordering to control problems in insurance and finance. Stochastics 86 (2), 330–340.

Bayraktar, E., Young, V.R., 2007. Minimizing the probability of lifetime ruin under borrowing constraints. Insurance Math. Econom. 41 (1), 196–221.

Bayraktar, E., Zhang, Y., 2015. Minimizing the probability of lifetime ruin under ambiguity aversion. SIAM J. Control Optim. 53 (1), 58–90.

Burghardt, G., Duncan, R., Liu, L., 2003. Deciphering drawdown. Risk Mag. (September), S16–S20.

Chekhlov, A., Uryasev, S., Zabarankin, M., 2005. Drawdown measure in portfolio optimization. Int. J. Theor. Appl. Finance 8 (1), 13–58.

Cherny, V., Obloj, J., 2013. Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model. Finance Stoch. 17 (4), 771–800.

Cvitanic, J., Karatzas, I., 1995. On Portfolio Optimization Under "Drawdown" Constraints. In: IMA Volumes in Mathematics and its Applications, vol. 65. pp. 35–46.

Elie, R., Touzi, N., 2008. Optimal lifetime consumption and investment under a drawdown constraint. Finance Stoch. 12 (3), 299–330.

Grossman, S.J., Zhou, Z., 1993. Optimal investment strategies for controlling drawdowns. Math. Finance 3 (3), 241–276.

Klass, M.J., Nowicki, K., 2005. The Grossman and Zhou investment strategy is not always optimal. Statist. Probab. Lett. 74 (3), 245–252.

Moore, K.S., Young, V.R., 2006. Optimal and simple, nearly optimal rules for minimizing the probability of financial ruin in retirement. N. Am. Actuar. J. 10 (4), 145–161.

Roche, H., 2006. Optimal consumption and investment strategies under wealth ratcheting. Preprint. Available at: http://ciep.itam.mx/~hroche/Research/MDCRESFinal.pdf.

Taylor, H.M., 1975. A stopped Brownian motion formula. Ann. Probab. 3 (2), 234–246.

Young, V.R., 2004. Optimal investment strategy to minimize the probability of lifetime ruin. N. Am. Actuar. J. 8 (4), 106–126.