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# An extension of ELECTRE to multi-criteria decision-making problems with multi-hesitant fuzzy sets

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ABSTRACT

In hesitant fuzzy sets (HFSs), which are generalized from fuzzy sets, the membership degree of an element to a set, for which decision-makers hesitate while considering several values before expressing their preferences concerning weights and data, can be assigned one or more possible precise values between zero and one. If two or more decision-makers assign an equivalent value, that value is only counted once. However, situations in which the same value is repeatedly assigned substantially differ from those in which the value appears only once. Therefore, multi-hesitant fuzzy sets (MHFSs) can be used to manage cases in which values are repeated in a single HFS. In this paper, a method for comparing multi-hesitant fuzzy numbers (MHFNs) is presented. Some outranking relations for MHFNs, which are based on traditional ELECTRE methods, are introduced, and several properties are analyzed. For ranking alternatives, we propose an outranking approach to multi-criteria decision-making (MCDM) problems similar to ELECTRE III, where weights and data are in the form of MHFNs. Finally, an example is given to illustrate the developed approach, and its validity and feasibility are demonstrated by a comparison analysis with other existing methods.



## 1. Introduction

In many cases, it is difficult for decision-makers to precisely express a preference regarding relevant alternatives under several criteria, especially when relying on inaccurate, uncertain, or incomplete information. Such problems are called multi-criteria decision-making (MCDM) problems. Zadeh's fuzzy sets (FSs), where the membership degree of an element to a set is represented by a real number between zero and one, are regarded as an important tool to solve MCDM problems because of their flexibility in describing uncertain information [4,54,59]. They are also used with fuzzy logic and approximate reasoning [27,58], pattern recognition [28,29], and intelligent systems [30].

Information regarding alternatives may be incomplete when they refer to a fuzzy concept. For example, the sum of the membership and non-membership degrees of an element in the universe can be less than one. Classical FS theory fails when attempting to manage an insufficient understanding of the membership degrees. Thus, Atanassov's intuitionistic fuzzy sets (IFSs) and interval-valued intuitionistic fuzzy sets (IVIFSs), both extensions of Zadeh's FSs, were introduced [1–3]. To date, IFSs, IVIFSs, and their extensions have been widely used to solve MCDM problems [7,23,24,44–47,60]. However, in actual

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decision-making problems, the membership degrees in FSs, IFSs, and IVIFSs can be assigned from more than a single real number or interval.

To manage situations in which people are hesitant to express their preference regarding the relevant alternatives in a decision-making process, hesitant fuzzy sets (HFSs), another extension of FSs, provide a useful reference. HFSs were originally defined by Torra, and allow the degree of membership to have different possible precise values between zero and one [37,38]. Recently, HFSs have been the subject of a great deal of research, and have been widely applied to MCDM problems. For example, some work on the aggregation operators of HFSs have been undertaken in previous studies, and the correlation coefficient, distance, and correlation measures for HFSs have been developed [9,14,15,50–53,55,61–64]. Farhadinia discussed novel score functions for HFSs, Zhang and Wei developed the E-VIKOR method to solve MCDM problems with HFSs, Zhang and Xu proposed the TODIM method based on distance measured functions with HFSs, and Zhu et al. proposed dual HFSs and outlined their operations and properties [16,65–69]. However, in any associated distance measure, two hesitant fuzzy numbers (HFNs) must be of equal length, and must be arranged in ascending order. Otherwise, it is necessary to add a specific value to the shorter of the two until they are both of equivalent length. To address these disadvantages, Wang et al. proposed an outranking approach with HFSs to solve MCDM problems [48]. Chen et al. proposed interval-valued hesitant fuzzy sets (IVHFSs) and some aggregation operators, and applied them to multi-criteria group decision-making (MCGDM) problems [10]. Peng et al. introduced an MCDM approach with hesitant interval-valued intuitionistic fuzzy sets (HIVIFSs), which is an extension of dual IVHFSs [31]. Having reviewed the extant research, Rodriguez et al. summarized the current state of HFSs, and proposed some directions for future research [35]. Based on these recommendations, Peng et al. developed an MCDM method based on TODIM and the Choquet integral with multi-valued hesitant fuzzy sets (MHFSs) [33].

However, three main disadvantages of the existing methods for employing HFSs have emerged. (1) Different aggregation operators are involved in different operations, and this can lead to different results. (2) Distance measures must satisfy certain conditions, as discussed earlier. In such cases, different methods of extension can produce different results. (3) Most existing methods mentioned above usually neglect the existence of repeated values in an HFN and consider the frequency of every possible value to be one by default. Situations in which the same value is repeatedly assigned substantially differ from those in which the value appears only once. For example, decision-makers may determine that the possible degrees of membership by which an alternative is assessed relative to the criterion "excellence" are 0.5, 0.6, and 0.6, which is expressed in the form of an HFN as {0.5, 0.6}. However, the nature of the evaluation {0.5, 0.6} substantially differs from that expressed in the form of an MHFS as {0.5, 0.6, 0.6}, which can lead to loss of information during the data collection process. Therefore, MHFSs were generalized from HFSs to avoid loss of information in situations where an equivalent membership value is repeatedly assigned. However, methods incorporating HFSs and MHFSs always involve operations and measures for the comparison of MHFSs whose impact on the final solution may be considerable.

Another method, denoted as the relational model, avoids these drawbacks. Relational models utilize outranking relations or priority functions for ranking the alternatives in terms of priorities among the criteria. Recently, relational models have been acknowledged to more accurately depict the actual decision-making process than other models. The elimination and choice translating reality (ELECTRE) methods originally developed by Benayoun and Roy are representative of this field [5,34]. Subsequently, ELECTRE I, II, III, IV, IS, and TRI were developed [5,17,34,41], which are extensions of ELECTRE. To date, ELECTRE methods have been successfully used in a wide variety of fields including biological engineering [13,18], energy sources [11,19], environmental studies [20,22], economics [6], value engineering [25], communication and transportation [36], personnel selection [32,56,57], and location selection problems [8,11,26]. For instance, Devi and Yadav developed an MCDM method based on the ELECTRE method to solve industrial plant location problems [12]. Hatami-Marbini and Tavana proposed an extension of the ELECTRE I method to solve group decision-making problems under a fuzzy environment [21]. Vahdani et al. proposed an extension of the ELECTRE methods to solve MCDM problems that have interval weights and data [40]. Vahdani and Hadipour presented a novel ELECTRE method to solve problems that have interval-valued fuzzy information [39]. Wang et al. developed an outranking method to solve MCDM problems with hesitant fuzzy linguistic term sets [49].

Previous studies of ELECTRE methods have focused on data characterized by a high degree of certainty, but, in some cases, precisely determining the exact value for each criterion is difficult. The research performed in this paper focuses on data characterized by a high degree of uncertainty as an extension of ELECTRE III, where these uncertainties are expressed using MHFSs. The proposed approach is based on ELECTRE methods to avoid the disadvantages associated with the operations and methods employed for comparing MHFSs. Furthermore, the proposed approach takes decision-makers' preferences into consideration. These are realized by choosing the appropriate thresholds of the given criteria.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and operations of HFSs are introduced. In Section 3, MHFSs are reviewed and the relevant method of comparing multi-hesitant fuzzy numbers (MHFNs) is presented. In Section 4, some outranking relations of MHFNs and some valuable properties are also analyzed. Subsequently, an outranking approach for MCDM problems with MHFNs is shown in Section 5. An illustrative example is provided to demonstrate the validity and feasibility of the proposed approach in Section 6, and conclusions are drawn in Section 7.

## 2. Preliminaries

In this section, the definition of HFSs is reviewed, and some operations and a comparison method for HFSs are presented that are used in the latter analysis.

**Definition 1** ([37,38]). Let $X$ be a reference set, and $E$ be an HFS given in terms of a function that will return a subset of $[0,1]$ when applied to $X$.

To simplify the representation, Xia and Xu expressed the HFS as a mathematical equation [50]:

$$E = \{\langle x, h_E(x)\rangle | x \in X\}. \tag{1}$$

Here, $h_E(x)$ is a set of values in $[0,1]$ denoting the possible degree of membership of the element $x \in X$ to the set $E$. The variable $h_E(x)$ is denoted as a hesitant fuzzy element (HFE) [50], and $H$ is given as the set of all HFEs. In particular, if $X$ has only a single element, $E$ is called an HFN, which can be denoted by $E = \{h_E(x)\}$. The set of all HFNs is represented by HFNS.

**Example 1.** Let $X = \{x_1, x_2, x_3\}$ and let $h_E(x_1) = \{0.1, 0.2\}$, $h_E(x_2) = \{0.2, 0.3\}$, and $h_E(x_3) = \{0.4, 0.5, 0.6\}$ be the HFEs of $x_i (i = 1,2,3)$ to a set $E$. $E$ can be considered an HFS, and can be denoted as follows:

$$E = \{\langle x_1, \{0.1, 0.2\}\rangle, \langle x_2, \{0.2, 0.3\}\rangle, \langle x_3, \{0.4, 0.5, 0.6\}\rangle\}.$$

Torra defined some operations involving HFNs [37,38] to which Xia and Xu added some new operations in addition to score functions [50].

**Definition 2** [50]. Let $h = \cup_{\gamma\in h}\{\gamma\}$, $h_1 = \cup_{\gamma_1\in h_1}\{\gamma_1\}$, and $h_2 = \cup_{\gamma_2\in h_2}\{\gamma_2\}$ be three HFNs. For $\lambda$ represents a scalar mathematical operator, and $\lambda \geqslant 0$, four operations can be defined as follows:

(1) exponentiation: $h^\lambda = \cup_{\gamma\in h}\{\gamma^\lambda\}$;
(2) multiplication: $\lambda h = \cup_{\gamma\in h}\left\{1 - (1-\gamma)^\lambda\right\}$;
(3) $\oplus$-union: $h_1 \oplus h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(4) $\otimes$-intersection: $h_1 \otimes h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\}$.

**Example 2.** Let $h_1 = \{0.1, 0.2\}$ and $h_2 = \{0.1, 0.3, 0.5\}$ be two HFNs, and let $\lambda = 2$. Then, the following results can be obtained:

(1) $h_1^2 = \{0.01, 0.04\}$;
(2) $2h_1 = \{0.19, 0.36\}$;
(3) $h_1 \oplus h_2 = \{0.19, 0.37, 0.55, 0.28, 0.44, 0.60\}$;
(4) $h_1 \otimes h_2 = \{0.01, 0.03, 0.05, 0.02, 0.06, 0.10\}$.

**Definition 3** [50]. Let $h \in HFNS$ and $s(h) = \frac{1}{l(h)}\sum_{\gamma\in h}\gamma$ be the score function of $h$, where $l(h)$ is the number of elements in $h$. For two HFNs $h_1$ and $h_2$, if $s(h_1) > s(h_2)$, then $h_1 > h_2$, and, if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

The disadvantage of using Definition 3 when comparing two HFNs is illustrated in the following example.

**Example 3.** Let $h_1 = \{0.5\}$, $h_2 = \{0.2, 0.8\}$, and $h_3 = \{0.2, 0.5, 0.8\}$ be three HFNs. Apparently, the relationship $h_1 \neq h_2 \neq h_3$ can be obtained. However, according to Definition 3, $s(h_1) = s(h_2) = s(h_3)$, and, thus, $h_1 = h_2 = h_3$, which is counterintuitive.

Farhadinia defined a new score function, which is described as follows [16].

**Definition 4** [16]. Let $h = \cup_{\gamma\in h}\{\gamma\} = \left\{\gamma_j \middle| j = 1,2,\ldots,l(h)\right\}$ be an HFN. Then, the score function of $h$ is defined as follows:

$$S(h) = \frac{\sum_{j=1}^{l(h)}\delta(j)\gamma_j}{\sum_{j=1}^{l(h)}\delta(j)}. \tag{2}$$

Here, $\{\delta(j)|\, j = 1,2,\ldots,l(h)\}$ is a positive-valued monotonic increasing sequence of the index $j$.

**Example 4.** Based on Example 3 and the novel score function given by Definition 4, $s(h_1) = 0.50$ and $s(h_2) = 0.70$, where $h_2$ becomes $\{0.2, 0.8, 0.8\}$ as required, and $s(h_3) = 0.60$. Then, $s(h_1) < s(h_3) < s(h_2)$ can be obtained, and $h_1 < h_3 < h_2$.

These results indicate that the score function in Definition 4 can overcome the counterintuitive results presented in Example 3. However, the new score function is always defined based on the assumption that the values in the relevant HFNs are arranged in an ascending order, and, if two HFNs differ in length, then the shorter one is sufficiently extended until both HFNs are of equal length. In this way, the extension method has the same disadvantage discussed earlier.

## 3. Multi-hesitant fuzzy sets

In this section, the definition of MHFSs, along with their comparison method, is introduced.

**Definition 5** [38]. Let $X$be a reference set, and MHFSs be defined as $E_M$ in terms of a function $H_{E_M}$ that returns a multi-subset of $[0,1]$ when applied to $X$.

Based on Definition 1, MHFSs can be expressed by the mathematical equation:

$$E_M = \{\langle x, H_{E_M}(x)\rangle | x \in X\}. \tag{3}$$

Here, $H_{E_M}(x)$ is a set of values in $[0,1]$ denoting the possible degrees of membership of the element $x \in X$ to the set $E_M$. In any $H_{E_M}(x)$, the values can be repeated multiple times. $H_{E_M}(x)$ is a multi-hesitant fuzzy element (MHFE), and $H_{E_M}$ is the set of all MHFEs. It is noteworthy that, if $X$ contains only a single element, $E_M$ is called a multi-hesitant fuzzy number (MHFN), briefly denoted by $E_M = \{H_{E_M}(x)\}$. The set of all MHFNs is represented by MHFNS. Any HFS is a special case of an MHFS.

**Example 5.** Let $X = \{x_1, x_2\}$ be a fixed set, and $h_E(x_1) = \{0.1, 0.1, 0.2\}$ and $h_E(x_2) = \{0.2, 0.2, 0.3\}$ be the HFEs of $x_i (i = 1, 2)$ to a set $E_M$. $E_M$ can be considered an MHFS, and can be denoted as follows:

$$E_M = \{\langle x_1, \{0.1, 0.1, 0.2\}\rangle, \langle x_2, \{0.2, 0.2, 0.3\}\rangle\}.$$

The operations given in Definition 2 can be applied to MHFNs.

**Example 6.** Defining three MHFNs as $H_1 = \{0.1, 0.2, 0.1, 0.3\},\ H_2 = \{0.2, 0.3, 0.3\}$, and $H = \{0.3, 0.4, 0.4, 0.5\}$, where, as in Example 2, $\lambda = 2$, the following results can be produced:

(1) $H^2 = \{0.09, 0.16, 0.16, 0.25\}$.
(2) $2 \cdot H = \{0.51, 0.64, 0.64, 0.75\}$.
(3) $H_1 \oplus H_2 = \{0.28, 0.37, 0.44, 0.51, 0.37, 0.51, 0.28, 0.37, 0.36, 0.44, 0.37, 0.44\}$.
(4) $H_1 \otimes H_2 = \{0.02, 0.03, 0.04, 0.06, 0.03, 0.06, 0.02, 0.03, 0.06, 0.09, 0.03, 0.09\}$.

**Definition 6.** Let $H \in MHFNs$. Then, $a(H) = \frac{1}{l(H)-1}\sum_{\gamma\in H}(s-\gamma)^2$ can be defined as an accuracy function of $H$, where $s$ is the score function defined in Definition 3 and $l(H)$ is the number of elements in $H$.

**Definition 7.** Defining $H_1, H_2 \in MHFNS$, the following comparison method can be obtained:

(1) if $s(H_1) > s(H_2)$, then $H_1 > H_2$;
(2) if $s(H_1) = s(H_2)$ and $a(H_1) < a(H_2)$, then $H_1 > H_2$;
(3) if $s(H_1) = s(H_2)$ and $a(H_1) = a(H_2)$, then $H_1 = H_2$.

This implies that the comparison laws in Definition 7 are also suitable for HFNs.

**Example 7.** Let $H_1 = \{0.1, 0.4, 0.4\},\ H_2 = \{0.1, 0.1, 0.7\}$, and $H_3 = \{0.2, 0.3, 0.4\}$ be three MHFNs. According to Definition 3, $s(H_1) = s(H_2) = s(H_3) = 0.3$ can be obtained and thus the best one(s) cannot be determined. However, based on Definitions 6 and 7, $a(H_1) = 0.03,\ a(H_2) = 0.12$, and $a(H_3) = 0.01$ can be obtained. Because $a(H_3) < a(H_1) < a(H_2)$, i.e., $H_3 > H_1 > H_2,\ H_3$ is the best.

## 4. Outranking relations on MHFNs

In ELECTRE methods, to allow the $i$-th criterion to be considered, the concordance index and discordance index must be constructed using three associated thresholds: the preference threshold $p_j$, the indifference threshold $q_j$, and the veto threshold $v_j$. Among these three thresholds, $p_j$ is used to establish the preference of one of two alternatives, $q_j$ represents the limit to which two alternatives can be regarded to be indifferent, and $v_j$ is assigned to introduce discordance into the outranking relations. In this paper, only a simple case in which the thresholds $p_j,\ q_j$, and $v_j$ are constants under each criterion, is considered. This simplification aids the illustration of the ELECTRE methods used. The thresholds can be generalized to functions that vary according to the value of the criterion $g_j(a_i)$, that is, in the case of variable thresholds $p_j(g_j(a_i)),\ q_j(g_j(a_i))$, and $v_j(g_j(a_i))$. Further details can be found in previous studies [5,34].

**Definition 8** ([5,34]). Let $G$ be a criteria set $G = \{g_1, \ldots, g_j, \ldots, g_m\}$, which is of the maximizing type, and let $B$ be the set of alternatives $B = \{a_1, \ldots, a_i, \ldots, a_n\}$. Two thresholds under the criterion $g_j$ have been specified to construct the fuzzy concordance index: $q_j$ and $p_j (0 \leqslant q_j < p_j)$. Let $a_1$ and $a_2$ be two alternatives, where $a_1,\ a_2 \in B$. The concordance index for a single criterion can then be defined on the basis of representing the degree of the majority criteria in favor of "$a_1$ is at least as good as $a_2$" as follows.

(1) If $a_1$ is better than $a_2$ or the degree to which $a_1$ is worse than $a_2$ does not exceed the indifference threshold for the criterion $g_j$, i.e., $g_j(a_1) + q_j \geqslant g_j(a_2)$, then $c_j(a_1, a_2) = 1$.
(2) If the degree to which $a_1$ is worse than $a_2$ exceeds the performance threshold for the criterion $g_j$, i.e., $g_j(a_1) + p_j \leqslant g_j(a_2)$, then $c_j(a_1, a_2) = 0$.
(3) Otherwise, the relationship is between these two extremes and is represented as a linear variation, i.e., if $g_j(a_1) + q_j < g_j(a_2) < g_j(a_1) + p_j$, then $c_j(a_1, a_2) = \frac{g_j(a_1) - g_j(a_2) + p_j}{p_j - q_j}$.

**Example 8.** Let $p = 0.2$ and $q = 0.1$

(1) If $a_1 = 0.3$ and $a_2 = 0.35$, then $a_1 + q > a_2$, so $c(a_1, a_2) = 1$.
(2) If $a_1 = 0.1$ and $a_2 = 0.3$, then $a_1 + p \leqslant a_2$, so $c(a_1, a_2) = 0$.
(3) If $a_1 = 0.25$ and $a_2 = 0.4$, then $a_1 + q < a_2 < a_1 + p$, so $c(a_1, a_2) = 0.5$.

**Definition 9** ([5,34]). The veto threshold $v_j (0 \leqslant q_j < p_j < v_j)$ is introduced based on Definition 8. The discordance index $d(a_1, a_2)$ is then defined on the basis of representing the degree of the minority criteria against "$a_1$ is at least as good as $a_2$" as follows.

(1) If the degree to which $a_2$ is better than $a_1$ does not exceed the preference threshold for the criterion $g_j$, i.e., $g_j(a_2) - g_j(a_1) \leqslant p_j$, then $d_j(a_1, a_2) = 0$.
(2) If the degree to which $a_2$ is better than $a_1$ exceeds the veto threshold for the criterion $g_j$, i.e., $g_j(a_2) - g_j(a_1) \geqslant v_j$, then $d_j(a_1, a_2) = 1$.
(3) Otherwise, the relationship is linear between the two extremes and is represented as a linear variation, i.e., if $p_j < g_j(a_2) - g_j(a_1) < v_j$, then $d_j(a_1, a_2) = \frac{g_j(a_2) - g_j(a_1) - p_j}{v_j - p_j}$.

It should be mentioned that, if a criterion exists for which the degree that the alternative $a_2$ performs better than the alternative $a_1$ exceeds the veto threshold, even if other criteria possibly favor the outranking of $a_1$ by $a_2$, then any outranking of $a_1$ by $a_2$ indicated by the concordance index can be overruled.

**Example 9.** Let $p = 0.01$ and $v = 0.02$.

(1) If $a_1 = 0.3$ and $a_2 = 0.4$, then $a_2 - a_1 \leqslant p$, so $d(a_1, a_2) = 0$.
(2) If $a_1 = 0.1$ and $a_2 = 0.3$, then $a_2 - a_1 \geqslant v$, so $d(a_1, a_2) = 1$.
(3) If $a_1 = 0.2$ and $a_2 = 0.35$, then $p < a_2 - a_1 < v$, so $d(a_1, a_2) = 0.5$.

**Definition 10** ([5,34]). Let $a_1$ and $a_2$ be two alternatives, where $a_1,\ a_2 \in B$. The binary relations can then be defined based on Definition 8 as follows.

(1) If $g_j(a_1) - g_j(a_2) \geqslant p_j$, then $a_1$ is strongly preferred to $a_2$, denoted by $P(a_1, a_2)$.
(2) If $q_j < g_j(a_1) - g_j(a_2) < p_j$, then $a_1$ is weakly preferred to $a_2$, denoted by $W(a_1, a_2)$.
(3) If $|g_j(a_1) - g_j(a_2)| \leqslant q_j$, then $a_1$ is indifferent to $a_2$, denoted by $I(a_1, a_2)$.

**Example 10.** Let $p = 0.02$ and $q = 0.01$.

(1) If $a_1 = 0.5$ and $a_2 = 0.3$, then $a_1 - a_2 \geqslant p$, so $a_1$ is strongly preferred to $a_2$.
(2) If $a_1 = 0.5$ and $a_2 = 0.35$, then $q < a_1 - a_2 < p$, so $a_1$ is weakly preferred to $a_2$.
(3) If $a_1 = 0.4$ and $a_2 = 0.3$, then $|a_1 - a_2| \leqslant q$, so $a_1$ is indifferent to $a_2$.

Following the rules of the ELECTRE method given by the outranking relations, a concordance index and a discordance index for MHFNs are defined as follows.

**Definition 11.** Let $H_1,\ H_2 \in MHFNS$, and $p$ and $q \quad (0 \leqslant q < p)$ be two thresholds. The concordance index can then be defined as follows:

$$r_{p,q}(H_1,H_2) = \frac{1}{l(H_1)} \sum_{\gamma_1 \in H_1} \min_{\gamma_2 \in H_2} \{c_{p,q}(\gamma_1,\gamma_2)\}. \tag{4}$$

Here, $l(H_1)$ is the number of elements in $H_1$ and $c_{p,q}(\gamma_1,\gamma_2)$ is the concordance index for the values $\gamma_1$ and $\gamma_2$ under thresholds $q$ and $p$.

It is fairly simple to ascertain that, if both $H_1$ and $H_2$ have only a single value, $r_{p,q}(H_1,H_2)$ will turn into a concordance index, as introduced in Definition 8.

According to Definition 10, the following properties can be easily obtained.

**Property 1.** *Let $H_1$ and $H_2$ be two MHFNs, and $q$ and $p$ $(0 \leqslant q < p)$ be two thresholds. Then, $0 \leqslant r_{p,q}(H_1,H_2) \leqslant 1$.*

**Definition 12.** The strong dominance relation, weak dominance relation, and indifferent relation of MHFNs can be defined as follows.

(1) If $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 1$, then $H_1$ strongly dominates $H_2$ ($H_2$ is strongly dominated by $H_1$), denoted by $H_1 >_S H_2$.
(2) If $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 0$, then $H_1$ is indifferent to $H_2$, denoted by $H_1 \sim H_2$.
(3) If $0 < r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) < 1$, then $H_1$ weakly dominates $H_2$ ($H_2$ is weakly dominated by $H_1$), denoted by $H_1 >_W H_2$.
(4) If $0 < r_{p,q}(H_2,H_1) - r_{p,q}(H_1,H_2) < 1$, then $H_2$ weakly dominates $H_1$ ($H_1$ is weakly dominated by $H_2$), denoted by $H_2 >_W H_1$.

**Example 11.** Let $p = 0.06$ and $q = 0.05$.

(1) If $H_1 = \{0.12, 0.12, 0.18\}$ and $H_2 = \{0.12, 0.12\}$, then $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 1$, so $H_1 >_S H_2$.
(2) If $H_1 = \{0.12, 0.12, 0.18\}$ and $H_2 = \{0.14, 0.14, 0.16\}$, then $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 0$, so $H_1 \sim H_2$.
(3) If $H_1 = \{0.125, 0.15, 0.15, 0.20, 0.24\}$ and $H_2 = \{0.125, 0.15, 0.15, 0.18\}$, then $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 0.9$, so $H_1 >_W H_2$.

**Property 2.** *Let $H_1$, $H_2 \in MHFNS$, and $p$ and $q$ $(0 \leqslant q < p)$ be two thresholds. $H_1 >_S H_2$ if and only if $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p$.*

**Proof.**

(1) Necessity: $H_1 >_S H_2 \Rightarrow \min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p$.
According to Definition 12, if $H_1 >_S H_2$, then $r_{p,q}(H_1,H_2) - r_{p,q}(H_2,H_1) = 1$. Because $0 \leqslant r_{p,q}(H_1,H_2) \leqslant 1$ and $0 \leqslant r_{p,q}(H_2,H_1) \leqslant 1, r_{p,q}(H_2,H_1) = 0$ can be obtained. Thus, $\frac{1}{l(H_2)} \sum_{\gamma_2 \in H_2} \min_{\gamma_1 \in H_1} c_{p,q}(\gamma_2,\gamma_1) = 0$ is obtained. As derived from Definition 8, $c_{p,q}(\gamma_2,\gamma_1) \in [0,1]$, so $c_{p,q}(\gamma_2,\gamma_1) = 0$. Hence, $\gamma_1 - \gamma_2 \geqslant p$ for any $\gamma_1 \in H_1$, $\gamma_2 \in H_2$. Therefore, $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p$ is certainly valid.
(2) Sufficiency: $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p \Rightarrow H_1 >_S H_2$.
Because $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p$, then $\gamma_1 - \gamma_2 \geqslant p$ for any $\gamma_1 \in H_1$, $\gamma_2 \in H_2$. As indicated by Definition 8, $c_{p,q}(\gamma_2,\gamma_1) = 0$ and $c_{p,q}(\gamma_1,\gamma_2) = 1$. Therefore, $\frac{1}{l(H_1)} \sum_{\gamma_1 \in H_1} \min_{\gamma_2 \in H_2} c_{p,q}(\gamma_2,\gamma_1) = 1$ and $\frac{1}{l(H_2)} \sum_{\gamma_2 \in H_2} \min_{\gamma_1 \in H_1} c_{p,q}(\gamma_1,\gamma_2) = 0$, which indicate that $r_{p,q}(H_1,H_2) = 1$ and $r_{p,q}(H_2,H_1) = 0$ based on Definition 11. Therefore, according to Definition 12, $H_1 >_S H_2$. □

**Property 3.** *Let $H_1, H_2, H_3 \in MHFNS$, and $p$ and $0 \leqslant q < p (0 \leqslant q < p)$ be two thresholds. If $H_1 >_S H_2$ and $H_2 >_S H_3$, then $H_1 >_S H_3$.*

**Proof.** According to Property 2, if $H_1 >_S H_2$, then $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p$.

If $H_2 >_S H_3$, then $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_3|\gamma_3 \in H_3\} \geqslant p$, so $\max\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_3|\gamma_3 \in H_3\} \geqslant p$.

Therefore, further derivations are obtained as follows:

$$\left.\begin{array}{l} \min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant p \\ \max\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_3|\gamma_3 \in H_3\} \geqslant p \end{array}\right\} \Rightarrow \max\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_3|\gamma_3 \in H_3\} \geqslant 2p \geqslant p.$$

Therefore, $H_1 >_S H_3$. □

**Property 4.** *Let $H_1, H_2, H_3 \in MHFNS$, and $p$ and $q$ $(0 \leqslant q < p)$ be two thresholds.*

(1) *The strongly dominant relations have the following properties.*
① *irreflexivity: $\forall H_1 \in MHFNS, H_1 \not>_S H_1$;*
② *asymmetry: $\forall H_1, H_2 \in MHFNS, H_1 >_S H_2 \Rightarrow \neg(H_2 >_S H_1)$;*

③ *transitivity:* $\forall H_1, H_2, H_3 \in MHFNS, H_1 >_S H_2, H_2 >_S H_3 \Rightarrow H_1 >_S H_3$.

(2) *The weakly dominant relations have the following properties.*

④ *irreflexivity:* $\forall H_1 \in MHFNS, H_1 \not>_W H_1$;

⑤ *asymmetry:* $\forall H_1, H_2 \in MHFNS, H_1 >_W H_2 \Rightarrow \neg(H_2 >_W H_1)$;

⑥ *non-transitivity:* $\exists H_1, H_2, H_3 \in MHFNS, H_1 >_W H_2, H_2 >_W H_3 \not\Rightarrow H_1 >_W H_3$.

(3) *The indifferent relations have the following properties.*

⑦ *reflexivity:* $\forall H_1 \in MHFNS, H_1 \sim H_1$;

⑧ *symmetry:* $\forall H_1, H_2 \in MHFNS, H_1 \sim H_2 \Rightarrow H_2 \sim H_1$;

⑨ *non-transitivity:* $\exists H_1, H_2, H_3 \in MHFNS, H_1 \sim H_2, H_2 \sim H_3 \not\Rightarrow H_1 \sim H_3$.

According to Definitions 11, 12, it is clear that properties ①–⑤, ⑦, and ⑧ are true, and ⑥ and ⑨ can be exemplified.

**Example 12.** Let $p = 0.06$ and $q = 0.05$. Properties ⑥ and ⑨ can be exemplified as follows.

(1) If $H_1 = \{0.125, 0.15, 0.2, 0.24\}$, $H_2 = \{0.125, 0.15, 0.15, 0.18\}$, and $H_3 = \{0.125, 0.15, 0.175, 0.21\}$ are three MHFNs, then $r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) = 0.875$, $r_{p,q}(H_1, H_3) - r_{p,q}(H_3, H_1) = 1$, and $r_{p,q}(H_2, H_3) - r_{p,q}(H_3, H_2) = 0.333$. Accordingly, $H_1 >_W H_2$, $H_2 >_W H_3$, but $H_1 >_S H_3$. This shows that weak dominance relations are non-transitive.

(2) If $H_1 = \{0.12, 0.12, 0.18\}$, $H_2 = \{0.14, 0.16\}$, and $H_3 = \{0.12, 0.14, 0.14\}$ are three MHFNs, then $r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) = 0$, $r_{p,q}(H_2, H_3) - r_{p,q}(H_3, H_2) = 0$, and $r_{p,q}(H_1, H_3) - r_{p,q}(H_3, H_1) = 0.333$. Accordingly, $H_1 \sim H_2$, $H_2 \sim H_3$, but $H_1 >_W H_3$. This shows that indifferent relations are non-transitive.

In the following, the strong opposition relation, weak opposition relation, and indifferent opposition relation are defined.

**Definition 13.** Let $H_1$, $H_2 \in MHFNS$, and $p$ and $p < \nu (p < \nu)$ be two thresholds. The discordance index for MHFNs can then be defined as follows:

$$t_{p,\nu}(H_1, H_2) = \frac{1}{l(H_1)} \sum_{\gamma_1 \in H_1} \min_{\gamma_2 \in H_2} \{d_{p,\nu}(\gamma_1, \gamma_2)\}. \tag{5}$$

It can be easily concluded that, when both $H_1$ and $H_2$ have only a single value, $t_{p,\nu}(H_1, H_2)$ becomes a discordance index, as introduced in Definition 9.

According to Definition 14, the following property is readily obtained.

**Property 5.** *Let $H_1, H_2 \in MHFNS$, and $p$ and $\nu (p < \nu)$ be two thresholds. Therefore,* $0 \leqslant t_{p,\nu}(H_1, H_2) \leqslant 1$.

**Definition 14.** The strong opposition relation, weak opposition relation, and indifferent opposition relation for MHFNs are defined as follows.

(1) If $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 1$, then $H_1$ strongly opposes $H_2$ ($H_2$ is strongly opposed by $H_1$), denoted by $H_1 >_{SO} H_2$.
(2) If $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 0$, then $H_1$ is indifferently opposed to $H_2$, denoted by $H_1 \sim_O H_2$.
(3) If $0 < t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) < 1$, then $H_1$ weakly opposes $H_2$ ($H_2$ is weakly opposed by $H_1$), denoted by $H_1 >_{WO} H_2$.
(4) If $0 < t_{p,\nu}(H_2, H_1) - t_{p,\nu}(H_1, H_2) < 1$, then $H_2$ weakly opposes $H_1$ ($H_1$ is weakly opposed by $H_2$), denoted by $H_2 >_{WO} H_1$.

**Example 13.** Let $p = 0.2$ and $\nu = 0.3$.

(1) If $H_1 = \{0.1, 0.2\}$ and $H_2 = \{0.5, 0.5, 0.7\}$, then $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 1$, so $H_1 >_{SO} H_2$.
(2) If $H_1 = \{0.2, 0.5\}$ and $H_2 = \{0.2, 0.2, 0.6\}$, then $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 0$, so $H_1 \sim_O H_2$.
(3) If $H_1 = \{0.2, 0.2, 0.5\}$ and $H_2 = \{0.45, 0.75\}$, then $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 0.333$, so $H_1 >_{WO} H_2$.

According to Definitions 9, 13, and 14, similar to Properties 2–4, the following properties are true.

**Property 6.** *Let $H_1, H_2 \in MHFNS$, and $p$ and $\nu (0 < p < \nu)$ be two thresholds. Then, $H_1 >_{SO} H_2$ if and only if* $\min\{\gamma_2 | \gamma_2 \in H_2\} - \max\{\gamma_1 | \gamma_1 \in H_1\} \geqslant \nu$.

**Proof.**

(1) Necessity: $H_1 >_{SO} H_2 \Rightarrow \min\{\gamma_2 | \gamma_2 \in H_2\} - \max\{\gamma_1 | \gamma_1 \in H_1\} \geqslant \nu$.
According to Definition 14, if $H_1 >_{SO} H_2$, then $t_{p,\nu}(H_1, H_2) - t_{p,\nu}(H_2, H_1) = 1$. Because $0 \leqslant t_{p,\nu}(H_1, H_2) \leqslant 1$ and $0 \leqslant t_{p,\nu}(H_2, H_1) \leqslant 1$, we can determine that $t_{p,\nu}(H_1, H_2) = 1$ and $t_{p,\nu}(H_2, H_1) = 0$. In this way, $\frac{1}{l(H_1)} \sum_{\gamma_1 \in H_1} \min_{\gamma_2 \in H_2} d_{p,\nu}(\gamma_1, \gamma_2) = 1$ was

determined. As indicated by Definition 9, $0 \leqslant d_{p,v}(\gamma_1, \gamma_2) \leqslant 1$, so $d_{p,v}(\gamma_1, \gamma_2) = 1$. For any $\gamma_1 \in H_1, \gamma_2 \in H_2$, we can obtain $\gamma_2 - \gamma_1 \geqslant v$. Therefore, $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v$ is certainly true.

(2) Sufficiency: $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v \Rightarrow H_1 >_{SO} H_2$.
Because $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v$, $\gamma_2 - \gamma_1 \geqslant v$ for any $\gamma_1 \in H_1$, $\gamma_2 \in H_2$. According to Definition 9, $d_{p,v}(\gamma_1, \gamma_2) = 1$ and $d_{p,v}(\gamma_2, \gamma_1) = 0$ are determined. Therefore, $\frac{1}{l(H_1)}\sum_{\gamma_1 \in H_1}\min_{\gamma_2 \in H_2} d_{p,v}(\gamma_1, \gamma_2) = 1$ and $\frac{1}{l(H_2)}\sum_{\gamma_2 \in H_2}\min_{\gamma_1 \in H_1} d_{p,v}(\gamma_2, \gamma_1) = 0$, which indicate that $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 1 - 0 = 1$. Therefore, $H_1 >_{SO} H_2$. □

**Property 7.** *Let $H_1, H_2, H_3 \in MHFNS$, and $p$ and $v(p < v)$ be two thresholds. If $H_1 >_{SO} H_2$ and $H_2 >_{SO} H_3$, then $H_1 >_{SO} H_3$.*

**Proof.** According to Property 6, if $H_1 >_{SO} H_2$, then $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v$, so $\max\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v$.

If $H_2 >_{SO} H_3$, then $\min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant v$.

Thus, further derivations are obtained as follows:

$$\left.\begin{array}{l}\max\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant v \\ \min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geqslant v\end{array}\right\} \Rightarrow \min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geqslant 2v \geqslant v.$$

Therefore, $H_1 >_{SO} H_3$. □

**Property 8.** *Let $H_1, H_2, H_3 \in MHFNS$, and $p$ and $v(0 < p < v)$ be two thresholds.*

(1) *The strictly opposed relations have the following properties.*
① *irreflexivity: $\forall H_1 \in MHFNS, H_1 \not>_{SO} H_1$;*
② *asymmetry: $\forall H_1, H_2 \in MHFNS, H_1 >_{SO} H_2 \Rightarrow \neg(H_2 >_{SO} H_1)$;*
③ *transitivity: $\forall H_1, H_2, H_3 \in MHFNS, H_1 >_{SO} H_2, H_2 >_{SO} H_3 \Rightarrow H_1 >_{SO} H_3$.*

(2) *The weakly opposed relations have the following properties.*
④ *irreflexivity: $\forall H_1 \in MHFNS, H_1 \not>_{WO} H_1$;*
⑤ *asymmetry: $\forall H_1, H_2 \in MHFNS, H_1 >_{WO} H_2 \Rightarrow \neg(H_2 >_{WO} H_1)$;*
⑥ *non-transitivity: $\exists H_1, H_2, H_3 \in MHFNS, H_1 >_{WO} H_2, H_2 >_{WO} H_3 \not\Rightarrow H_1 >_{WO} H_3$.*

(3) *The indifferently opposed relations have the following properties.*
⑦ *reflexivity: $\forall H_1 \in MHFNS, H_1 \sim_O H_1$;*
⑧ *symmetry: $\forall H_1, H_2 \in MHFNS, H_1 \sim_O H_2 \Rightarrow H_2 \sim_O H_1$;*
⑨ *non-transitivity: $\exists H_1, H_2, H_3 \in MHFNS, H_1 \sim_O H_2, H_2 \sim_O H_3 \not\Rightarrow H_1 \sim_O H_3$.*

According to Definitions13, 14, it is clear that properties ①–⑤, ⑦, and ⑧ are true, and ⑥ and ⑨ can be exemplified.

**Example 14.** Let $p = 0.15$ and $v = 0.2$. Properties ⑥ and ⑨ can be exemplified as follows.

(1) If $H_1 = \{0.1, 0.1, 0.2\}$, $H_2 = \{0.3, 0.4\}$, and $H_3 = \{0.5, 0.6, 0.6\}$ are three MHFNs, then $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 0.5$, $t_{p,v}(H_2, H_3) - t_{p,v}(H_3, H_2) = 0.5$, and $t_{p,v}(H_1, H_3) - t_{p,v}(H_3, H_1) = 1$. Accordingly, we have $H_1 >_{WO} H_2$, $H_2 >_{WO} H_3$, but $H_1 >_{SO} H_3$. Thus, the weak opposition relations are non-transitive.

(2) If $H_1 = \{0.1, 0.1\}$, $H_2 = \{0.25, 0.25\}$, and $H_3 = \{0.3, 0.3, 0.4\}$ are three MHFNs, then $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 0$, $t_{p,v}(H_2, H_3) - t_{p,v}(H_3, H_2) = 0$, and $t_{p,v}(H_1, H_3) - t_{p,v}(H_3, H_1) = 1$. Accordingly, $H_1 \sim_O H_2$ and $H_2 \sim_O H_3$, but $H_1 >_{SO} H_3$. This shows that the indifferent opposition relations are non-transitive.

## 5. An ELECTRE approach for MCDM problems with MHFNs

MCDM ranking and selection problems with multi-hesitant fuzzy information consist of a group of alternatives, denoted as $A = \{a_1, a_2, \ldots, a_n\}$. All alternatives are evaluated based on criteria, denoted by $C = \{c_1, c_2, \ldots, c_m\}$. We denote $a_{ij}$ as the value of the alternative $a_i$ with respect to the criterion $c_j$, where $a_{ij} = \left\{\gamma_{ij}^k,\ k = 1, 2, \ldots, l(a_{ij})\right\}$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ are in the form of MHFNs, and $l(a_{ij})$ represents the number of elements in $a_{ij}$. The weight vector corresponding to the criteria is given as $w = (w_1, w_2, \ldots, w_m)$, where $w_j$ is in the form of MHFNs. This method is only suitable if there is a small quantity of decision-makers. The decision-makers can evaluate these alternatives based on the given criteria, and a single decision-maker can assign multiple values to any $a_{ij}$. In particular, in the case in which two or more decision-makers assign an equivalent value, the frequency of this repeated value will remain unchanged in MHFNs.

This approach is an integration of MHFNs and the outranking method used to handle the MCDM problems mentioned above.

***Step 1***. Normalize the decision matrix $R = (a_{ij})_{n\times m}$.

For MCDM problems, the most common criteria involve maximizing and minimizing types. To unify all criteria, it is necessary to normalize the value of the alternative $a_i$ with respect to the criterion $c_j$, i.e., $a_{ij}$. However, it should be remarked that, if all the criteria are of the maximizing type and represent the same unit of measurement, they need not be normalized. Suppose that the matrix $R = (a_{ij})_{n\times m}$, where $a_{ij} = \left\{\gamma_{ij}^1, \gamma_{ij}^2, \ldots, \gamma_{ij}^k\right\}$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m;$ $k = 1,2,\ldots,l(a_{ij}))$ are MHFNs, can be normalized to the corresponding matrix $\tilde{R} = (\tilde{a}_{ij})_{n\times m}$. Here, $\tilde{a}_{ij} = \left\{\tilde{\gamma}_{ij}^1, \tilde{\gamma}_{ij}^2, \ldots, \tilde{\gamma}_{ij}^k\right\}$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m;\ k = 1,2,\ldots,l(a_{ij}))$. For the maximizing criteria, the normalization formula is as follows:

$$\tilde{\gamma}_{ij}^k = \gamma_{ij}^k, k = 1,2,\ldots,l(a_{ij});$$

and, for the minimizing criteria, it is as follows:

$$\tilde{\gamma}_{ij}^k = 1 - \gamma_{ij}^k, k = 1,2,\ldots,l(a_{ij}).$$

Apparently, the normalization values $\tilde{a}_{ij} = \left\{\tilde{\gamma}_{ij}^1, \tilde{\gamma}_{ij}^2, \ldots, \tilde{\gamma}_{ij}^k\right\}$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m)$ are also MHFNs.

***Step 2***. Determine the weighted normalized matrix.

According to the weights of the criteria and the operations in Definition 2, the weighted normalized decision matrix can be constructed using the following formula:

$$\tilde{a}_{ij}^* = \tilde{a}_{ij} \otimes w_j\ (i = 1,2,\ldots,n;\ j = 1,2,\ldots,m). \quad (6)$$

Here, $w_j$ is the weight of the $j$-th criterion.

***Step 3***. Determine the concordance set of subscripts.

The concordance set of subscripts, which should satisfy the constraint $\tilde{a}_{ij}^* >_S \tilde{a}_{kj}^*$ or $\tilde{a}_{ij}^* >_W \tilde{a}_{kj}^*$ or $\tilde{a}_{ij}^* \sim \tilde{a}_{kj}^*$, is represented as follows:

$$O_{ik} = \left\{j | r_{p,q}(\tilde{a}_{ij}^*, \tilde{a}_{kj}^*) - r_{p,q}(\tilde{a}_{kj}^*, \tilde{a}_{ij}^*) \geqslant 0\right\}(i,k = 1,2,\ldots,n). \quad (7)$$

Here, $r_{p,q}(\tilde{a}_{ij}^*, \tilde{a}_{kj}^*)$ represents the concordance index between $\tilde{a}_{ij}^*$ and $\tilde{a}_{kj}^*$, and can be calculated using Eq. (4) in Definition 11.

***Step 4***. Determine the concordance matrix.

Using the weight vector $w$ associated with the criteria, the concordance index $C(a_i, a_k)$ is represented as follows:

$$C(a_i, a_k) = s(c^*(a_i, a_k)). \quad (8)$$

Here, $c^*(a_i, a_k) = \sum_{j\in O_{ik}} w_j \oplus \sum_{j\in\left\{j|\tilde{a}_{kj}^* >_W \tilde{a}_{ij}^*\right\}} w_j \cdot r_{p,q}(\tilde{a}_{kj}^*, \tilde{a}_{ij}^*)$, where $s(\cdot)$ is the score function defined in Definition 3.

Therefore, the concordance matrix $C$ is as follows:

$$C = \begin{pmatrix} - & c_{12} & \cdots & c_{1n} \\ c_{21} & - & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & - \end{pmatrix}. \quad (9)$$

***Step 5***. Determine the credibility index of outranking relations.

$$\sigma(a_i, a_k) = C(a_i, a_k) \cdot \prod_{j=1}^{m} \delta_j(a_i, a_k). \quad (10)$$

Here,

$$\delta_j(a_i, a_k) = \begin{cases} \frac{1 - t_{p,v}(\tilde{a}_{ij}^*, \tilde{a}_{kj}^*)}{1 - C(a_i, a_k)} & \text{if } t_{p,v}(\tilde{a}_{ij}^*, \tilde{a}_{kj}^*) > C(a_i, a_k) \\ 1 & \text{otherwise} \end{cases},$$

where $t_{p,v}(\tilde{a}_{ij}^*, \tilde{a}_{kj}^*)$ represents the discordance index between $\tilde{a}_{ij}^*$ and $\tilde{a}_{kj}^*$, and can be calculated using Eq. (5) in Definition 13.

***Step 6***. Determine the ranking of the alternatives' indices.

The ranking of the alternatives' indices is defined in two preorders using descending and ascending distillations. Let $\lambda = \max_{a_i, a_k \in A} \sigma(a_i, a_k), \lambda - \kappa(\lambda)$ be a credibility value such that $\kappa(\lambda)$ is sufficiently close to $\lambda$ (more details concerning the values of $\kappa(\lambda)$ can be found in [20]). Therefore, $S$ can be defined as follows:

$$S(a_i, a_k) = \begin{cases} 1, & \text{if } \sigma(a_i, a_k) > \lambda - \kappa(\lambda) \\ 0, & \text{otherwise} \end{cases}. \quad (11)$$

According to the matrix $S(a_i, a_k)$, the final qualification score for each alternative is the number of alternatives that are outranked by $a_i$ minus the number of alternatives that outrank $a_i$.

The descending distillation process is implemented by first retaining the alternative with the highest qualification score, and then applying the same procedure to the remaining alternatives. The ascending distillation process is similar to descending distillation, except that the process is based on the lowest qualification score rather than the highest.

***Step 7***. Rank all the alternatives.

## 6. Illustrative example

In this section, an example was adapted from a previous work by Wei [43]. In this example, the school of management in a Chinese university is planning to recruit some outstanding teachers from overseas to strengthen academic capabilities and enhance the quality of teaching at the university. The university's president and human resource officer make up the panel of decision-makers responsible for the recruitment. They performed a strict evaluation for five alternatives denoted as $a_1, a_2, a_3, a_4, a_5$ according to the following four criteria: morality, research capability, teaching skills, and educational background, here denoted as $c_1, c_2, c_3, c_4$, and their corresponding weights were $w_1 = \{0.45, 0.3\}$, $w_2 = \{0.3, 0.25\}$, $w_3 = \{0.2, 0.2\}$, and $w_4 = \{0.10, 0.20\}$. The evaluation of the five candidates $a_i$ $(i = 1, 2, 3, 4, 5)$ was performed with MHFNs by two decision-makers using the criteria $c_k$ $(k = 1, 2, 3, 4)$. A given decision-maker could assign several values to each candidate based on the criteria. In particular, in the case in which both decision-makers assigned the same value, the frequency of the repeated values will be the same as that in the statistical results. A multi-hesitant fuzzy decision matrix $R = (a_{ij})_{5\times 4}$ was constructed as shown below:

$$R = \begin{pmatrix} \{0.4, 0.5, 0.7\} & \{0.5, 0.5, 0.8\} & \{0.6, 0.6, 0.9\} & \{0.5, 0.6\} \\ \{0.6, 0.7, 0.8\} & \{0.5, 0.6\} & \{0.6, 0.7, 0.7\} & \{0.4, 0.5\} \\ \{0.6, 0.8\} & \{0.2, 0.3, 0.5\} & \{0.6, 0.6\} & \{0.5, 0.7\} \\ \{0.5, 0.5, 0.7\} & \{0.4, 0.5\} & \{0.8, 0.9\} & \{0.3, 0.4, 0.5\} \\ \{0.6, 0.7\} & \{0.5, 0.7\} & \{0.7, 0.8\} & \{0.3, 0.3, 0.4\} \end{pmatrix}$$

### *6.1. Illustration of the proposed approach*

The procedures used to identify the optimal alternative using the method proposed here are as follows.

***Step 1***. Normalize the data in decision matrix $R = (a_{ij})_{5\times 4}$.
Because all the criteria are of the maximizing type and have the same measurement unit, there is no need for normalization, and $\tilde{R} = (\tilde{a}_{ij})_{5\times 4} = (a_{ij})_{5\times 4}$.

***Step 2***. Determine the weighted normalized matrix.
For instance, based on the operations in Definition 2 and Eq. (6), the weighted normalized value $\tilde{a}^*_{53}$ can be calculated as follows:

$$\tilde{a}^*_{53} = \tilde{a}_{53} \otimes w_3 = \{0.7, 0.8\} \otimes \{0.2, 0.2\} = \{0.14, 0.14, 0.16, 0.16\}.$$

Similarly, the weighted normalized matrix $R^*$ can be determined as shown below:

$$D^* = \begin{pmatrix} \{0.12,0.18,0.15,0.225,0.21,0.315\} & \{0.125,0.15,0.125,0.15,0.20,0.24\} & \{0.12,0.12,0.12,0.12,0.18,0.18\} & \{0.05,0.10,0.06,0.12\} \\ \{0.18,0.27,0.21,0.315,0.24,0.36\} & \{0.125,0.15,0.15,0.18\} & \{0.12,0.12,0.14,0.40,0.14,0.14\} & \{0.04,0.08,0.05,0.10\} \\ \{0.18,0.27,0.24,0.36\} & \{0.05,0.06,0.09,0.075,0.125,0.15\} & \{0.12,0.12,0.12,0.12\} & \{0.05,0.10,0.07,0.14\} \\ \{0.15,0.225,0.15,0.225,0.21,0.315\} & \{0.10,0.12,0.125,0.15\} & \{0.16,0.16,0.18,0.18\} & \{0.03,0.06,0.04,0.08,0.05,0.10\} \\ \{0.18,0.27,0.21,0.315\} & \{0.125,0.15,0.175,0.21\} & \{0.14,0.14,0.16,0.16\} & \{0.03,0.06,0.03,0.06,0.04,0.08\} \end{pmatrix}$$

***Step 3***. Determine the concordance set of subscripts.
Let $q_j = 0.05, p_j = 0.06$, and $\upsilon_j = 0.07$ be the thresholds for all criteria $c_j (j = 1, 2, 3, 4)$. According to Eq. (7), because $\tilde{a}^*_{12} >_W \tilde{a}^*_{22}, \tilde{a}^*_{13} >_S \tilde{a}^*_{23}$, and $\tilde{a}^*_{14} >_W \tilde{a}^*_{24}$, $O_{12} = \{2, 3, 4\}$. Similarly, the concordance set of subscripts can be determined as follows:

$$O = (O_{ik}) = \begin{pmatrix} - & 2,3,4 & 2,3 & 1,2,3,4 & 2,3,4 \\ 1 & - & 1,2 & 1,2,3,4 & 1,3,4 \\ 1,3,4 & 3,4 & - & 1,4 & 1,3,4 \\ 1,3 & 3 & 2,3 & - & 3,4 \\ 1,3 & 1,2,3 & 1,2,3 & 1,2,3 & - \end{pmatrix}.$$

***Step 4***. Determine the concordance matrix.
According to Eq. (8), the concordance index $c_{25}$ can be calculated as follows:

$$c^*(a_2, a_5) = w_1 \oplus w_3 \oplus w_4 \oplus w_2 \cdot r_{p,q}(\tilde{a}^*_{22}, \tilde{a}^*_{52})$$
$$= \{0.669, 0.578, 0.669, 0.578, 0.657, 0.564, 0.657, 0.564, 0.705, 0.625, 0.705, 0.625, 0.665, 0.654\};$$
$$c_{25} = s(c^*(a_2, a_5)) = 0.637.$$

Similarly, the concordance matrix can be determined as shown below:

$$C = \begin{pmatrix} - & 0.579 & 0.545 & 0.692 & 0.579 \\ 0.556 & - & 0.599 & 0.692 & 0.637 \\ 0.575 & 0.499 & - & 0.524 & 0.703 \\ 0.533 & 0.492 & 0.554 & - & 0.420 \\ 0.567 & 0.666 & 0.638 & 0.666 & - \end{pmatrix}.$$

***Step 5***. Determine the credibility index.
Based on Step 4 and Eq. (10), the credibility index matrix can be determined as follows:

$$\sigma = \begin{pmatrix} - & 0.579 & 0.545 & 0.692 & 0.579 \\ 0.556 & - & 0.599 & 0.692 & 0.637 \\ 0.575 & 0.499 & - & 0.524 & 0.703 \\ 0.533 & 0.492 & 0.554 & - & 0.42 \\ 0.567 & 0.666 & 0.638 & 0.666 & - \end{pmatrix}.$$

***Step 6***. Determine the ranking of the alternatives' indices.
According to Step 5, $\lambda = \max_{a_i, a_j \in A} \sigma(a_i, a_j) = 0.703$. If $\kappa(\lambda) = 0.15$ [20], then the following is true:

$$S(a_i, a_j) = \begin{pmatrix} - & 1 & 0 & 1 & 1 \\ 1 & - & 1 & 1 & 1 \\ 1 & 0 & - & 0 & 0 \\ 0 & 0 & 1 & - & 0 \\ 1 & 1 & 1 & 1 & - \end{pmatrix}.$$

Therefore, we can derive the descending distillation as $\{a_2\} \rightarrow \{a_5\} \rightarrow \{a_1, a_3, a_4\}$, the ascending distillation as $\{a_2, a_5\} \rightarrow \{a_3\} \rightarrow \{a_1\} \rightarrow \{a_4\}$, and the final ranking as $\{a_2\} \rightarrow \{a_5\} \rightarrow \{a_3\} \rightarrow \{a_1\} \rightarrow \{a_4\}$.

***Step 7***. Rank all the alternatives.

This shows that the best alternative is $a_2$, and the worst alternative is $a_4$.

### *6.2. Comparative analysis and discussion*

A comparative study was performed to confirm the feasibility of the proposed decision-making method. The analysis included three classes of other methods. The first class was comprised of methods that use aggregation operators [42,50,55,61,62]. The second class was comprised of methods based on distance measures [14,52,66,67]. The method described by Wang et al. was in a class by itself [48]. The results of all three classes of methods were compared to the results of the proposed method.

These three classes of methods provide no clarification on the means of resolving situations in which repeated values exist in the evaluation information of alternatives, and the criteria weights are expressed by MHFNs. Under these conditions, a comparative analysis based on an equivalent illustrative example was performed, where each value was only counted once in the decision matrix $R = (a_{ij})_{5\times 4}$. The criteria weight vector $w = (0.375, 0.275, 0.200, 0.150)$ can be calculated according to the score function in Definition 3. When the three classes of methods and the proposed approach were applied to the modified decision-making information, the results were obtained as follows.

**Case 1**. Comparison of the proposed approach to methods that use aggregation operators.

We include five methods proposed in previous studies that developed aggregation operators to aggregate the hesitant information [42,50,55,61,62]. The score function was then calculated and used to determine the final ranking order of all the alternatives. The results of these methods and the proposed method are listed in Table 1.

As shown in Table 1, the proposed approach and the approach described by Wei [42] in which the weighted averaging operator is used and the prioritization among the criteria is $c_1 \succ c_2 \succ c_3 \succ c_4$. Both provided an equivalent ranking with respect to the lowest and highest ranked candidates, and the best alternative was always $a_2$. However, the results of the proposed method were different from those produced by the methods described by Xia and Xu [50], Yu [55], Zhu et al. [61], and Zhang et al. [62].

There are three possible explanations for these differences. First, the different operations and aggregation operators involved in these other methods can be used to interpret the differences in the final rankings to some extent. Second, different aggregation operators are used to address different relationships of the aggregated arguments. The methods described by Wei [42], Xia and Xu [50], Yu [55], and Zhang et al. [62] involved weighted averaging operators that weight the hesitant fuzzy values, and indicate the overall influence of all data. The weighted geometry operators described by Wei [42], Yu [55], Zhu et al. [61], and Zhang et al. [62] may be infeasible for situation in which extreme values are involved, and this is a vital shortcoming for them. The ordered weighted averaging operator described by Zhang et al. weighted the ordered positions of the hesitant fuzzy values [62]. The effectiveness of the geometric Bonferroni means described by Zhu et al. cannot be restricted by extreme values because the importance of each argument and the conjunctions among them have been considered in the aggregation process [61]. Nevertheless, decision-makers cannot make choices among those operators mentioned above, which share similar characteristics.

Moreover, if aggregation operators are used, the number of operations and the magnitudes of the results will increase exponentially if more HFNs are involved in the operations. The deterioration caused by these complexities may limit the application of hesitant fuzzy aggregation operators. Therefore, to resolve the MCDM problem described in Section 6, the proposed approach not only produces reasonable and credible results but also requires only simple computation procedures.

**Case 2**. Comparison of the proposed approach to methods based on distance measures.

We include two methods described by Farhadinia [14] and by Xu and Xia [52] that calculate the distance between each actual alternative and an ideal alternative, which was used to determine the final ranking. In addition, the E-VIKOR and TODIM methods described by Zhang and Wei [66] and by Zhang and Xu [67], respectively, both of which are based on distance, were also used to determine the final ranking of all the distances. These results are listed in Table 2.

According to the results presented in Table 2, the proposed approach and the approach described by Zhang and Wei [66] both provided an equivalent ranking with respect to the lowest, second lowest, and highest ranked candidates, but different from those produced using the methods described by Farhadinia [14], Xu and Xia [52], and Zhang and Xu [67], for which the best alternative was found to be $a_5$.

Two conclusions can be drawn from these results. First, all four methods measure the distance under the condition that all HFNs must be arranged in ascending order and be of equal length. If the two HFNs being compared have different lengths, then the value of the shorter HFN must be increased until both are equal. However, in such cases, different methods of extension can produce different results. Second, the distance measurements are subject to different reference points. In Zhang and Xu's method [67], each alternative can be determined as the reference point in TODIM. The methods described by Farhadinia [14], Xu and Xia [52], and Zhang and Wei [66] all involved an ideal alternative in the decision-making process. As shown here, two cases may lead to different rankings.

Unlike methods that use distance measures, which present various disadvantages in the decision-making process, the proposed approach does not take distances into account. The proposed approach is more suitable for accommodating MCDM problems with multi-valued hesitant fuzzy information.

**Table 1**
Comparison of the proposed method with methods using aggregation operators.

| Methods | Ranking of alternatives |
|---|---|
| Wei [42] | $a_2 \succ a_5 \succ a_3 \succ a_1 \succ a_4$ |
| Xia and Xu [50] | $a_5 \succ a_4 \succ a_1 \succ a_2 \succ a_3$ |
| Yu [55] | $a_1 \succ a_5 \succ a_2 \succ a_4 \succ a_3$ or $a_5 \succ a_1 \succ a_2 \succ a_4 \succ a_3$ |
| Zhu et al. [61] | $a_5 \succ a_4 \succ a_1 \succ a_2 \succ a_3$ |
| Zhang et al. [62] | $a_5 \succ a_2 \succ a_1 \succ a_4 \succ a_3$ or $a_5 \succ a_1 \succ a_2 \succ a_4 \succ a_3$ |
| Proposed method | $a_2 \succ a_1 \succ a_5 \succ a_3 \succ a_4$ |

**Table 2**
Comparison of the proposed method with methods based on distance measures.

| Methods | Ranking of alternatives |
|---|---|
| Farhadinia [14] | $a_5 \succ a_2 \succ a_1 \succ a_4 \succ a_3$ |
| Xu and Xia [52] | $a_5 \succ a_2 \succ a_1 \succ a_3 \succ a_4$ |
| Zhang and Wei [66] | $a_2 \succ a_5 \succ a_1 \succ a_3 \succ a_4$ |
| Zhang and Xu [67] | $a_5 \succ a_1 \succ a_2 \succ a_3 \succ a_4$ |
| Proposed method | $a_2 \succ a_1 \succ a_5 \succ a_3 \succ a_4$ |

**Case 3**. Comparison of the proposed approach to the method described by Wang et al. [48].

When the method described by Wang et al. [48] was used to solve the MCDM problem, the final ranking was $a_4 \prec a_1 \prec a_3 \prec a_5 \prec a_2$. This is consistent with the results of the proposed approach. Although the development of the method was based on reliable theories and was not subject to the disadvantages of other methods, it is not able to manage repetitive values in HFSs. As such, this method cannot be used to directly accommodate MHFSs in MCDM problems.

As indicated by the comparative analyses presented above, the proposed method of addressing MCDM problems with MHFNs demonstrates the following advantages.

The MHFNs used in this paper can express the evaluation information more flexibly. They can retain the completeness of the original data or the inherent thoughts of decision-makers by taking into account repetitive values in HFSs, which is a prerequisite for accurate final outcomes. The main advantages of the approach proposed here are its ability to accommodate preference information expressed by MHFNs effectively, and its ability to accommodate criteria weights in the form of MHFNs.

The proposed outranking method for MHFNs is different from existing methods, which always involve operations and measures whose impact on the final solution may be considerable. The proposed method can overcome these disadvantages. This can prevent loss of data and distortion of the preference information initially provided, resulting in final outcomes that more closely correspond to those in actual decision-making processes.

## 7. Conclusions

HFSs are useful for managing decision-making problems that are defined under uncertainties for which decision-makers hesitate while considering several values before expressing their preferences concerning weights and data. MHFSs are applicable to cases in which some HFS values are repeated. In this paper, a comparison method for MHFNs is discussed. Some outranking relations with MHFNs are proposed, and their properties, derived from ELECTRE III, are presented in detail. An outranking method that can overcome the disadvantages of traditional methods is proposed to manage MCDM problems where the weights and data are in the form of MHFNs. The primary advantages of the developed approach over other methods are not only its ability to retain the preference information expressed by MHFNs, but also its expression of criteria weights by MHFNs. This can avoid loss of data and distortion of the preference information initially provided, resulting in final outcomes that more closely correspond to those in actual decision-making processes. Future research may address the means of establishing optimal values of indifference, preference, and veto thresholds in the ELECTRE methods using a specified model under a multi-hesitant fuzzy environment.

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