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Interfaces with Other Disciplines

Theory of integer-valued data envelopment analysis

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Abstract

Conventional data envelopment analysis (DEA) models assume real-valued inputs and outputs. In many occasions, some inputs and/or outputs can only take integer values. In some cases, rounding the DEA solution to the nearest whole number can lead to misleading efficiency assessments and performance targets. This paper develops the axiomatic foundation for DEA in the case of integer-valued data, introducing new axioms of "natural disposability" and "natural divisibility". We derive a DEA production possibility set that satisfies the minimum extrapolation principle under our refined set of axioms. We also present a mixed integer linear programming formula for computing efficiency scores. An empirical application to Iranian university departments illustrates the approach.

Keywords: Data envelopment analysis; Efficiency; Mixed Integer linear programming; Production theory

1. Introduction

Data envelopment analysis (DEA) is a mathematical programming approach for evaluating performance of decision making units (DMUs) that convert multiple inputs into multiple outputs. Conventional DEA models assume real-valued inputs and outputs. However, there are many occasions in which some inputs and/or outputs must only take integer values. For example, in efficiency evaluation of university departments, such inputs as the number of professors and such outputs as the number of published articles are restricted to the whole numbers. While the rounding of performance targets to the nearest whole number does not necessarily make a big difference for large departments, for small departments it can be a major issue. For example, suppose a department has 3 full professors, and the DEA analysis suggests the efficient level of professors is 2.4. Such result raises a dilemma: there is no evidence that 2 professors would suffice to meet the educational and scientific objectives of the department, but rounding 2.4 up to 3 does not save any resources even though the efficiency score of the department is only 0.8.

The need to deal with integer-valued data in DEA naturally occurs when one uses categorical or ordinal data (Banker and Morey, 1986; Kamakura, 1988; Rousseau and Semple, 1993; among others), but restricting to the whole numbers can be important even when the input–output variables are defined on the interval or ratio scales. Lozano and Villa (2006) were the first to address the issue at a more general level, proposing a mixed integer linear programming (MILP) DEA model to guarantee the required integrality of the computed targets. However, this pioneering article has two major shortcomings. First, the theoretical foundation of Lozano and Villa's model is ambiguous. Clearly, assuming integer-valued inputs and outputs immediately violates the standard convexity, free disposability and returns to scale properties of DEA. Thus, Lozano and Villa's model is not consistent with the minimum extrapolation principle (Banker et al., 1984), which is the foun-

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dation of all DEA models. Second, Lozano and Villa's MILP formulation for computing efficiency scores can lead to overestimated efficiency results, as shown below by numerical examples.

This paper tackles both these problems. We develop a new axiomatic foundation for integer-valued DEA models, and show that the production possibility set proposed by Lozano and Villa (2006) is consistent with the proposed set of axioms. We also proposed a modification of the classic Farrell input efficiency measure, and derive a MILP formulation for computing it.

The rest of the paper is organized as follows. We start by introducing the new notions of "natural disposability" and "natural divisibility" in the next section. In Section 3 we derive the associated DEA production sets that satisfy the fundamental minimum extrapolation principle (Banker et al., 1984). Section 4 generalizes the method to the hybrid case where both real and integer-valued inputs and outputs are present. In Section 5 we adapt the Farrell input efficiency measure to the integer DEA setting, and show how the efficiency score can be computed by solving a MILP problem, which differs from that of Lozano and Villa (2006) in two important respects. An application to the efficiency evaluation of 42 departments of the Islamic Azad University, Karaj Branch (IAUK) in Iran illustrates the method in Section 6. Section 7 presents our concluding remarks and suggests avenues for future research.

2. Axioms

In DEA, each observed DMU is characterized by a pair of non-negative input and output vectors $(\mathbf{X}_j, \mathbf{Y}_j) \in \mathbb{R}_+^{m+s}$, $j \in J = \{1, \ldots, n\}$. The classic Charnes et al. (1978) DEA model assumes that the underlying production possibility set, denoted by $T = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbb{R}_+^m \text{ can produce } \mathbf{y} \in \mathbb{R}_+^s\}$, satisfies the following axioms:

(A1) *Envelopment*: $(\mathbf{X}_j, \mathbf{Y}_j) \in T\ \forall j \in J$.
(A2) *Free disposability*: $(\mathbf{x}, \mathbf{y}) \in T$ and $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}_+^{m+s}$, $\mathbf{y} \geqslant \mathbf{v}$, $\Rightarrow (\mathbf{x} + \mathbf{u}, \mathbf{y} - \mathbf{v}) \in T$.
(A3) *Constant returns to scale*: $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\lambda\mathbf{x}, \lambda\mathbf{y}) \in T\ \forall \lambda \in \mathbb{R}_+$.
(A4) *Convexity*: $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in T, (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \lambda(\mathbf{x}, \mathbf{y}) + (1 - \lambda)(\mathbf{x}', \mathbf{y}'), 0 \leqslant \lambda \leqslant 1 \Rightarrow (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in T$.

According to the *minimum extrapolation principle* (Banker et al., 1984), the DEA production possibility set (PPS) is the intersection of all sets $S \subset \mathbb{R}_+^{m+s}$ that satisfy the maintained axioms. Under the maintained assumptions (A1)–(A4), the minimum extrapolation PPS can be explicitly stated as

$$T_{\text{DEA}} = \left\{ (\mathbf{x}, \mathbf{y}) \middle| \mathbf{x} \geqslant \sum_{j=1}^{n} \mathbf{X}_j \lambda_j;\ \mathbf{y} \leqslant \sum_{j=1}^{n} \mathbf{Y}_j \lambda_j;\ \boldsymbol{\lambda} \geqslant 0 \right\}.$$

Many variations of axioms (A2)–(A4) have been presented in the literature. Relaxation of (A2) leads to models of weak disposability (e.g. Kuosmanen, 2005) and congestion (e.g. Cherchye et al., 2001). Relaxation of (A3) leads to models of variable and non-increasing (decreasing) returns to scale (e.g. Seiford and Thrall, 1990). Relaxation of (A4) leads to free disposable hull (Deprins et al., 1984) and free replicable hull models (Tulkens, 1993). All these variants assume real-valued data.

Consider a situation where all input–output data are restricted to be integer-valued (in Section 4 we broaden the scope to the more general setting that allows for both integer and real-valued input–output variables). Formally, we impose an additional axiom

(A5) *Integrality*: $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_+^{m+s}$.

It is easy to see that (A5) contradicts axioms (A2)–(A4). Therefore, the earlier attempts to deal with integer-valued data in DEA are not axiomatically sound.

To deal with integer-valued data in a systematic fashion, an alternative set of axioms is needed. To this end, we note that when (A3) holds, the convexity postulate (A4) can be harmlessly replaced by a weaker postulate

(B4) *Additivity*: $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in T \Rightarrow (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}') \in T$.

Additivity axiom (B4) is consistent with (A5). Moreover, axioms (A3) and (B4) together imply (A3) and (A4) (e.g. Arrow and Hahn, 1971). Therefore, the main problem with the integer data actually concerns the disposability and scaling properties (A2) and (A3). We propose to substitute these axioms by the following alternative properties:

(B2) *Natural disposability*: $(\mathbf{x}, \mathbf{y}) \in T$ and $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_+^{m+s}$, $\mathbf{y} \geqslant \mathbf{v}$, $\Rightarrow (\mathbf{x} + \mathbf{u}, \mathbf{y} - \mathbf{v}) \in T$.
(B3) *Natural divisibility*: $(\mathbf{x}, \mathbf{y}) \in T$ and $\exists \lambda \in [0, 1] : (\lambda\mathbf{x}, \lambda\mathbf{y}) \in \mathbb{Z}_+^{m+s} \Rightarrow (\lambda\mathbf{x}, \lambda\mathbf{y}) \in T$.

The notions of *natural disposability* and *natural divisibility* are new variants of the standard free disposability (A2) and non-increasing returns to scale axioms. Intuitively, natural disposability differs from free disposability only in that the disposed input–output quantities (vector $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}_+^{m+s}$) must be integer-valued. Similarly, natural divisibility introduces an additional restriction that downsizing a production plan must result an integer-valued input–output vector. For example, if two units of input can produce four units of output, then one unit of input can produce two outputs, and four units of input can produce eight units of output. However, it is not possible to produce just one unit of output because that would require a half unit of input, and by assumption, inputs cannot be halved. By construction, axioms (B2) and (B3) are consistent with integrality (A5).

Our next objective is to characterize a PPS that satisfies the minimum extrapolation principle subject to the properties (A1), (A5) and (B2)–(B4). From the outset, this is a challenging task because properties (B2)–(B4) can be applied sequentially one after another. In the spirit of Bogetoft et al. (2000), one could try to construct the PPS by using the admissible addition, scaling and disposability properties (B2)–(B4) sequentially, but this will generally require an infinite sequence of operations.

To illustrate this, consider a simple numerical example in a single-input single-output case with only a single observation $(X, Y) = (17, 7)$. By applying natural disposability (B2), we already obtain an infinite number of new points. Point $(18, 6)$ is one of them. Applying natural divisibility (B3), we obtain point $(3, 1) = (18, 6)/6$. By additivity (B4), we have $(17, 7) + (3, 1) = (20, 8)$. Applying natural divisibility again, we get $(5, 2) = (20, 8)/4$. By additivity, $(5, 2) + (3, 1) = (8, 3)$. The procedure can be continued indefinitely.

This example aptly illustrates that a sequential application of the axioms can generate new feasible points that are not achievable by applying the axioms just once. For example, point $(8, 3)$ is not directly achievable from observation $(17, 7)$ by any of the axioms, but it can be achieved through a sequential application of feasible operations. It is also clear from this example that there exist an infinite number of admissible operations, so it is not possible to enumerate all feasible points in a finite time through such sequential algorithm. Even if one is interested in checking feasibility of a given unobserved integer-valued input–output vector, one may not be able to verify infeasibility in a finite number of operations, because the additivity axiom (B4) can be used for generating an arbitrarily large sum vector $(\mathbf{X}, \mathbf{Y})$ which can be subsequently scaled down to the evaluated point by applying the natural divisibility property (B3).

3. Main result

Lozano and Villa (2006) proposed to measure efficiency relative to the production possibility set that consists of integer-valued production plans. If all inputs and outputs are integer-valued, their reference technology can be formally stated as

$$T_{\text{IDEA}} = \left\{ (\mathbf{x},\mathbf{y}) \in \mathbb{Z}_+^{m+s} \,\middle|\, \mathbf{x} \geqslant \sum_{j=1}^{n} \mathbf{X}_j\lambda_j;\ \mathbf{y} \leqslant \sum_{j=1}^{n} \mathbf{Y}_j\lambda_j;\ \lambda_j \geqslant 0\ \forall j \right\}.$$

This set consists of all integer-valued input–output vectors that are contained by T_{DEA}. However, this set is inconsistent with the minimum extrapolation principle of DEA under the standard set of axioms (A1)–(A5); for example, T_{IDEA} is not convex. Fortunately, we can establish the minimum extrapolation interpretation for T_{IDEA} under our adapted set of axioms.

Theorem 1. *Under axioms* (A1), (A5), (B2), (B3), and (B4), T_{IDEA} *is the minimum extrapolation production possibility set.*

Importantly, this theorem sets the intuitive PPS proposed by Lozano and Villa (2006) on a firmer theoretical ground. We next generalize this result to a hybrid setting where there are both integer and real-valued inputs and outputs.

4. Generalization

Consider next the more general setting where only some of the inputs and outputs are deemed to satisfy the integrality assumption (A5), while the others are not. Following Lozano and Villa (2006), we partition the set of input variables as $I = I^I \cup I^{NI}$ and the set of output variables as $O = O^I \cup O^{NI}$, where subsets I^I and O^I are subject to the integrality condition (A5) and subsets I^{NI} and O^{NI} are real-valued. Subsets I^I and I^{NI}, as well as O^I and O^{NI}, are assumed to be mutually disjoint, and $|I^I| = p \leqslant m$ and $|O^I| = q \leqslant s$. Based on the preceding notations, every feasible activity which is characterized by a pair of non-negative input and output vectors $(\mathbf{x}, \mathbf{y})$ can be written as $\mathbf{x} = \begin{pmatrix} \mathbf{x}^I \\ \mathbf{x}^{NI} \end{pmatrix}, \mathbf{y} = \begin{pmatrix} \mathbf{y}^I \\ \mathbf{y}^{NI} \end{pmatrix}$.

In this hybrid setting, Lozano and Villa (2006) proposed a DEA PPS that can be stated as

$$T_{\text{HIDEA}} = \left\{ \begin{pmatrix} \mathbf{x}^I & \mathbf{y}^I \\ \mathbf{x}^{NI}, & \mathbf{y}^{NI} \end{pmatrix} \middle| (\mathbf{x}^I, \mathbf{y}^I) \in \mathbb{Z}_+^{p+q}; \begin{pmatrix} \mathbf{x}^I \\ \mathbf{x}^{NI} \end{pmatrix} \geqslant \sum_{j=1}^{n} \begin{pmatrix} \mathbf{X}_j^I \\ \mathbf{X}_j^{NI} \end{pmatrix} \lambda_j; \begin{pmatrix} \mathbf{y}^I \\ \mathbf{y}^{NI} \end{pmatrix} \leqslant \sum_{j=1}^{n} \begin{pmatrix} \mathbf{Y}_j^I \\ \mathbf{Y}_j^{NI} \end{pmatrix} \lambda_j; \lambda_j \geqslant 0 \ \forall j \right\}.$$

The axiomatic foundation established in Theorem 1 can be extended to this PPS by applying different sets of axioms to the subsets (I^I, O^I) and (I^{NI}, O^{NI}), respectively.

Theorem 2. *If subsets* (I^{NI}, O^{NI}) *satisfy axioms* (A2)–(A4), *subsets* (I^I, O^I) *satisfy* (A5), (B2), (B3), and (B4), *and axiom* (A1) *is jointly satisfied by the observed data, then* T_{HIDEA} *is the minimum extrapolation production possibility set.*

5. Efficiency measurement

We now turn to efficiency estimation of DMUs. It is worth to note that the common efficiency measures (including the radial Farrell input and output measures, the additive Pareto–Koopmans efficiency measures, and the directional distance functions) all assume continuous real-valued data. For brevity, we here restrict attention to the classic Farrell input efficiency measure, defined as

$$\text{Eff}(\mathbf{x}_0, \mathbf{y}_0) = \min\{\theta | (\theta \mathbf{x}_0, \mathbf{y}_0) \in T\},$$

where vector $(\mathbf{x}_0, \mathbf{y}_0)$ refers to the (observed or hypothetical) DMU under evaluation. Applying this measure directly to T_{IDEA} can yield counter-intuitive results because T_{IDEA} is a nonmonotonic and nonconvex set of disconnected points. For example, it is possible that $(\mathbf{x}_0, \mathbf{y}_0)$ that is strictly dominated by another point in T_{IDEA} is assigned the efficiency score 1 associated with full efficiency. The following numerical example illustrates.

Consider a two-input single-output case where there are two efficient DMUs A and B with $(X_1^A, X_2^A, Y^A) = (7, 1, 1)$ and $(X_1^B, X_2^B, Y^B) = (2, 4, 1)$. Let us evaluate efficiency of DMU 0 with $(x_1^0, x_2^0, y^0) = (9, 4, 1)$. Fig. 1 illustrates the example graphically. The thick black piece-wise linear frontier represents the DEA CRS frontier. Projection path from DMU 0 towards the origin is indicated by a broken line. Some feasible input vectors are indicated by black circle, and some infeasible points are indicated by white circles.

The example illustrates the need to modify the input efficiency measure in the case of integer DEA. No feasible integer-valued point coincides on the line segment between DMU 0 and the origin. The example also illustrates that simply rounding the DEA CRS efficiency score can give misleading results. Solving the DEA CRS model, we obtain the input efficiency of 0.553, and the reference point $(\hat{x}_1, \hat{x}_2) = (4.979, 2.213)$. If we round to the nearest integer (5, 2) (point indicated by a white circle), we end up to an infeasible point. If we round both inputs upward, we end up to a feasible point (5, 3), but this yields an efficiency score of 0.75 which is unnecessarily high. Note that the rounding error of one unit of input can have a major impact on the efficiency score. In the example of Fig. 1, suppose we evaluate efficiency of DMU C with inputs (4, 4) producing 1 unit of output. Rounding the DEA efficiency score to the nearest integer would give the reference

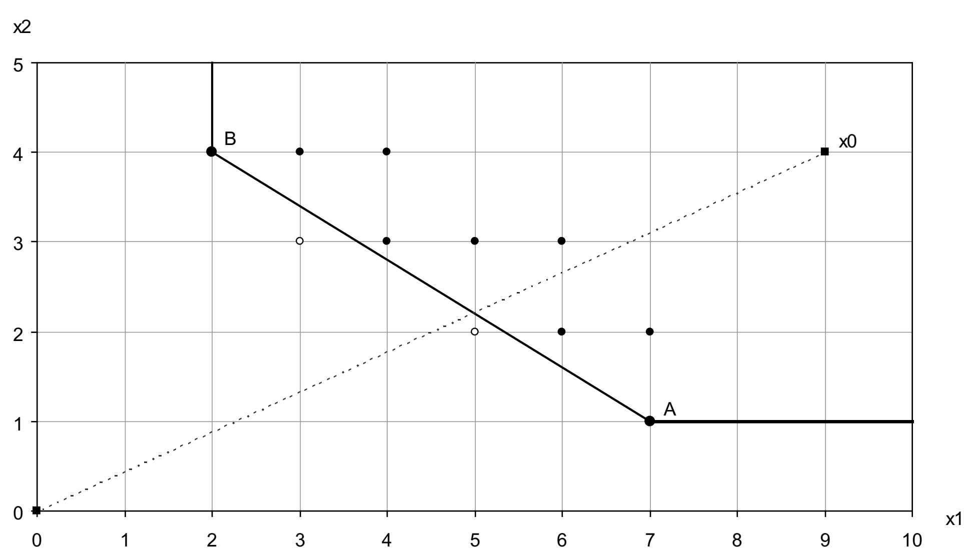

Fig. 1. Illustration of the numerical example.

point (3, 3), and the associated input efficiency of 0.75. However, point (3, 3) is not feasible. In fact, no strictly dominating feasible input vector exists in this example.

To measure efficiency improvement potential in integer variables, a modified input efficiency measure is needed. We propose to modify the Farrell input efficiency measure as

$$\text{Eff}^{+}(\mathbf{x}_0,\mathbf{y}_0)=\min\{\theta\in\mathbb{R}_+|\exists(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in T:\tilde{\mathbf{x}}^I\in\mathbb{Z}_+^p;\theta\mathbf{x}_0\geqslant\tilde{\mathbf{x}};\mathbf{y}_0\leqslant\tilde{\mathbf{y}}\}.$$

This modified measure gauges radial distance to the monotonic hull of the production possibility set, requiring that the reference point $(\theta\mathbf{x}_0,\mathbf{y}_0)$ is such that $(\theta\mathbf{x}_0^I,\mathbf{y}_0^I)$ is integer-valued. This preserves the usual interpretation of the Farrell measure as a downward scaling potential in inputs at the given output level, and guarantees that DMUs assigned the efficiency score one are weakly efficient in the Pareto–Koopmans sense.

The modified input efficiency scores relative to the general T_{HIDEA} reference technology can be computed by solving the following MILP problem, which is computable by standard MILP algorithms and solver software

$$\begin{aligned}
\text{Eff}^{+}(\mathbf{x}_0,\mathbf{y}_0)=\min_{\theta,\boldsymbol{\lambda},\tilde{\mathbf{x}},\mathbf{s}}\quad & \theta-\varepsilon\left(\sum_{r=1}^{s}s_r^{+}+\sum_{i=1}^{m}s_i^{-}+\sum_{i=1}^{p}s_i^{I}\right),\\
\text{s.t.}\quad & y_{ro}+s_r^{+}=\sum_{j=1}^{n}y_{rj}\lambda_j,\quad r\in O,\\
& \theta x_{io}-s_i^{-}=\sum_{j=1}^{n}x_{ij}\lambda_j,\quad i\in I^{NI},\\
& \tilde{x}_i-s_i^{-}=\sum_{j=1}^{n}x_{ij}\lambda_j,\quad i\in I^{I},\\
\text{(IDEA)}\qquad & \theta x_{io}-s_i^{I}=\tilde{x}_i,\quad i\in I^{I},\\
& \tilde{x}_i\in\mathbb{Z}_+,\quad i\in I^{I},\\
& \lambda_j\geqslant 0,\quad j\in J,\\
& s_r^{+}\geqslant 0,\ s_i^{-}\geqslant 0,\ s_j^{I}\geqslant 0,\quad r\in O,\ i\in I,\ j\in I^{I}.
\end{aligned}$$

Symbol ε denotes a non-Archimedean infinitesimal, variables s_r^{+},s_i^{-},s_r^{I} represent the nonradial slacks, and $\tilde{\mathbf{x}}\in\mathbb{Z}_+^p$ is the integer-valued reference point for inputs $\boldsymbol{I}^I$.

Theorem 3. *For any non-negative data* $(\mathbf{x}_0,\mathbf{y}_0)$, $(\mathbf{X},\mathbf{Y})$, *the optimal* θ^* *of model* (*IDEA*) *is equal to the modified Farrell input efficiency measure* $\text{Eff}^{+}(\mathbf{x}_0,\mathbf{y}_0)$ *defined with respect to the* T_{HIDEA} *reference technology.*

It is worth to note that model (IDEA) distinguishes between two types of input slacks: slacks s_i^{-} represent the difference between the convex combination $\sum_{j=1}^{n}x_{ij}\lambda_j$ and the reference point $(\tilde{\mathbf{x}},\theta\mathbf{x}_0^{NI})\in\mathbb{Z}_+^p\times\mathbb{R}_+^{s-p}$, while input slacks s_i^I represent the difference between the reference point and the projection $(\theta\mathbf{x}_0)\in\mathbb{R}_+^s$ in the subset $\boldsymbol{I}^I$. In the example of Fig. 1, the reference point of DMU 0 is $\tilde{\mathbf{x}}=(6,2)$ with slacks $\mathbf{s}^{-}=\left(\frac{2}{3},0\right)$. Radial input efficiency is $\theta=\frac{2}{3}$ and the nonradial slack $\mathbf{s}^I=\left(0,\frac{2}{3}\right)$.

It is worth to emphasize that model (IDEA) differs from the MILP formulation by Lozano and Villa (2006) in one very important respect. While our model (IDEA) only requires that vector $\left(\sum_{j=1}^{n}\mathbf{X}_j^I\lambda_j,\sum_{j=1}^{n}\mathbf{Y}_j^I\lambda_j\right)$ dominates our integer-valued reference point $(\tilde{\mathbf{x}},\mathbf{y}_0^I)\in\mathbb{Z}_+^{p+q}$ (consistent with Theorem 1), Lozano and Villa's MILP problem impose a more stringent requirement that $\left(\sum_{j=1}^{n}\mathbf{X}_j^I\lambda_j,\sum_{j=1}^{n}\mathbf{Y}_j^I\lambda_j\right)$ itself must be integer-valued. As a result, the intensity weights λ_j need not be optimal.

Consider again the example of Fig. 1. The reference point $\tilde{\mathbf{x}}=(6,2)$ is achievable as a convex combination of observations, but this requires assigning a positive weight to the inefficient DMU 0. Suppose the example includes a third input, which is real-valued, and assume $x_{A,3}=x_{B,3}=2$ and $x_{0,3}=3$. It is easy to see that assigning a positive weight to the inefficient DMU 0 will directly increase the input efficiency measure above the value of $\frac{2}{3}$ found above. Therefore, Lozano and Villa's MILP formulation will overestimate the efficiency score in this modified example. The application of the next section demonstrates that the two MILP formulations can yield substantially different results even when all inputs and outputs are integer-valued.

6. Application

We next illustrate the integer DEA model by applying it to the real-world data of 42 university departments of IAUK. These data are used for the internal performance assessment by the university. The input variables are the number of post graduate students (x_1), the number bachelor students (x_2), and the number of master students (x_3). The output variables are the number of graduations (y_1), the number of scholarships (y_2), the number of research products (y_3), and the level of manager satisfaction (y_4). Note that all variables have integer structure and y_4 is an ordinal-scale variable. The full data are presented in the first eight columns of Table 1.

For comparison, four alternative models were computed: (1) model (IDEA) described above, (2) Lozano and Villa's integer DEA model, (3) the conventional DEA CRS model, and (4) the free disposable hull (FDH) model. We used Lingo 7 software on an Intel 4, 256 Mbytes RAM, 2 GHz PC. The computational times for the DEA CRS and FDH models were negligible, and for the integer DEA models just a few seconds per MILP problem.

The obtained radial input efficiency scores are presented in the last four columns of Table 1. The results of our integer DEA model come close to those of Lozano and Villa's and DEA CRS models, but there are some notable differences, particularly with DMUs 42, 20, and 41. It is worth to stress that the non-Archimedean ε does not play any role in these results: all models were computed using the two-stage method where the radial input efficiency component (θ) is minimized in the first stage and the non-radial slacks (**s**) are maximized in the second stage. In general, the efficiency scores of our integer DEA model must be always larger than those of DEA CRS model, but smaller than those of Lozano and Villa's model. Table 1 confirms this. The FDH model suggests generally much higher efficiency level than the other two models. This suggests that the additivity and natural divisibility axioms can enhance the discriminatory power of the model considerably.

Table 1
Data and efficiency scores

DMU	x_1	x_2	x_3	y_1	y_2	y_3	y_4	Integer DEA	Lozano & Villa	DEA CRS	FDH
1	0	261	0	225	1	1	3	0.881	0.881	0.880	1.000
2	0	170	56	213	2	0	3	0.964	0.977	0.956	1.000
3	0	281	70	326	2	0	3	0.943	0.947	0.940	1.000
4	0	138	33	159	1	0	2	0.942	0.964	0.941	1.000
5	164	0	0	52	1	0	3	1.000	1.000	1.000	1.000
6	291	815	0	1014	2	2	2	0.918	0.918	0.917	1.000
7	0	0	61	50	0	0	4	1.000	1.000	1.000	1.000
8	113	95	0	73	0	0	2	0.495	0.513	0.487	1.000
9	0	727	0	675	3	0	3	0.928	0.931	0.928	1.000
10	0	773	0	697	2	0	3	0.902	0.904	0.902	1.000
11	0	0	66	46	0	0	3	0.758	0.758	0.758	0.924
12	346	197	0	132	0	0	1	0.266	0.274	0.264	0.629
13	0	988	0	812	8	10	2	0.883	0.885	0.882	1.000
14	0	0	34	32	0	0	2	1.000	1.000	1.000	1.000
14	0	795	0	601	6	2	2	0.758	0.764	0.758	1.000
16	0	672	0	591	6	12	2	1.000	1.000	1.000	1.000
17	0	166	0	166	7	0	4	1.000	1.000	1.000	1.000
18	0	761	0	761	0	3	2	1.000	1.000	1.000	1.000
19	193	124	0	293	0	0	3	1.000	1.000	1.000	1.000
20	484	0	0	361	0	0	1	0.893	0.998	0.892	1.000
21	0	517	0	434	0	4	2	0.880	0.880	0.879	1.000
22	0	584	0	492	1	4	2	0.875	0.875	0.874	1.000
23	0	682	0	565	2	3	2	0.840	0.841	0.840	0.985
24	0	565	0	423	1	2	2	0.758	0.758	0.756	1.000
25	0	603	0	433	1	3	2	0.740	0.740	0.738	0.969
26	0	373	0	332	1	1	1	0.890	0.895	0.890	1.000
27	0	347	0	328	2	3	3	0.997	0.997	0.996	1.000
28	0	0	70	51	0	3	4	1.000	1.000	1.000	1.000
29	0	328	0	170	0	1	3	0.540	0.543	0.539	0.796
30	0	267	0	123	0	0	3	0.468	0.498	0.466	0.622
31	262	0	0	219	3	0	3	1.000	1.000	1.000	1.000
32	0	1023	0	794	2	0	4	0.776	0.780	0.776	1.000
33	366	995	0	1111	2	2	3	0.817	0.819	0.816	1.000
34	0	266	15	238	3	4	3	0.951	0.955	0.949	1.000
35	172	375	0	547	4	3	3	1.000	1.000	1.000	1.000
36	0	460	0	385	4	8	3	1.000	1.000	1.000	1.000
37	223	0	535	232	14	6	4	1.000	1.000	1.000	1.000
38	0	1202	58	1158	12	0	3	0.923	0.924	0.922	1.000
39	0	1025	61	394	4	1	3	0.365	0.367	0.364	1.000
40	0	0	69	50	0	2	4	0.971	0.986	0.971	1.000
41	314	0	0	204	0	0	1	0.780	0.834	0.777	0.834
42	371	0	0	226	0	0	1	0.730	0.868	0.729	1.000

Table 2 presents the integer-valued input targets (i.e., $\tilde{\mathbf{x}}$) of our model (IDEA), Lozano and Villa's integer DEA model, and the conventional DEA CRS model. Consider first the input targets obtained by simply rounding the DEA CRS targets. For 12 departments (29% of the sample), rounding the DEA CRS benchmark to the nearest integer results a different target point from that of our integer DEA model. The result was somewhat better when we rounded the DEA CRS benchmarks upward: 8 departments (or 19% of the sample) were projected to a different target point. Especially for small departments, rounding of the conventional DEA benchmarks can give over-optimistic (or pessimistic) performance goals.

The interpretation of our integer DEA model can be illustrated by considering a specific university department, say DMU 13. The integer DEA efficiency score of this DMU was 0.883, obtained with the intensity weights $\lambda_{16}^{*} = 0.7376378$, $\lambda_{17}^{*} = 0.510961$, and $\lambda_{18}^{*} = 0.3827820$ (the weights of all other DMUs are equal to zero). These weights yield the reference input vector $\sum_{j=1}^{42} \mathbf{x}_j \lambda_j = (0, 871.809, 0)$, which dominates our integer-valued input target $\tilde{\mathbf{x}} = (0, 872, 0)$; see Table 2. Theorem 1 implies that $\tilde{\mathbf{x}} = (0, 872, 0)$ is a feasible target. Note that Lozano and Villa's MILP formulation yields input target (0, 874, 0), with two units higher target value for input 2. The result is due to the fact that it is impossible to find intensity weights λ that satisfy both $\sum_{j=1}^{42} \mathbf{x}_j \lambda_j = (0, 872, 0)$ and constraints $\sum_{j=1}^{42} \mathbf{y}_j \lambda_j \in \mathbb{Z}_{+}^{4}, \sum_{j=1}^{42} \mathbf{y}_j \lambda_j \geqslant (812, 8, 10, 12)$ simultaneously.

Table 2
Input targets according to Integer DEA, Lozano & Villa's and DEA CRS models

DMU	Integer DEA			Lozano & Villa			DEA CRS		
	x_1^*	x_2^*	x_3^*	x_1^*	x_2^*	x_3^*	x_1^*	x_2^*	x_3^*
1	0	230	0	0	230	0	0	229.6	0
2	0	163	54	0	166	54	0	162.6	53.6
3	0	264	66	0	266	66	0	264.1	65.8
4	0	130	31	0	133	31	0	129.8	31
5	164	0	0	164	0	0	164	0	0
6	267	747	0	267	748	0	266.8	747.2	0
7	0	0	61	0	0	61	0	0	61
8	55	47	0	58	48	0	55.1	46.3	0
9	0	675	0	0	677	0	0	675.1	0
10	0	697	0	0	699	0	0	697	0
11	0	0	50	0	0	50	0	0	50
12	92	52	0	94	54	0	91.2	51.9	0
13	0	872	0	0	874	0	0	871.7	0
14	0	0	34	0	0	34	0	0	34
14	0	603	0	0	607	0	0	602.5	0
16	0	672	0	0	672	0	0	672	0
17	0	166	0	0	166	0	0	166	0
18	0	761	0	0	761	0	0	761	0
19	193	124	0	193	124	0	193	124	0
20	432	0	0	483	0	0	431.9	0	0
21	0	455	0	0	455	0	0	454.3	0
22	0	511	0	0	511	0	0	510.2	0
23	0	573	0	0	574	0	0	572.6	0
24	0	428	0	0	428	0	0	427.1	0
25	0	446	0	0	446	0	0	445.1	0
26	0	332	0	0	334	0	0	332	0
27	0	346	0	0	346	0	0	345.4	0
28	0	0	70	0	0	70	0	0	70
29	0	177	0	0	178	0	0	176.6	0
30	0	125	0	0	133	0	0	124.5	0
31	262	0	0	262	0	0	262	0	0
32	0	794	0	0	798	0	0	793.9	0
33	299	812	0	299	815	0	298.7	812.2	0
34	0	253	14	0	254	14	0	252.3	14.2
35	172	375	0	172	375	0	172	375	0
36	0	460	0	0	460	0	0	460	0
37	223	0	535	223	0	535	223	0	535
38	0	1109	53	0	1110	53	0	1108	53.4
39	0	374	22	0	376	22	0	373.1	22.2
40	0	0	67	0	0	68	0	0	67
41	245	0	0	262	0	0	244.0	0	0
42	271	0	0	322	0	0	270.4	0	0

Although the efficiency scores of our integer DEA model come on the average very close to those obtained by Lozano and Villa's model, the differences can be rather substantial for benchmarking and target setting. The benchmarks obtained by the two integer DEA models are different for 29 DMUs (69% of the sample). For some DMUs (in particular DMUs 42, 20, 32, 41, and 9) the differences are rather substantial, up to 51 units. An anonymous reviewer of this journal correctly pointed out that the input targets may not be always unique, and thus there may exist alternative, equally valid input targets. In practice, we expect such alternative input targets to be unlikely in the present setting. Moreover, this is by no means a special feature of the proposed model: all DEA models (including that of Lozano and Villa) are subject to alternate optima in one form or another, which should be born in mind when interpreting the results. Finally, we would like to emphasize that this application only includes integer-valued data; we would expect the differences to be even greater in a hybrid setting involving both integer and real-valued variables.

7. Conclusions

The conventional axioms of convexity and free disposability fail if DMUs are restricted to operate with integer-valued input and output quantities. In this paper we have presented an axiomatic foundation for a DEA model that assumes subsets of input and output variables to be integer-valued. After modifying the notions of convexity and free disposability axioms by introducing the new notions of "natural disposability" and "natural divisibility", we showed that the production possibility set proposed by Lozano and Villa (2006) satisfies the minimal extrapolation principle under the new set of axioms. We also modified the Farrell input efficiency measure to take into account the possibility of integer-valued inputs and outputs, and presented a MILP formulation for computing it. Our MILP formulation differs from that of Lozano and Villa (2006) in that we do not restrict the convex combination $\left(\sum_{j=1}^{n}\mathbf{X}_j^I\lambda_j, \sum_{j=1}^{n}\mathbf{Y}_j^I\lambda_j\right)$ to be integer-valued, but only require dominance by this vector.

An empirical efficiency evaluation of 42 university departments further illustrated the importance of dealing with integer-valued data, and the differences resulting form alternative model formulations. The application demonstrates that the MILP formulation by Lozano and Villa can underestimate the production possibility set, leading to overestimated efficiency assessment and inefficient targets. The application also showed that simply rounding the DEA CRS results to the nearest integer-valued point results as infeasible performance targets for many DMUs.

This paper has restricted to the DEA setting characterized by constant returns to scale (additivity and natural divisibility of integer-valued inputs and outputs), similar to Lozano and Villa (2006). Later work by Lozano and Villa (2007) has extended the approach to variable returns to scale technology and alternative non-radial and additive efficiency measures. Unfortunately, the axiomatic foundation presented in this paper does not easily generalize to the variable returns to scale setting where the additivity and natural divisibility axioms are not valid. We consider the development of the axiomatic basis for integer DEA in the variable returns to scale environment as an interesting challenge for future research.

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Appendix. Proofs of Theorems

Theorem 1

Denote the true minimum extrapolation set subject to axioms (A1), (A5), (B2), (B3), and (B4), by T_{true}. We need to show that $T_{\text{IDEA}} = T_{\text{true}}$. Since axioms (A2)–(A4) imply (but are not implied by) (B2)–(B4), we must have

(i) $T_{\text{true}} \subset T_{\text{DEA}}$,

Moreover, it is straightforward to verify that

(ii) $T_{IDEA} = T_{DEA} \cap \mathbb{Z}_+^{m+s}$.

Consider an arbitrary $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_+^{m+s}$ such that $(\mathbf{x}, \mathbf{y}) \notin T_{\text{IDEA}}$. Equality (ii) implies that $(\mathbf{x}, \mathbf{y}) \notin T_{\text{DEA}}$. By (i), $(\mathbf{x}, \mathbf{y}) \notin T_{\text{true}}$. Thus,

(iii) $(\mathbf{x},\mathbf{y}) \notin T_{\mathrm{IDEA}} \Rightarrow (\mathbf{x},\mathbf{y}) \notin T_{\mathrm{true}}$.
We next need to show that the converse also holds.Consider now an arbitrary $(\mathbf{x},\mathbf{y}) \in \mathbb{Z}_{+}^{m+s}$ such that $(\mathbf{x},\mathbf{y}) \in T_{\mathrm{IDEA}}$. Therefore, $\exists \boldsymbol{\lambda} \geqslant \mathbf{0}$ such that

(iv) $\mathbf{x} \geqslant \sum_{j=1}^{n} \mathbf{X}_j \lambda_j,$

(v) $\mathbf{y} \leqslant \sum_{j=1}^{n} \mathbf{Y}_j \lambda_j.$

We next multiply both sides of inequalities (iv) and (v) by some real number $r \in \Re_{+}$ such that

(vi) $r\mathbf{x} \geqslant \sum_{j=1}^{n} \mathbf{X}_j (r\lambda_j),$

(vii) $r\mathbf{y} \leqslant \sum_{j=1}^{n} \mathbf{Y}_j (r\lambda_j)$

and that $r\lambda_j \in \mathbb{Z}_{+} \forall j \in J$. To see that such multiplication is always possible, suppose $\lambda_j \in \Re_{+}$ has $k \in \mathbb{Z}_{+}$ decimal digits. Multiplying by $r = 10^{k+1} \in \mathbb{R}_{+}$ will ensure that the product $r\lambda_j$ is an integer. Multiplier r can be arbitrarily large.Since $r\lambda_j$ is an integer, we can obtain the point $\left(\sum_{j=1}^{n} \mathbf{X}_j (r\lambda_j), \sum_{j=1}^{n} \mathbf{Y}_j (r\lambda_j)\right)$ by applying $r\lambda_j$ times the additivity axiom (B4) to each DMU j. Applying the natural disposability axiom (B2) to $\left(\sum_{j=1}^{n} \mathbf{X}_j (r\lambda_j), \sum_{j=1}^{n} \mathbf{Y}_j (r\lambda_j)\right)$, we obtain the point $(r\mathbf{x}, r\mathbf{y})$ (see inequalities (vi) and (vii)). Finally, point $(\mathbf{x},\mathbf{y})$ is obtained from point $(r\mathbf{x}, r\mathbf{y})$ by simply applying the natural divisibility axiom (B3). Thus, we have shown that
(viii) $(\mathbf{x},\mathbf{y}) \in T_{\mathrm{IDEA}} \Rightarrow (\mathbf{x},\mathbf{y}) \in T_{\mathrm{true}}$.
Combining (iii) and (viii), we have $T_{\mathrm{IDEA}} = T_{\mathrm{true}}$. □

Theorem 2

Note first that axioms (B2) and (B3) are integrality restricted special cases of axioms (A2) and (A3). Moreover, it is known that the set of axioms {(A2), (A3), (B4)} is equivalent to the set of axioms A = {(A2), (A3), (A4)} (Arrow and Hahn, 1971). Therefore, the set of axioms B = {(A5), (B2), (B3), (B4)} is just an integrality restricted special case of A. Therefore, the same intensity weights λ apply to both subsets (I^{NI}, O^{NI}) and (I^{I}, O^{I}). The minimum extrapolation result for the case of (I^{NI}, O^{NI}) with axioms A has been formally proved by Banker et al. (1984), and the case of (I^{I}, O^{I}) with axioms B was proved in Theorem 1 above. □

Theorem 3

Inserting set T_{HIDEA} to the modiefied Farrell measure $\mathrm{Eff}^{+}(\mathbf{x}_0, \mathbf{y}_0)$ yields

$$\mathrm{Eff}^{+}(\mathbf{x}_0,\mathbf{y}_0) = \min\{\theta \in \mathbb{R}_{+} | \exists (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in T_{\mathrm{HIDEA}} : \tilde{\mathbf{x}}^{I} \in \mathbb{Z}_{+}^{p}; \theta \mathbf{x}_0 \geqslant \tilde{\mathbf{x}}; \mathbf{y}_0 \leqslant \tilde{\mathbf{y}}\}$$

$$\text{(ix)} = \min_{\theta,\lambda,\tilde{\mathbf{x}}} \left\{ \theta \middle| \exists \tilde{\mathbf{x}} \in \mathbb{Z}_{+}^{p} : \theta \mathbf{x}_0^{I} \geqslant \tilde{\mathbf{x}}; \begin{pmatrix} \tilde{\mathbf{x}} \\ \theta \mathbf{x}_0^{NI} \end{pmatrix} \geqslant \sum_{j=1}^{n} \begin{pmatrix} \mathbf{X}_j^{I} \\ \mathbf{X}_j^{NI} \end{pmatrix} \lambda_j; \begin{pmatrix} \mathbf{y}_0^{I} \\ \mathbf{y}_0^{NI} \end{pmatrix} \leqslant \sum_{j=1}^{n} \begin{pmatrix} \mathbf{Y}_j^{I} \\ \mathbf{Y}_j^{NI} \end{pmatrix} \lambda_j; \lambda_j \geqslant 0 \ \forall j \right\}$$

We can equivalently express (ix) as the MILP problem

$$\min_{\theta,\lambda} \quad \theta,$$

$$\text{s.t.} \quad y_{ro} \leqslant \sum_{j=1}^{n} y_{rj}\lambda_j, \quad r \in O,$$

$$\theta x_{io} \geqslant \sum_{j=1}^{n} x_{ij}\lambda_j, \quad i \in I^{NI},$$

$$\theta x_{io} \geqslant \tilde{x}_i \geqslant \sum_{j=1}^{n} x_{ij}\lambda_j, \quad i \in I^{I}.$$

(x) $\tilde{x}_i \in \mathbb{Z}_{+}, \quad i \in I^{I},$
$\lambda_j \geqslant 0, \quad j \in J.$

The inequality constraints of (x) can be equivalently expressed as equalities if we introduce slack variables: the constraints of (x) are equivalent to those of model (IDEA). Note in particular that the third constraint of (x) involves two inequalities; thus two different sets of slack variables are necessary. Finally, adding the slack variables to the objective function, multiplied by a non-Archimedean infinitesimal, does not influence the value of the objective function. Hence, problem (x) is equivalent to problem (IDEA). □

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