Received 5 February 2013
Revised 26 April 2013
Accepted 26 April 2013

Solvency capital requirement for insurance products via dynamic cash flow matching under lattice models

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Abstract

Purpose – The purpose of this paper is to propose a framework based on cash flow matching for computing the Solvency Capital Requirement under Solvency II.

Design/methodology/approach – The time horizon of the insurance liabilities is typically longer than the maturities of bonds available in the market. With the assumption that a collection of bonds will be available for purchase in the future, the authors study the cash flow matching program under interest rate lattice models.

Findings – The solution can be interpreted as the worst-case cost and the economic capital can be found accordingly.

Originality/value – The paper illustrates the methodology of computing the Solvency Capital Requirement using a dynamic cash flow matching framework under lattice models. The proposed method is particularly useful for insurance products with a typical long time horizon when most duration matching techniques are not easily applicable.

Keywords Capital, Cash flow, Financial management, Insurance, Modelling, Solvency capital requirement, Cash flow matching

Paper type Research paper

1. Introduction

Long term risks are common in the insurance industry. Under Solvency II, insurance companies are required to report the solvency capital requirement (SCR) regarding the liabilities in their balance sheets. Under the regulation, insurance companies can use internal models to provide estimates as long as the estimates are compatible with a standard formula, i.e. the value-at-risk (VaR) at 99.5 percent confidence level in a one year time horizon.

Long term liabilities are in general difficult to appraise. In the context of risk management, it is also difficult to estimate the VaR, not to mention constructing a portfolio to hedge the long term risk. Duration matching approach, or many of its modified approaches (Fong and Vasicek, 1984; Shiu, 1987, 1988; Reitano, 1996; Hürlimann, 2002), are generally used for interest rate risk management. However, they are not feasible due to the lack of long term bonds to discount the liabilities and compute the duration.

In this paper, we extend the cash flow matching framework proposed by Iyengar and Ma (2009). Cash flow matching (Kocherlakota *et al.*, 1988, 1990) is a class of bond

The Journal of Risk Finance
Vol. 14 No. 4, 2013
pp. 344-352

1526-5943
DOI 10.1108/JRF-02-2013-0007

immunization technique (Redington, 1952; Fisher and Weil, 1971; Hiller and Schaack, 1990; Zipkin, 1992). Under the traditional cash flow matching models, a stream of liabilities is matched perfectly by cash flows generated from a bond portfolio. The resulting portfolio is thus truly immune from interest rate changes. Various extensions of the cash flow matching technique have been studied. Hiller and Eckstein (1993) consider a stochastic programming approach in the spirit of mean variance portfolio optimization (Markowitz, 1952; Cheng, 1962). Ronn (1987) also incorporates tax effect in the model.

Instead of taking a static approach in Iyengar and Ma (2009), we take a dynamic approach under lattice models. Various lattice models (Ho and Lee, 1986; Black *et al.*, 1990; Hull and White, 1994) can be incorporated in the framework. The main contribution of this paper is to provide a framework for long term interest rate risk management when other popular techniques such as duration matching are not feasible.

The rest of the paper begins with a motivating example to illustrate the weaknesses of the existing methodology, followed by our proposed methodology. We then illustrate our framework for computing SCR using the Hull-White model (Hull and White, 1993).

2. Motivating example

We consider a typical insurance product for individuals in Hong Kong, who were once reported to have the longest life expectancy in the world. According to the Hong Kong Life Table 2011, the life expectancy of a new born male is 79.7 years.

If an insurance company sells a deferred whole life annuity to an individual aged 40, with payment starting after the retirement age of 65, the expected present value is 6.675 per payment of one unit, assuming a fixed interest rate of 3 percent. Table I shows that the expected present value is very sensitive to the interest rate chosen. It is straight forward to consider the duration matching approach to manage the interest rate risk. However, the duration of this whole life annuity is 32.8 years which is greater than the duration of most bonds available in the market. We therefore propose another approach for long term interest rate risk management.

The lesson learnt from the savings and loans (S&L) crisis in the 1980s is that interest rate risk should not be neglected. Insurance companies should pay extra attention to interest rate risk as the time horizon of typical insurance products is generally very long term. For the whole life annuity product we considered, it involves not only interest rate risk but also longevity risk, which is the risk of paying out more than expected due to longer life expectancy of the policy holder. Our proposed framework can deal with risks other than interest rate risk which conforms to Solvency II (Figure 1).

Bond index	Name	Maturity	Coupon rate (%)	Current price
1	T-Bill	Six months	0	95.861
2	T-Note	One year	4.5	96.149
3	T-Note	Two years	4.5	92.705
4	T-Note	Five years	4.5	84.224
5	T-Bond	Ten years	5.0	77.626
6	T-Bond	30 years	5.0	63.924

Table I.
Details of the Treasury bonds

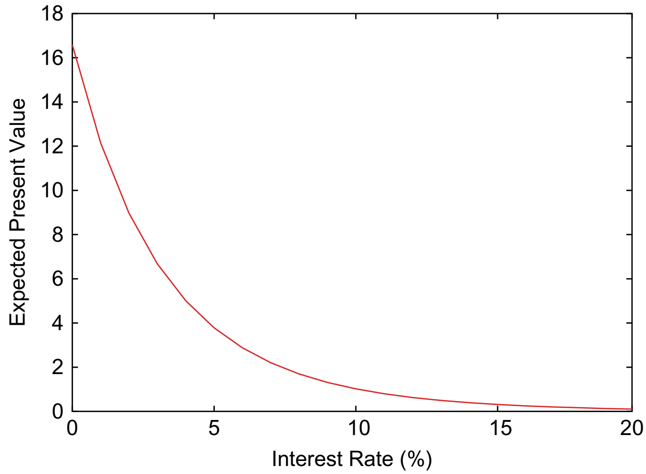

Figure 1.
Interest rate sensitivity of the expected present value of a deferred whole life annuity

3. Methodology

3.1 Notation and assumption

We assume that the cash flows occur at discrete instants of time $t = \tau_0, \tau_1, \ldots, \tau_N$. As in Iyengar and Ma (2009), we assume that a collection of bonds with the same preselected characteristics including the bond issuer, maturities, and coupon rates is available for purchase at all time $t = \tau_0, \tau_1, \ldots, \tau_N$. For example, we could restrict ourselves to Treasury zero-coupon bonds with maturities six months, one year, two years, five years, ten years, and 30 years. We will then assume that we can always purchase Treasury zero-coupon bonds with maturities six months, one year, two years, five years, ten years, and 30 years from the primary market at all time $t = \tau_0, \tau_1, \ldots, \tau_N$.

We use the following notation:

l_i = the liability payment at time τ_i.

M = the number of bonds in our collection.

$c_{i,j}^{(k)}$ = the cash flow at time τ_j from bond k purchased at time τ_i.

$\mathbf{c}_{i,j}$ = the vector $\left(c_{i,j}^{(1)}, \ldots, c_{i,j}^{(M)}\right)^T$.

$p_i^{(k)}$ = the price of bond k at time τ_i.

$\mathbf{p}_i$ = the vector $\left(p_i^{(1)}, \ldots, p_i^{(M)}\right)^T$.

$x_i^{(k)}$ = number of shares of bond k purchased at time τ_i.

$\mathbf{x}_i$ = the vector $\left(x_i^{(1)}, \ldots, x_i^{(M)}\right)^T$.

w = total cost to match the liabilities.

We assume that the stream of liabilities l_i is known at time τ_0. In the context of actuarial application, l_i can be the expected liabilities according to the mortality. The vectors of bond prices $\mathbf{p}_i$ are random variables for $i > 0$. Throughout this paper, we also assume that all bonds are non-callable and default-free.

The vectors $\mathbf{x}_i$, $i = 0, 1, \ldots, N$, and the cost w are the decision variables in the model. The initial bond portfolio is given by $\mathbf{x}_0$. For $i > 0$, $\mathbf{x}_i$ denotes the reinvestment strategy. As in the classical approach, we assume that once we buy the bond we hold it to maturity. Iyengar and Ma (2009) take a static approach in that $\mathbf{x}_i$ is determined at time zero regardless of the interest rate in the future. We take a dynamic approach to consider $\mathbf{x}_i$ which depends on the interest rate in the future and we accomplish this under lattice models.

3.2 Classical cash flow matching

Using the notation given in Section 3.1, the classical cash flow matching problem can be formulated as the following LP:

$$\begin{aligned} &\min \ w \\ &\text{s.t.} \ l_0 + \mathbf{p}_0^T \mathbf{x}_0 \le w, \\ &l_i - \mathbf{c}_{0,i}^T \mathbf{x_0} \le 0, \qquad i = 1, \ldots, N, \\ &\mathbf{x_0} \ge 0. \end{aligned} \tag{1}$$

The classical cash flow matching problem involves computing the minimum cost portfolio with no shortfall over time, i.e. $l_i - \mathbf{c}_{0,i}^T \mathbf{x}_0 \le 0$ for all i. However, if the liabilities span a time horizon longer than the maturities of the available bonds, the cash flow matching problem (1) will be infeasible.

In order to match the liabilities with a longer time horizon than the maturities of bonds available for purchase, we need to make assumptions on the future interest rates. We will consider the cash flow matching problem under lattice models.

3.3 Cash flow matching under lattice models

We form a lattice model for the cash flow matching problem. Let (i, j) denote the node at time τ_i and $j \in j_i$ where J_i are the index system for the lattice model at time τ_i. We assume that the price of the bonds can be found under the lattice model and we let $\mathbf{p}_{i,j}$ denote the price vector at node (i, j). As opposed to the static strategy considered in Iyengar and Ma (2009), we allow the investment strategies $\mathbf{x}_i$ to vary in different nodes (i, j). Thus, we let $\mathbf{x}_{i,j}$ to denote the investment strategy at node (i, j). Under the lattice model, the set of indices of all kth preceding node of (i, j) is denoted by $J_k(i, j)$ which is a subset of J_{i-k}. The dynamic cash flow matching problem can then be formulated:

$$\begin{aligned} &\min \ w \\ &\text{s.t.} \ l_0 + \mathbf{p}_{0,j}^T \mathbf{x}_{0,j} \le w, && j \in J_0, \\ &l_i - \mathbf{p}_{i,j}^T \mathbf{x}_{i,j} \le \sum_{k=0}^{i=1} \min_{jk \in J_{i-k}(i,j)} \mathbf{c}_{k,i}^T \mathbf{x}_{k,jk}, && j \in J_i, i = 1, \ldots, N, \\ &\mathbf{x}_{i,j} \ge 0, && j \in J_i, i = 0, \ldots, N. \end{aligned} \tag{2}$$

The minimum part is to ensure that the liability as well as the cost at node (i, j) can be paid by the minimum cash flow over all the preceding nodes. It can be converted into the following optimization problem:

$$
\begin{aligned}
&\min\ w\\
&\text{s.t. } l_0 + \mathbf{p}_{0,j}^T \mathbf{x}_{0,j} \le w, && j \in J_0,\\
&l_i - \mathbf{p}_{i,j}^T \mathbf{x}_{i,j} \le \sum_{k=0}^{i=1} m_{k,i,j}, && j \in J_i, i = 1, \ldots, N,\\
&m_{k,i,j} \le \mathbf{c}_{k,i}^T \mathbf{x}_{k,j_k}, && j_k \in J_{i-k}(i,j), \quad j \in J_i, k = 0, \ldots, i-1, i = 1, \ldots, N,\\
&\mathbf{x}_{i,j} \ge 0, && j \in J_i, i = 0, \ldots, N.
\end{aligned} \tag{3}
$$

The optimization problem (3) is an LP and therefore the implementation of the model is simple in practice.

3.4 Applications

In the context of risk management, economic capital is the amount of capital needed in order to absorb unexpected losses with respect to a particular type of risk. If we define a_0 to be the present value of the stream of liabilities under the term structure implied by the lattice model, a_0 is then interpreted as the expected loss while the optimal solution w from equation (3) is interpreted as the worst-case loss. Therefore, the interest rate risk economic capital for the stream of liabilities is $w - a_0$.

4. Numerical example

We consider the whole life annuity in Section 2 and illustrate how we can compute the SCR under Solvency II. We take a bottom-up approach and consider the two main risks affecting the whole life annuity which are longevity risk and interest rate risk. We use our framework for the interest rate risk.

4.1 Treasury bonds and liabilities

We assume that the collection of bonds available for investment at each time instant is given by the Treasury bonds shown in Table I. Recall that $c_{i,j}^{(k)}$ is the cash flow at time τ_j from bond k purchased at time τ_i and thus can be defined according to Table I with $\tau_i = 0.5i$ and $\Delta\tau = \tau_{i+1} - \tau_i = 0.5$. The stream of liabilities of the whole life annuity in Section 2 can be set as the expected value of the payment according to the Hong Kong Life Table 2011. In particular, if kp_x is the probability of an individual aged x of surviving k years, then:

$$
l_k = \begin{cases} {}_{\frac{k}{2}}p40, & k = 62, 64, \ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{4}
$$

The liabilities are shown in Figure 2.

4.2 Lattice models

In the literature, there are many interest rate lattice models (Ho and Lee, 1986; Black *et al.*, 1990; Hull and White, 1993). As an illustration, we consider the Hull-White one-factor model for the interest rates as in Iyengar and Ma (2009). In this model, the short rate $r(t)$ follows the stochastic differential equation:

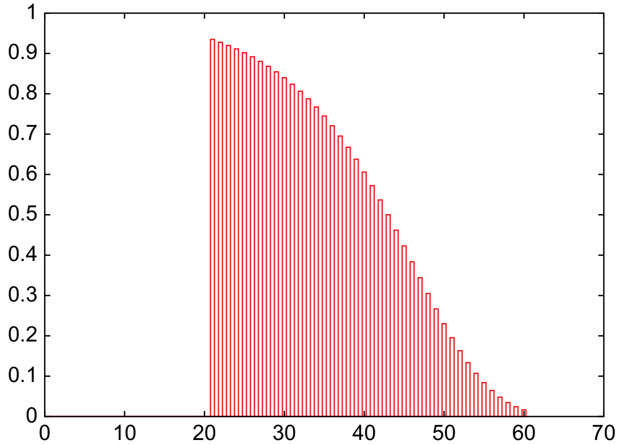

Figure 2.
Liabilities against time

$$dr(t) = \alpha\left[\mu(t) - r(t)\right]dt + \sigma dW(t), \tag{5}$$

where α and σ are constants, $W(t)$ is a standard Brownian motion, and $\mu(t)$ is a deterministic function fitting the initial term structure. The function $\mu(t)$ is given by (Cairns, 2004):

$$\mu(t) = \frac{1}{\alpha}\frac{\partial}{\partial t}F(t) + F(t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}), \tag{6}$$

where $F(t)$ is the current forward rate. At time t, the price of a zero-coupon bond with face value 1 and maturing at time T is:

$$P(t, T) = e^{A(t,T)-B(t,T)r(t)}, \tag{7}$$

where:

$$B(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha},$$

$$A(t, T) = \log\frac{P(0, T)}{P(0, t)} + B(t, T)F(t) - \frac{\sigma^2}{4\alpha^3}(1 - e^{-\alpha(T-t)})^2(1 - e^{-2\alpha t}).$$

The solution to equation (5) is:

$$r(t) = F(t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma\int_0^t e^{-\alpha(t-s)}dW(s). \tag{8}$$

Then we construct a trinomial tree following Hull and White (1994). Because of the mean reverting properties of the Hull-White model, the number of different nodes at each time will not exceed $2 \times 0.184/(\alpha\Delta t) + 1$.

We let $R(t)$ denote the Δt rate at time t. The analytic result of the Hull-White trinomial tree for the bond prices are then given by equation (7) where:

$$r(t) = \frac{R\Delta t + A(t, t+\Delta)}{B(t, t+\Delta)}. \quad (9)$$

As an illustration, we assume that the model matches the present term structure of interest rates with:

$$F(t) = A + Be^{-Ct}. \quad (10)$$

as given in Iyengar and Ma (2009). We assume that $A = 0.08$, $B = 0.005$, $C = 0.3$, $\alpha = 0.2$, and $\sigma = 0.01$. The bond price at time $t = 0$ maturing at time T is then given by:

$$P(0, T) = e^{-\int_0^T F(t)dt} = e^{(B/C)(e^{-CT}-1)-AT}. \quad (11)$$

4.3 SCR under Solvency II

According to the Fifth Quantitative Impact Study (QIS 5) proposed in 2011 for SCR calculation, any internal model should provide an estimate compatible with a standard formula, i.e. VaR at 99.5 percent confidence level in a one year time horizon. Since a decrease in interest rate causes an increase in the value of the life annuity, we want to set up a trinomial tree such that the 99.5 percent critical value of $R(1)$ is covered. Note that for small Δt, $R(1)$ is normal with mean equal to $F(1) + (\sigma^2/2\alpha^2)(1 - e^{-\alpha})^2$ and variance equal to $\sigma^2 \int_0^1 e^{-2\alpha(1-s)} ds = (\sigma^2/2\alpha^2)(1 - e^{-2\alpha})$. The critical $R(1)$ is then $\hat{R}(1) = F(1) + (\sigma^2/2\alpha^2)(1 - e^{-\alpha})^2 - 2.5758\sqrt{(\sigma^2/2\alpha)(1 - e^{-2\alpha})}$. We find the maximum Δt such that $R(0) + \Delta R \max(j_{\min}, -1/\Delta t) \leq \hat{R}(1)$ with Δt divides $\Delta\tau$.

After finding w and a_0, we can compute the SCR with respect to the interest rate risk, namely SCR_1. The whole life annuity is also subject to longevity risk. According to QIS 5, the worst-case cost is found by considering a 25 percent decrease in the mortality rate. We refer the SCR with respect to the longevity risk as SCR_2. The overall SCR is computed by:

$$SCR = \sqrt{SCR2_1^2 + 0.25 \times SCR_1 \times SCR_2 + SCR_2^2}, \quad (12)$$

where the correlation between the life underwriting risk (longevity risk) and market risk (interest rate risk) is 0.25 under the Solvency II standard procedure.

4.4 Results

The present value of the whole life annuity, a_0, under the Hull-White term structure is 1.531. Table II shows the SCR with respect to the interest rate for the whole life annuity for different confidence levels computed using our methodology. The SCR for this annuity with respect to the longevity risk is 0.464. Therefore, under QIS 5, the SCR is equal to $\sqrt{0.464^2 + 0.25 \times 0.464 \times 0.856 + 0.856^2} = 1.024$.

Table II.
SCR for the whole life annuity

Confidence (%)	Δt	w	SCR_1
95	1/6	2.081	0.549
99	1/12	2.368	0.836
99.5	1/14	2.387	0.856

5. Conclusion

In this paper, we illustrate the methodology of computing the SCR using a dynamic cash flow matching framework under lattice models. The proposed method is particularly useful for insurance products with a typical long time horizon when most duration matching techniques are not easily applicable. The methodology involves solving a linear programming problem which is easy to implement in practice. Future research direction may include solving the cash flow matching problem under a lattice model for both the interest rate and the mortality rate.

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