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Portfolio optimization in an upside potential and downside risk framework☆

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ABSTRACT

The lower partial moment (LPM) has been the downside risk measure that is most commonly used in portfolio analysis. Its major disadvantage is that its underlying utility functions are linear above some target return. As a result, the upper partial moment (UPM)/lower partial moment (LPM) analysis has been suggested by Holthausen (1981. *American Economic Review*, *v71*(1), 182), Kang et al. (1996. *Journal of Economics and Business, v48*, 47), and Sortino et al. (1999. *Journal of Portfolio Management*, *v26*(1,Fall), 50) as a method of dealing with investor utility above the target return. Unfortunately, they only provide dominance rules rather than a portfolio selection methodology. This paper proposes a formulation of the UPM/LPM portfolio selection model and presents four utility case studies to illustrate its ability to generate a concave efficient frontier in the appropriate UPM/LPM space. This framework implements the full richness of economic utility theory be it [Friedman and Savage (1948). *Journal of Political Economy*, *56*, 279; Markowitz, H. (1952). *Journal of Political Economy*, *60*(2), 151; Von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior.* (3rd ed., 1953), Princeton University Press], and the prospect theory of (Kahneman and Tversky (1979). *Econometrica*, *47*(2), 263).

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The methods and techniques proposed in this paper are focused on the following computational issues with UPM/LPM optimization.

- Lack of positive semi-definite UPM and LPM matrices.
- Rank of matrix errors.
- Estimation errors.
- Endogenous and exogenous UPM and LPM matrices.

1. Introduction

The mean-lower partial moment (μ,LPM) model has been attractive to decision makers because it does not require any distributional assumptions and it is a necessary and sufficient condition for investors with various classes of Von Neumann and Morgenstern (1944) (hereafter, vNM) utility functions which is equivalent to expected utility-maximization under risk aversion.[1] Because it does not make any distributional assumption, it has been particularly useful in the management of derivative portfolios (Merriken, 1994; Huang, Srivastava, & Raatz, 2001; Pedersen, 2001; and Jarrow & Zhao, 2006).

However, LPM has traditionally been challenged by academic researchers because of the computational complexity of the asymmetric Co-LPM matrix used in μ-LPM portfolio analysis and the persistent belief that it is an ad-hoc method that is not grounded in capital market equilibrium theory and in expected utility maximization theory.[2]

A major challenge to the use of any portfolio theory formulation that does not use mean-variance analysis is by Markowitz (2010). His position is even with non-normal security distributions, the mean-variance criterion is still a useful approximation of the expected utility of the investor. In other words, any alternative to mean-variance portfolio theory has to rest on a solid foundation of utility theory. It is not sufficient for the portfolio framework to simply be a nonparametric approach. The UPM/LPM framework is powerful because it is a nonparametric approach and it implements the full richness of economic utility theory be it Friedman and Savage (1948), Markowitz (1952), Von Neumann and Morgenstern (1944), or the prospect theory of Kahneman and Tversky (1979). Markowitz (2010) does end up supporting the geometric mean-semivariance portfolio theory model in his paper because of its utility theory foundation. Semivariance is the only risk measure other than variance that is accorded any support by Markowitz (1959, 2010).

While our focus is not on LPM and capital market theory, the discussion in Hogan and Warren (1974), Bawa and Lindenberg (1977), Harlow and Rao (1989), Leland (1999), and Pedersen and Satchell (2002) makes it pretty clear that LPM is not an ad-hoc model that is ungrounded in capital market theory. We are interested in solving the computational complexities of the μ-LPM and its well-known utility maximization limitation of assuming a linear utility function above the target return.[3] By solving the μ-LPM computational problem, we are able to introduce the upper partial moment-lower partial moment (UPM/LPM) portfolio selection model which extends the expected utility maximization capabilities of the LPM model.

The paper continues with a discussion of mean-LPM and UPM/LPM portfolio analysis and their place in expected utility theory. Next, we offer a formulation for testing UPM/LPM portfolio optimization problems and discuss the historic issue of exogenous and endogenous LPM matrices. Next, four empirical problems are discussed which include: (1) Lack of positive semi-definite UPM and LPM matrices; (2) Rank of matrix errors; (3) Estimation errors; and (4) Endogenous and Exogenous UPM

[1] See Frowein (2000) for the necessary and sufficient conditions.

[2] See Grootveld and Hallerbach (1999) for a discussion of empirical issues and Pedersen and Satchell (2002) for a discussion of the theoretical foundation of μ-LPM in capital market theory.

[3] LPM utility functions are interested only in downside or below target return risk. It assumes a risk-neutral investor for above-target returns. See Fishburn (1977), Fishburn and Kochenberger (1979) and Kaplan and Siegel (1994a, 1994b).

and LPM matrices. The paper offers discussions and solutions for each of these problems. Finally, the paper will present four utility theory case problems to illustrate the ability of the UPM/LPM model to generate a concave efficient frontier in the appropriate UPM/LPM space.

2. UPM/LPM, stochastic dominance and expected utility maximization

Stochastic dominance is the analysis of the cumulative probability function of two securities and is used to determine investor preference for one asset over the other. Bawa (1975) provided proofs that in terms of expected utility theory, first degree stochastic dominance (FSD) contains all utility functions, second degree stochastic dominance (SSD) contains all risk averse utility functions and third degree stochastic dominance (TSD) includes all utility functions with decreasing absolute risk averse utility functions.[4]

The foundation of the congruence of LPM with expected utility maximization goes back to Porter (1974) who found that the SSD set contains all mean-semivariance (from a target return) dominant portfolios.

In terms of stochastic dominance, Bawa (1975) shows that the mean-variance is neither a necessary nor a sufficient condition for SSD and that mean-variance is consistent with Von Neumann–Morgenstern utility theory only if the utility function is quadratic.[5] By contrast, the TSD admissible set contains all mean-target semivariance (LPM degree 2) rules. Therefore, Bawa concludes that mean-semivariance can be used as a suitable approximation to the TSD rule.[6]

Fishburn extends Bawa's analysis to his Theorem 3 that states that the LPM is a family of utility functions denoted by the LPM degree a such that:

FSD contains all mean-LPM(a,t) decision rules for all $a \geq 0$.
SSD contains all mean-LPM(a,t) decision rules for all $a \geq 1$.
TSD contains all mean-LPM(a,t) decision rules for all $a \geq 2$.

where a denotes the degree of the LPM measure and t is the target return. Fishburn notes that a decision maker's preferences may satisfy a mean-risk utility model without also satisfying the Von Neumann and Morgenstern axioms for expected utility. Therefore, he provides a proof (Theorem 2) that shows that the mean-LPM(a,t) model is congruent with expected utility theory. In addition, he studies its compatibility with 24 investment managers with Von Neumann–Morgenstern (vNM) utility functions that have appeared in the literature: Grayson (1960), Green (1963), Swalm (1966), and Halter and Dean (1971). All of the utility functions can be described in terms of below-target utility. However, the above target utility for the mean-LPM (a,t) model is assumed to be linear. In 9 out of 24 utility functions the above-target utility is approximately linear as shown in Fig. 1. The remaining 15 exhibit a variety of concave and convex forms above the target with a tendency toward concavity or risk aversion.

Fishburn and Kochenberger (1979) extend the two-piece vNM (above and below target) expected utility analysis to 28 empirically assessed vNM utility functions that have appeared in the literature.[7] They find that the majority of the below-target functions are risk seeking (convex) and the majority of the above-target functions are risk averse (concave).

	Concave above target	Convex above target	
Convex below target	13	5	18
Concave below target	3	7	10
Total	16	12	28

[4] The analysis consists of the integration of the cumulative probability function Bawa (1975) and Levy (1998).
[5] Kroll, Levy, and Markowitz (1984) do provide a strong counter-argument supporting mean-variance.
[6] Ogryczak and Ruszczynski (2001) provide the proof that n-degree mean-semideviation models and stochastic dominance models are equivalent.
[7] The utility cases came from Swalm (1966) 13 cases, Halter and Dean (1971) 2 cases, Grayson (1960) 8 cases, Green (1963) 3 cases, and Barnes and Reinmuth (1976) 2 cases.

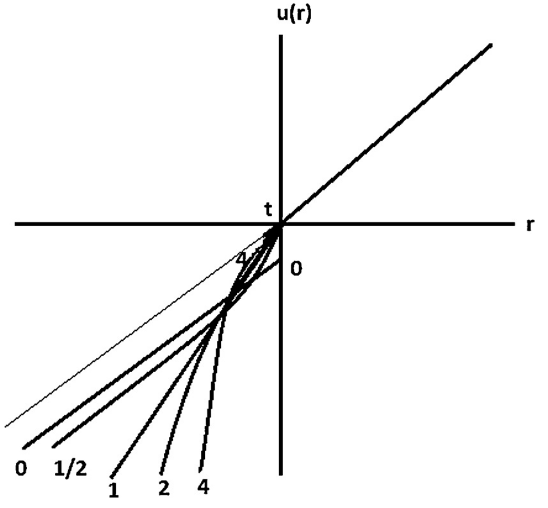

Fig. 1. Plots of LPM utility functions for five values of $a = 0$, 1/2, 1, 2, and 4. Note that all utility functions are linear above the target return *t*. Source: Fishburn (1977).

The finding of convex–concave and concave–convex utility functions in more than 70% of the cases supports the Kahneman and Tversky (1979) reflection effect, which suggests that above-target risk aversion is often accompanied by below-target risk seeking and above-target risk seeking is often accompanied by below-target risk aversion.[8] In these reported utility cases, (only 3 of 28), *the frequently invoked assumption that individuals are everywhere risk averse (concave–concave) does not hold*.

Holthausen (1981), looking at Fishburn's μ-LPM(a,t) results, wonders why the criteria should continue to use the expected mean which includes all returns from below the target. This is redundant because the downside returns are already included in the LPM(a,t) calculation.[9] Another concern is that the mean-LPM(a,t) model imposes risk neutrality above the target return. Given the results from Fishburn (1977) and Fishburn and Kochenberger (1979) showing linear, convex and concave functions above the target, Holthausen proposes the a–β–t model, or upper partial moment/lower partial moment (UPM/LPM) model where β is the coefficient for the above-target utility (see Fig. 2). Investors can be classified as risk seeking ($a < 1$, $\beta > 1$), risk neutral ($a = 1$, $\beta = 1$) and risk averse ($a > 1$, $\beta < 1$). He demonstrates that this model is congruent with expected utility theory.

Holthausen (1981) provides a proof for the following relationships:

FSD contains all UPM/LPM(a,β,t) rules for all $a \geq 0$, $\beta \geq 0$.
SSD contains all UPM/LPM(a,β,t) rules for all $a \geq 1$, $0 \leq \beta \leq 1$.
TSD contains all UPM/LPM(a,β,t) rules for all $a \geq 2$, $0 \leq \beta \leq 1$.

[8] We propose that upside risk seeking and upside risk aversion are more appropriately denoted as "potential seeking" and "potential averse". These terms more accurately describe the upside behavior of investors.

[9] We can give you a simple example why UPM/LPM is important. Let's assume that the target return is 3%. Now, a 2% return will enter the LPM calculation as a below-target return. It will also enter into the calculation of the mean return. So when computing a mean/LPM ratio, that particular return is being counted twice, both in the numerator and the denominator, and it represents a positive outcome in the mean calculation and a negative outcome in the LPM calculation. UPM/LPM analysis avoids this conflict Musiol and Muhlig (2003).

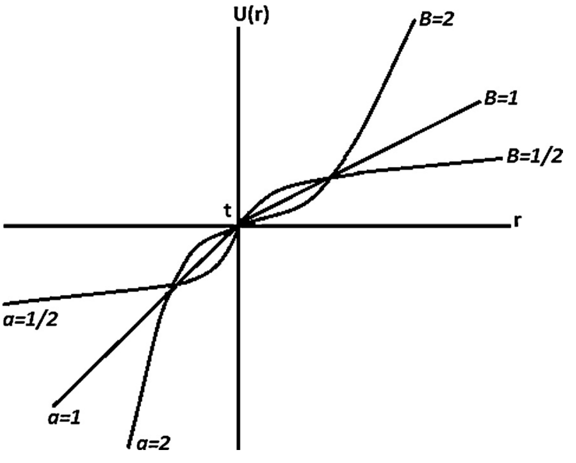

Fig. 2. Plots of the UPM/LPM Utility Function for $k = 2$ and various α, β values. The prospect theory S-shaped utility function can be seen in $\alpha = 1/2$, $\beta = 1/2$. Markowitz reverse S-shaped utility is $\alpha = 2$, $\beta = 2$, and risk/potential neutral investors are $\alpha = 1$, $\beta = 1$. Source: Holthausen (1981).

One major result of Holthausen's paper is the proof that the a-β-t model is congruent with a Von Neumann–Morgenstern utility function of the form:

$$U(x) = \begin{array}{ll} (x-t)^{\beta} & \text{for all } x \geq t \\ -k(t-x)^{a} & \text{for all } x \leq t \end{array}$$

where is k is a constant that indicates a "kink" in the utility function at the target return whenever $k > 1$. Thus, the Holthausen a–β–t model is fully congruent with expected utility theory with the restriction that the above-target utility has to be concave or linear.

Note that $\beta > 1$ is included only in the FSD rules. The SSD and TSD rules do not include $\beta \geq 1$ which is an area of interest because Fishburn (1977), Fishburn and Kochenberger (1979), and Kahneman and Tversky (1979) have noted risk-seeking (convex) utility functions above the target return.

As a result, there should be some interest in non-concave utility functions exhibited by the experimental results of Kahneman and Tversky (1979) prospect theory (S-shaped) and the Friedman and Savage (1948), Markowitz (1952), and Kahneman and Tversky (1979) non-concave functions (reverse S-shaped).

Levy and Levy (2002), Wakker (2003), Post and Levy (2005), Baltussen, Post, and van Vliet (2006), Post, van Vliet and Levy (2008) find most of the empirical evidence supports the reverse S-shaped utility functions found in their stochastic dominance criteria.

The concept of UPM/LPM analysis has appeared elsewhere in the literature. Bawa, Brown, and Klein (1981) proposed their asymmetric response model for asset pricing where the market returns are separated into above target and below target returns. Harlow and Rao (1989) used this model to develop a generalized mean-LPM model of asset pricing. Pedersen and Satchell (2000) use the asymmetric response model to improve the small sample estimation of the Sharpe, Treynor and Jensen performance measures. Kang, Brorsen, and Adam (1996) and Sortino, Van Der Meer and Plantinga (1999) have extended UPM/LPM analysis to a risk dominance criteria and portfolio performance measures, respectively.[10]

[10] More recently, Mauser, Saunders and Seco (2006) proposed a linear programming (LP) formulation for a UPM/LPM degree 1 otherwise known as the Omega ratio. UPM/LPM Degree 1 is not very interesting as the utility function is linear throughout its range. See Fig. 2 for $\alpha = 1$, $\beta = 1$. Konno, Tanaka and Yamamoto (2011) propose a LP formulation for a conditional value at risk model using CVaR upper/CVaR lower. This formulation is uninteresting because it assumes a UPM/LPM degree of one. See Tee (2009) for a proof that VaR is equivalent to LPM degree 0. Markowitz (2010) rejects the use of both VaR and CVaR.

However, no matter the shape of investor utility, the traditional everywhere risk averse or otherwise, the Holthausen (1981) a–β–t (or UPM/LPM a–c–τ in this paper) model provides a very flexible framework for portfolio analysis. First, it has been shown to be congruent with the Von Neumann–Morgenstern expected utility model and it can handle the reverse S-shaped and S-shaped utility functions of Friedman and Savage (1948), Markowitz (1952) and Kahneman and Tversky (1979).

The next section of the paper addresses the need for a formulation and solution algorithm for the UPM/LPM model in order to provide a useful methodology for future research in this area.

3. The UPM/LPM (a–c–τ) algorithm

There are two approaches to the μ-LPM portfolio selection problem: endogenous and exogenous LPM matrix formulations. The endogenous matrix approach was introduced by Markowitz (1959) and Hogan and Warren (1972, 1974). The exogenous matrix approach is embodied by Francis and Archer (1979), Nawrocki (1991, 1992), Estrada (2008) and Cumova and Nawrocki (2011). The Markowitz (1959) and the Hogan and Warren (1972) papers introduce the asymmetric cosemivariance or Co-LPM (a = 2) matrix and the μ-LPM(2,t) portfolio selection formulation. However, it is an endogenous matrix where the cosemivariance matrix cannot be computed until we know the portfolio allocations. This formulation is not closed form and can only be solved by iterative or Monte Carlo techniques. The second approach is to compute the exogenous cosemivariance matrix directly from the data and use it in a closed form solution to compute the portfolio allocations. Nawrocki (1991, 1992), Estrada (2008) and Cumova and Nawrocki (2011) have found that this exogenous matrix approach is a very close approximation to the endogenous matrix approach. This paper employs the exogenous matrix for UPM/LPM analysis.

The cosemivariance is analogous to the covariance found in traditional mean-variance analysis. It is presented as a measure of the relationship between stocks in Markowitz (1959), Hogan and Warren (1972, 1974) and Francis and Archer (1979). The cosemivariance between a security and the market (capital market theory) is developed in Hogan and Warren (1974) and the Co-LPM between a security and the market appears in Bawa and Lindenberg (1977), Nantell and Price (1979) and Harlow and Rao (1989)[11]. Markowitz (1959) and Hogan and Warren (1972) demonstrate that mean-semivariance is equivalent to mean-variance analysis under the restrictive assumption of a symmetric distribution.

As the UPM is the mirror image of the LPM, we assume that all issues pertaining to the LPM model also apply to the UPM model.[12] Given that assumption, the μ-LPM(α,τ) model may be extended to the "upside potential"-downside risk UPM/LPM(a–c–τ) model which may be formulated with Co-LPM (CLPM) and Co-UPM (CUPM) matrices as follows.[13]

In an endogenous matrix formulation, Markowitz (1959) computes the portfolio semivariance assuming K portfolio returns, $\sum w_i R_{it}$ for t = 1,2,. . .,T periods, below the target return. The portfolio semivariance is computed using only the below target portfolio returns which in this case is a target return of zero.

$$r_t = \Sigma w_i R_{it} \tag{1}$$

where r_t is the portfolio return for period t and R_{it} is the return for security i in period t for T periods. The portfolio semivariance, S, is computed using only returns below the target. Assume there are K portfolio returns below target.

$$S = 1/T \sum_{k=1}^{K} r_k^2 \tag{2}$$

[11] Harlow and Rao (1989) define an n-degree co-lower partial moment between two assets as (6).

[12] We find with the upper partial moment that investor behavior is more accurately described as potential seeking or potential aversion rather than risk seeking or risk averse on the up-side. Upside potential reflects the potential of greater gains. "Upside risk" is not a clear expression of this expectation.

[13] Eqs. (1)–(3) for a = 2 are from Markowitz (1959, pp. 195–196).

where T is the total number of observations. Once we know the allocations, w_i, we can compute the portfolio returns that are below target. Only now we can compute the cosemivariance, s.

$$s_{ij} = 1/T \sum_{k=1}^{K} R_{ik} R_{jk} \tag{3}$$

The approach in this paper is to compute the UPM and LPM matrices as exogenous matrices directly from the data, such that the problem formulation is:

$$\begin{aligned} &\text{Maximize} \\ &E(\text{UPM}_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E(\text{CUPM}_{ij}) \\ &\text{where} \\ &E(\text{CUPM}_{ij}) = 1/K \sum_{t=1}^{K} [\text{Max}\{0, (R_{it} - \tau)\}]^{c-1} (R_{jt} - \tau) \end{aligned} \tag{4}$$

Minimize

$$E(\text{LPM}_{portf}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E(\text{CLPM}_{ij}) \tag{5}$$

where

$$E(\text{CLPM}_{ija}) = 1/K \sum_{t=1}^{K} [\text{Max}\{0, (\tau - R_{it})\}]^{a-1} (\tau - R_{jt}) \tag{6}$$

subject to:

$$\sum_{i=1}^{n} w_i = 1 \tag{7}$$

where K is the total number of observations (not just the returns below target), τ is the target return, w is the security weight in the portfolio, a is the degree of the LPM and c is the degree for the UPM. The above multi-objective optimization problem may be alternatively formulated as a minimization of downside risk (LPM) for a certain target "b" level of upside potential (UPM-upside partial moment).

$$\begin{aligned} &\text{Minimize} \\ &E(\text{LPM}_{portf}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E(\text{CLPM}_{ij}) \end{aligned} \tag{8}$$

subject to:

$$\begin{aligned} &b = E(\text{UPM}_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E(\text{CUPM}_{ij}) \\ &\text{where} \\ &E(\text{CUPM}_{ij}) = 1/K \sum_{t=1}^{K} [\text{Max}\{0, (R_{it} - \tau)\}]^{c-1} (\tau - R_{jt}) \end{aligned} \tag{9}$$

$$\sum_{i=1}^{n} w_i = 1 \quad (10)$$

We consider these formulations to be more useful than the endogenous matrix approach as they represent closed form solutions of the problem and have demonstrated close approximation to the endogenous matrix model. Grootveld and Hallerbach (1999) object to Eqs. (5) and (8) because there is no guarantee that the resulting matrix is positive semi-definite and therefore, potentially not solvable with current algorithms. This is a major reason that previous mean-semivariance algorithms such as those suggested by Ang (1975), Markowitz, Todd, Xu and Yamane (1993) and Mausser, Saunder, and Seco (2006) have utilized a weighted semivariance/LPM/UPM formulation that ignores the interrelationships between the securities. The portfolio LPM calculation (5) and (8) represents an asymmetric CLPM matrix which may be converted to a symmetric matrix in order to guarantee that the matrix will be positive semi-definite.[14] Let L represent the asymmetric CLPM matrix, it is possible to transform it without losing its characteristics by $(L^T + L)/2$. This transformation also has to be done with the CUPM matrix (9). Again, let U represent the asymmetric CUPM matrix, it is possible to transform it without losing its characteristics by $(U^T + U)/2$.[15] This conversion is not always required as the augmented Lagrangian algorithm used in this study can solve matrices that are not positive semi-definite. However, a solution by the Markowitz critical line algorithm (CLA) requires the conversion and proof. In addition, we have used a genetic search algorithm to solve the asymmetric matrix formulation.

In the UPM/LPM portfolio model, the desirable property of mean-downside risk model, i.e. minimizing deviations only below the target return, remains unchanged. So, we are not minimizing the returns above the target. With the exponent $a < 1$ we can express downside risk seeking, $a = 1$ risk neutrality, and $a > 1$ risk aversion behavior on the part of the investor. Risk aversion means the further returns fall below the target return, the more we dislike them. On the other hand, risk seeking behavior means that the further returns fall below the target return, the more we prefer them. To clarify, whenever $a < 1$ and security A has greater magnitude of losses than security B, the computed LPM for A will be less than the computed LPM for B. Assuming equal mean returns, the lower the value of the risk seeking coefficient, $a < 1$, the more we will prefer A to B. The exponent c, on the other hand, describes potential gains on the upside. Therefore c can describe potential seeking/potential aversion behavior on the upside of the distribution (UPM) where $c > 1$ is potential seeking and $c < 1$ is potential averse.

The major difference from the traditional mean-downside risk model is the replacement of the expected portfolio return maximization with the maximization of the expected upside return potential (UPM-upside partial moment). The UPM measure captures the upside return deviations from benchmark. Therefore, the expected upside partial moment $E(\mathrm{UPM})^{1/c}$ of a portfolio can be interpreted as a expected return potential of portfolio relative to a benchmark.

$$E(\mathrm{UPM})^{1/c} = \sqrt[c]{\frac{1}{K}\sum_{k=1}^{K} \max[(R_k - \tau); 0]^{c}} \quad (11)$$

Similar to the downside risk LPM calculation, the UPM could also be expressed as an expected upside deviation from benchmark multiplied by the related probability:

$$\mathrm{UPM} = \frac{1}{K}\sum_{k=1}^{K} [(R_k - \tau)^c | R_k > \tau] \cdot P(R_k > \tau) \quad (12)$$

The UPM contains important information about how often and how far investor wishes to exceed the benchmark, which the mean return ignores.

[14] There have been other attempts to convert the asymmetric LPM matrix to symmetric. Huang et al. (2001) used the symmetric LPM matrix suggested by Nawrocki (1991) and demonstrate that it is positive semi-definite and generates concave efficient frontiers.

[15] The general proof is: $x'Ax = x'Ax)' = x'A'x$. Thus, $x'Ax = x'[(A + A')/2]x$.

We do not consider the upside deviation from benchmark to be risk, therefore, we label it as an upside return potential. As in the LPM calculation, different exponent values for c represent different investor behaviors: potential seeking, potential neutrality or potential aversion above the benchmark return. Potential seeking means the higher the returns above the target return, the greater the investor utility. The potential aversion describes a conservative strategy on the upside.[16] Because of the maximization of the UPM, the exponent $c<1$ represents potential aversion, $c=1$ potential neutrality, and $c>1$ potential seeking. Hence, the often criticized utility neutrality above the benchmark that is inherent in the μ-LPM(a,t) model and the potential aversion inherent in the mean-variance model is eliminated. The exponent c does not have to be the same value as the penalizing exponent a; so we can combine different strategies, for example, a risk aversion $a=4$ and a potential aversion $c=0.5$ might represent a conservative investor.

Most investors consider protection against losses as more important than exposure to gain, so the a-exponent will usually be higher than the c-exponent.

3.1. Utility functions using UPM/LPM analysis

At this point, it should be clear that this portfolio model involves a broad spectrum of utility functions (see Fig. 3). The variability of the below benchmark returns is similar to Fishburn's utility functions employing the LPM measure. However, the upper part of the return distribution exhibits variable investor behavior and is not limited to the potential neutrality ($c=1$) imposed by the Fishburn's LPM utility functions.

The two objectives of maximizing the return and minimizing the risk can be viewed either as a multi-objective optimization problem, or the objectives can be combined using a utility function. Then, the expected utility of the portfolio can be interpreted as a risk-adjusted expected return, since it is computed by subtracting a *risk penalty* from the expected return.[17]

$$E(U) = \text{expected return} - h \times \text{expected risk}$$

To obtain the efficient portfolio, expected utility has to be maximized for a given parameter $h>0$, which represents the investor's risk tolerance, i.e. investor's *marginal rate of substitution of expected value for expected risk*. By computing efficient portfolios for different values of the h-parameter, we can generate an efficient frontier.

In case of expected return potential (UPM) and downside risk LPM, expected utility, as a risk-adjusted expected return potential, is computed by subtracting a downside risk penalty from the expected return potential. In order to obtain efficient portfolios, we have to solve the following formulation based on (8) and (9):

Maximize:

$E(U)$ = expected return potential of the portfolio, and $h\cdot$ expected downside risk of the portfolio[18], or

$$E(U) = E(\text{UPM}) - h \times E(\text{LPM}) \tag{13}$$

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[16] For example, such a strategy could utilize a short call or a short put and their dynamic replication with stock and bonds.

[17] For a proof, see: Markowitz (1959, p. 287). See, for example, Womersley and Lau (1996). However, note that in this paper the expected return is replaced by the expected return potential (UPM). h is the slope of the utility curve at the optimal portfolio point on the efficient frontier. This follows the traditional Markowitz (1959) formulation.

[18] E(UPM) and E(LPM) are the portfolio values from Eqs. (8) and (9). Grootveld and Hallerbach (1999) complain that Eqs. (13) and (14) are not equivalent. This is because Eq. (13) represents exogenous CLPM/CUPM matrices and Eq (14) represents endogenous CLPM/CUPM matrices. As an anonymous referee has pointed out, the only time the exogenous and endogenous matrices will be equivalent is when all R_{it} are equal to R_{pt} which does not hold in practice. However, we have noted that Estrada (2008) and Cumova and Nawrocki (2011) have found that (13) is a close approximation to (14). We remind the reader that Markowitz (2010) has argued that mean-variance analysis is useful because it is a good approximation of investor utility. As Huang et al. (2001) note, the key to an appropriate portfolio utility model is the ability of the UPM/LPM model to generate a concave efficient frontier in the appropriate UPM/LPM space which we will demonstrate later in this paper.

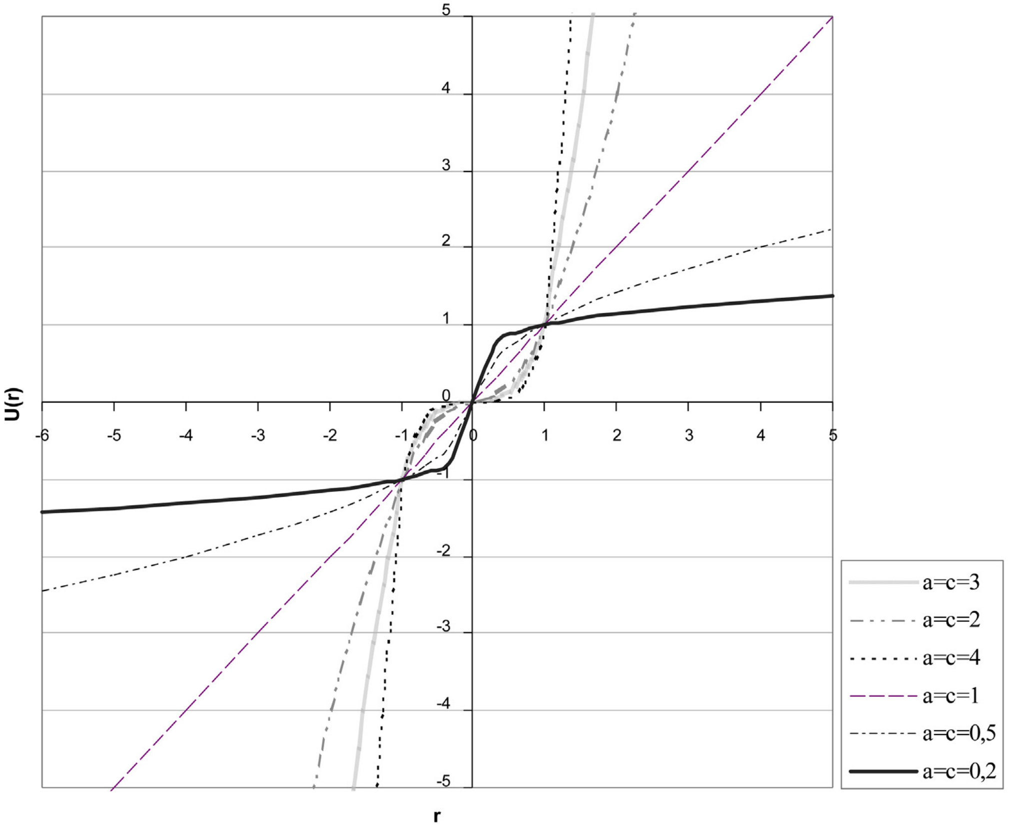

Fig. 3. Utility functions of UPM–LPM portfolio model for different exponents ($\tau = 0$; $h = 1$). The bold line ($a = c = 0.2$) represents upside potential aversion and downside risk-seeking behavior consistent with prospect theory.

Based on Holthausen (1981), the expected utility function is the expected value of the utility function:

$$U = \max[(r - t); 0]c - h \times \max[(t - r); 0]^a \tag{14}$$

where

$$U(r) = (r - t)^c, \quad \text{for all} \quad r \geq \tau \text{ and} \tag{15}$$

$$U(r) = -h \times (t - r)^a, \quad \text{for all} \quad r \leq \tau \text{ and } h > 0 \tag{16}$$

3.2. Comparison of UPM/LPM utility function and the traditional quadratic utility function

In Fig. 4, we see that $a = 2$, $c = 0.5$ is one of combinations of a, c, and τ that can approximate the traditional quadratic utility curve. This approximated utility differs only for greater values than the zenith of the (μ, σ)—utility function $r \leq 1/2k$. Beyond this point the (μ, σ)—utility decreases with increasing final wealth, which is irrational. However, with the utility function based on the (UPM, LPM) model, investor utility increases with the increasing final wealth.

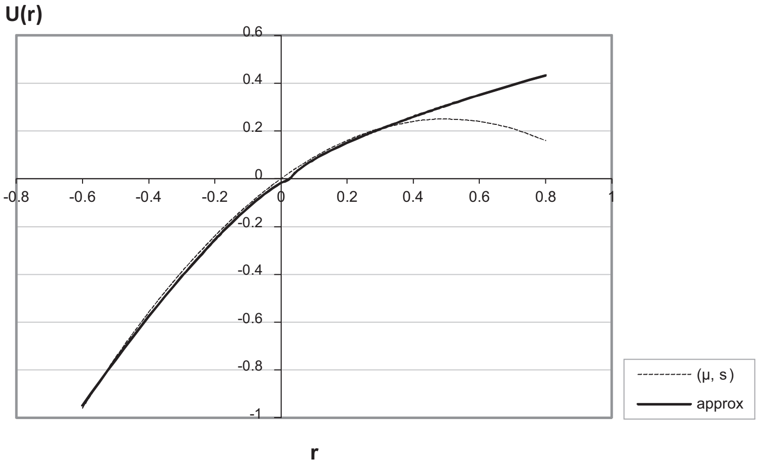

Fig. 4. Quadratic utility function for μ,σ $[U(r)=r+ar^2]$ and the approximate utility function based on UPM/LPM(2,0.5,τ).

4. Computational problems with UPM/LPM(a–c–τ) portfolio selection model

A successful algorithm has to demonstrate its ability to deal with the following problems:

- Grootveld and Hallerbach (1999) note that the asymmetric Co-LPM matrix is not always positive semi-definite and may return negative LPM values.
- Rank of matrix errors (more securities than observations).
- Estimation error when estimating the inputs from historic data.
- The original Markowitz (1959) formulation of mean-semivariance analysis used an endogenous cosemivariance matrix which does not provide a closed form solution. However, Estrada (2008) and Cumova and Nawrocki (2011) found that the exogenous cosemivariance matrix is a reasonable approximation of the endogenous matrix and provides a closed form solution.

4.1. Negative LPM values

The large variety of concave and convex functions that are available in UPM/LPM analysis is a source of the matrix problem. Even with our conversion to symmetric matrices, the UPM and LPM matrices are not positive semi-definite and convex for all non-concave utility functions. The problem as formulated contains a nonlinear objective function subject to nonlinear constraints. The Markowitz critical line algorithm is not appropriate because the Kuhn–Tucker conditions, the basis for the critical line, are not sufficient conditions for a global optimum. The nonlinearity causes nonlinear critical lines hence; the computation of corner portfolios is more difficult. This problem may be solved by the following optimization algorithms that solve nonlinearly constrained optimization problems. These are applicable to the non-convex, non-positive semi-definite functions:

- Augmented Lagrangian and exact penalty functions (Bertsekas, 1982).
- Sequential linear and quadratic programming methods.
- Feasible sequential quadratic programming.
- Reduced-gradient methods.

The standard reduced-gradient algorithm[19] is implemented in CONOPT, GRG2, LSGRG2 software programs which are available on the NEOS-server for optimization (http://www-neos.mcs.anl.gov). This algorithm works reliably even for large-scale problems. It works by solving a sequence of sub-problems. Each sub-problem, or major iteration, solves a linearly constrained minimization (maximization) problem. The constraints for this linear sub-problem are the non-linear constraints converted to linear using a Taylor expansion series containing first order terms.

A large-scale implementation of the Bertsekas (1982) augmented Lagrangian approach to optimization can be found in the LANCELOT and MINOS program packages on the NEOS-server. This algorithm also works reliably for large scale problems.

Negative LPM values noted by Grootveld and Hallerbach (1999) result from three sources: (1) using an algorithm which requires a matrix that is positive semi-definite, (2) the CLPM Eq. (6) can return a negative value whenever $a < 1$, and (3) when there is a rank of matrix error (more securities than observations).

Note that Eq. (6) applies for $a > 1$. However, whenever $0 < a < 1$, (6) can result in anomalies with negative portfolio risk. Therefore, we propose the following computational equation which is operative for all values of a:[20]

$$E(\mathrm{CLPM}_{ij}) = 1/K \sum_{t=1}^{K} \mathrm{Max}[0, (\tau - r_{it})]^{a/2} \cdot |\tau - r_{jt}|^{a/2} \cdot \mathrm{sign}(\tau - r_{jt}) \tag{17}$$

4.2. Rank of matrix error

The rank of matrix error is the source of many anomalies associated with portfolio selection algorithms. Essentially, it occurs whenever there are more securities in the analysis than the number of observations used to estimate the inputs. Markowitz (1959) notes that this is a "degrees of freedom" problem similar to multiple regression. In addition, Klein and Bawa (1977) identify it as a major source of estimation error. Very simply, algorithms that are sensitive to the rank of matrix error will return non-optimal solutions or will not reach an optimum solution (singular matrix error). Markowitz (1987) derives the conditions necessary for an algorithm to be robust relative to rank of matrix error and goes on to show that the critical line algorithm is not sensitive to the rank of matrix error. While the critical line algorithm may not be appropriate for this optimization, the algorithms used in this study have been shown to have a low degree of sensitivity to rank of matrix error, however, it is best to minimize the error by maintaining sufficient positive degrees of freedom. In addition, the rank of matrix error has not been shown to be a problem until more than 50 securities are used in the analysis.[21] Since most portfolio optimization applications do not use more than 50 securities, this problem is minimal but should be noted whenever large security universes are used.

4.3. Estimation risk

Even with the improved algorithms, estimation error remains a problem. Grootveld and Hallerbach (1999) provide evidence that the estimation error for the optimization inputs is more prevalent for LPM analysis because it uses only part of the distribution. Kaplan and Siegel (1994a, 1994b) also note the small sample problem when using LPM measures.

There have not been very many studies on estimating LPM from data. Nawrocki (1992) found that when the rank of matrix error is minimized, the LPM measure for different degrees stabilizes around 48 monthly observations. The variance and beta needed sample sizes larger than 72 months to stabilize. Josephy and Aczel (1993) derived a sample estimator for the semivariance and showed that it was

[19] Lasdon and Warren (1978, pp. 335–362).

[20] Because the UPM is a mirror image of the LPM, this correction also applies to the CUPM computation whenever $0 < c < 1$. Eqs. (6) and (19) are mathematically equivalent when $a \geq 1$. However, (17) will not return negative values whenever $a < 1$.

[21] Cumova and Nawrocki (2011).

asymptotically unbiased and strongly consistent, thus possessing qualities of a 'good' estimator. They recommend the use of the semivariance whenever the underlying distribution is skewed. Bond and Satchell (2002) found that the variance is a more volatile risk measure than semivariance when return distributions are asymmetric. When return distributions are symmetric, then the variance is the more efficient measure. As asymmetric distributions are cited as the most common reason for utilizing LPM risk measures, these results would support the use of LPM measures.

In actual investment practice and in academic empirical studies, bootstrapping techniques such as Efron and Tibshirani (1993) may be employed. Sortino and Forsey (1996) demonstrate the use of bootstrapping with the LPM measure. Bootstrapping simulation is used in this study to minimize estimation error. Other methods include James-Stein estimators proposed by Jobson and Korkie (1980) and Michaud's (1989) re-sampled frontier technique. Elton and Gruber (1973) and Elton, Gruber, and Urich (1978) demonstrate that overall mean models implemented in simple heuristics reduce the degree of estimation risk. In real-world practice, a UPM/LPM heuristic based on the overall mean model of Elton, Gruber, and Padberg (1976) may prove to be the most useful application of UPM/LPM analysis.

4.4. Exogenous and endogenous Co-LPM and Co-UPM matrices

Estrada (2008) provides a good description of exogenous and endogenous cosemivariance (Co-LPM) matrices that will also affect Co-UPM matrices. Essentially, the cosemivariance matrix as formulated by Markowitz (1959) cannot be computed until we know the portfolio returns which we do not know until we know the portfolio allocations. Therefore, the mean-semivariance problem is not a closed-form solution because the cosemivariance matrix is endogenous. Estrada (2008) finds that a computed exogenous cosemivariance matrix can closely approximate the endogenous matrix. Thus, any closed form UPM/LPM optimization algorithm does rest on this exogenous matrix approximation. On the other hand, the endogenous matrix is not part of our information set when we start the analysis. Currently, the only algorithms we have to estimate UPM/LPM efficient portfolios are mean-variance (covariance) and mean-LPM (Co-LPM) algorithms. Therefore, the empirical test in this paper compares optimal UPM/LPM portfolios to optimal mean-variance and mean-LPM portfolios to test the usefulness of the UPM/LPM exogenous matrix algorithm. In all cases the appropriate covariance, CLPM, and CUPM matrix is used in a closed form optimization.

5. Four UPM–LPM utility cases

The next section of the paper presents four utility cases with different concave and convex functions to illustrate the use of the UPM/LPM model.

5.1. Concave efficient frontiers

Hogan and Warren (1972, 1974), Bawa (1975), Bawa and Lindenberg (1977), Harlow and Rao (1989) provide proofs that μ-LPM efficient frontier is concave. Pedersen and Satchell (2002) provide the proof for μ-semistandard deviation frontiers. Huang et al. (2001) provide empirical support for the exogenous matrix model generating concave frontiers. As an extension of the Hogan and Warren (1972) corollary, we state that the UPM/LPM efficient frontier is concave when the UPM of efficient portfolios in the UPM/LPM space is concave as a function of the LPM. We demonstrate this concave result in the next section through an empirical example. The issue is whether the UPM/LPM algorithms can generate a concave efficient frontier in the UPM/LPM(a,c,τ) space as per Hogan and Warren (1972). The only other algorithms that we have to test the UPM/LPM model against are the mean-variance (μ-σ) and mean-semivariance (μ-LPM degree 2) algorithms. In order to compare different algorithms in the appropriate UPM/LPM space, the efficient frontiers and appropriate statistics are computed using portfolio returns (equation 1) generated by the algorithms. Therefore, there are no exogenous-endogenous matrix issues with these results.

Table 1
Generated summary statistics for 12 assets used in the four utility cases: stocks (S), covered calls (CC), and protective puts (PP).

Input parameters for the various securities (S) stocks, (CC) covered calls, and (PP) protective puts in the optimization

Asset	Mean	Std.Dev	SemiDev	Skewness
S1	0.185	0.382	0.187	0.166
S2	0.079	0.230	0.133	−0.060
S3	0.215	0.462	0.210	0.380
S4	0.175	0.351	0.160	0.413
CC1	0.179	0.195	0.132	−2.937
CC2	0.095	0.115	0.081	−2.916
CC3	0.179	0.217	0.139	−2.050
CC4	0.146	0.160	0.111	−3.074
PP1	0.052	0.241	0.128	1.372
PP2	0.022	0.147	0.096	0.913
PP3	0.065	0.296	0.135	1.294
PP4	0.059	0.239	0.109	1.518

Correlation matrix for the 12 assets

	S1	S2	S3	S4	CC1	CC2	CC3	CC4	PP1	PP2	PP3	PP4
S1	1.00	0.57	0.73	0.46	0.72*	0.50	0.52	0.32	0.93*	0.49	0.69	0.42
S2	0.57	1.00	0.59	0.28	0.51	0.78*	0.50	0.26	0.48	0.93*	0.52	0.22
S3	0.73	0.59	1.00	0.51	0.49	0.51	0.71*	0.35	0.70	0.51	0.95*	0.48
S4	0.46	0.28	0.51	1.00	0.41	0.27	0.31	0.68*	0.39	0.22	0.51	0.94*
CC1	0.72	0.51	0.49	0.41	1.00	0.65	0.61	0.55	0.42*	0.32	0.34	0.26
CC2	0.50	0.78	0.51	0.27	0.65	1.00	0.56	0.27	0.32	0.49*	0.39	0.21
CC3	0.52	0.50	0.71	0.31	0.61	0.56	1.00	0.31	0.37	0.36	0.45*	0.24
CC4	0.32	0.26	0.35	0.68	0.55	0.27	0.31	1.00	0.13	0.20	0.30	0.38*
PP1	0.93	0.48	0.70	0.39	0.42	0.32	0.37	0.13	1.00	0.48	0.72	0.42
PP2	0.49	0.93	0.51	0.22	0.32	0.49	0.36	0.20	0.48	1.00	0.49	0.19
PP3	0.69	0.52	0.95	0.51	0.34	0.39	0.45	0.30	0.72	0.49	1.00	0.50
PP4	0.42	0.22	0.48	0.94	0.26	0.21	0.24	0.38	0.42	0.19	0.50	1.00

5.2. Summary of assets used in four UPM–LPM utility cases

To help present these cases with significantly non-normal security distributions, 12 assets were utilized. The first four assets are stocks. The second four assets are the result of writing covered call options on the first four stocks. The last four assets represent the purchase of covered puts on the first four stocks.

A Monte Carlo simulation is run to estimate 3000 security returns for the four stocks using a normal distribution random number generator and the parameters and correlations are presented in Table 1. The returns for the covered call and protective put positions are computed using the methodology proposed by Bookstaber and Clarke (1983, 1984).[22] Four random numbers (Wiener process) are generated for each stock for one iteration and then the Cholesky matrix (factorization) is used to obtain correlated returns. With every stock return, the call strategy returns and put strategy returns are computed applying Black–Scholes formula. From the obtained return series, the additional correlations between the put and call return series are computed.

As can be seen, the assets present a wide spectrum of high and low returns, high and low standard deviations, and positive and negative skewness in order to demonstrate the properties of the UPM/LPM model. The same optimization algorithm (the Augmented Lagrangian by Bertsekas, 1982) was used to compute mean-variance, mean-LPM, and UPM/LPM efficient frontiers.[23]

[22] This allows the Monte Carlo simulation to start with a Gaussian distribution and then positive and negative skewness distributions are added with the put and call positions, respectively. Huang et al. (2001) and Coval and Shumway (2001) also provide methods for computing portfolio returns for optioned portfolios.
[23] All non-linear formulations use the basic model described by Eqs. (8)–(10), (13) and (17).

5.3. Case 1: Downside risk aversion ($a = 2$) and upside potential seeking ($c = 3$)

This is probably a very common case where investors wish to reduce downside risk while at the same time preserving as much of the upside returns as economically feasible. This means that the investor is risk averse below the target return and upside potential seeking above the target. The utility function expressing such preferences is the Markowitz (1952) reverse S-shaped function, which means that it is concave below the target return and convex above the target.

In the utility function, where return deviations from some reference point are evaluated by some exponent, risk aversion is represented by an exponent $a > 1$, and upside seeking is represented by $c > 1$. In this example, we assume that the degree of risk aversion is identified $a = 2$, and degree of upside potential seeking is $c = 3$. The minimum target return is set equal to the risk free return of 3%. The utility function *U(r)* is defined as:

$$U(r) = (r - 0.03)^3, \quad \text{for all } r \geq 0.03, \text{ and} \tag{18}$$

$$U(r) = -h \times (0.03 - r)^2, \quad \text{for all } r < 0.03 \text{ or} \tag{19}$$

$$U(r) = \max[(r - 0.03); 0]^3 - h \times \max[(0.03 - r); 0]^2 \tag{20}$$

The efficient frontier of portfolios with maximal expected utility we obtain by varying the slope – h-parameter – by the maximization of the EU*(r)* using input data from Table 1.[24] The efficient frontiers are computed using the ($UPM_{3,3}$, $LPM_{2,3}$), (μ,$LPM_{2,3}$), and (μ, σ) portfolio models.

Using the ($UPM_{3,3}$, $LPM_{2,3}$) coordinate system (Fig. 5), we see that the UPM/LPM(2,3,3) frontier represents a successful result where the UPM is a concave function of the LPM. Note that the other frontiers are not concave in the UPM/LPM(2,3,3)space.

5.4. Case 2: Downside risk aversion ($a = 2$) and upside potential aversion ($c = 0.5$)

As investors differ in their preferences, it is possible that many of them are downside risk averse and upside potential averse. Such a preference corresponds with a conservative investment strategy where the investor is everywhere risk averse. This strategy would try to concentrate returns toward some target return (t). The implied utility function is concave as assumed in the classical theory of expected utility. In the following example, we assume that the degree of upside potential aversion is $c = 0.5$ and the degree of downside risk aversion is $a = 2$[25]. The minimum target return (t) is set equal to risk free return of 3%.

Assume the following utility function:

$$U(r) = \max[(r - 0.03); 0]^{0.5} - h \times \max[(0.03 - r); 0]^2 \tag{21}$$

The general UPM and LPM measures we define according to the assumed utility function as $UPM_{c;t}$ and $LPM_{a;t}$ or $UPM0_{0.5;3}$, and $LPM_{2;3}$. The first part of the equation of the expected utility function corresponds with the applied *UPM* and the second part with the *LPM* measure.

We obtain the efficient frontier of portfolios with maximal expected utility by varying the slope – h-parameter – in the EU*(r)* optimization formulation. *Again, the* UPM/LPM(2,0.5,3) *frontier is concave in the* UPM/LPM(2,0.5,3) *space.* The ($UPM_{0.5;3}$, $LPM_{2;3.}$), (μ,σ), and (μ,$LPM_{2;3}$*)* portfolios provide roughly the same efficient frontiers except in the higher risk area where the μ,LPM(2,3) frontier is not concave. This result makes sense as the utility function for UPM/LPM(2,0.5,3) approximates the quadratic function inherent in (μ,σ) formulation. An alternative strategy, ($UPM_{3;3}$, $LPM_{2,3}$), where the investor has strong emphasis on seeking upside potential generates portfolios with significantly higher risk and a non-concave efficient frontier. (Fig. 6)

[24] The efficient frontiers were generated using the Augmented Lagrangian algorithm developed by Bertsekas (1982). The LANCELOT program package on the NEOS server (http://www-neos.mcs.anl.gov) provides access to this algorithm for large scale optimizations.

[25] See the method for estimation of degree of risk aversion and potential seeking in the appendix of this paper.

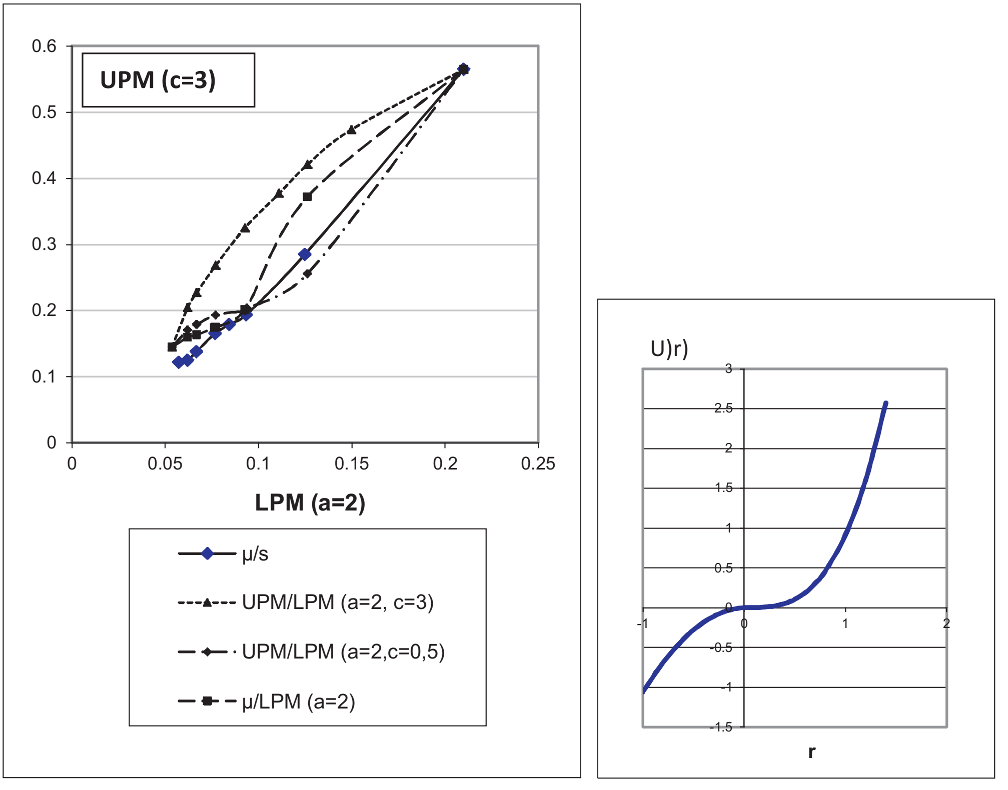

Fig. 5. Case 1 concave–convex utility function and efficient frontier graph for UPM/LPM ($a = 2$, $c = 3$). The Frontier graph contains frontiers for mean-variance (μ/σ), UPM/LPM ($a = 2$, $c = 0.5$), and μ/LPM ($a = 2$) for comparison purposes. Graph coordinates are for UPM/LPM ($a = 2$, $c = 3$). This is the Markowitz (1952) reverse S-shaped utility function.

5.5. Case 3: Downside risk seeking ($a = 0.9$) and upside potential aversion ($c = 0.5$)

If the investor's main concern is not to fall short but without regard of the amount, and to exceed the target return without regard of the amount, then the appropriate utility function is risk seeking below the target, and upside potential averse above the target. Such preferences are not unusual as confirmed by many experimental studies[26]. In addition, there is a strong correspondence with utility functions contained in prospect theory. (Tversky,1995). These preferences indicate a tendency by investors to make risk-averse choices in gains and risk-seeking choices in losses. Such investors are very risk-averse for small losses but will take on investments with small probabilities of very large losses.

Assume the degree of the downside risk seeking to be slightly below risk neutrality $a = 0.9$, as this is the most common finding in the Swalm's (1966) experimental study. The degree of upside potential aversion is assumed to be $c = 0.5$, and the minimum target return is unchanged at 3%.

This implies the following utility function:

$$U(r) = \max\left[(r - 0.03); 0\right]^{0.5} - h \times \max\left[(0.03 - r); 0\right]^{0.9} \tag{22}$$

The resulting utility function is very close to being linear below the target (slightly convex) and concave above the target. We compute new ($UPM_{0.5;3}$, $LPM_{0.9;,3.}$) and (μ,$LPM_{0.9;3}$) portfolios while

[26] Swalm (1966) found that the predominant pattern below $\tau = 0$ is a slight amount of convexity, so that $0 < a < 1$ is descriptive for most of the utility curves found in that study.

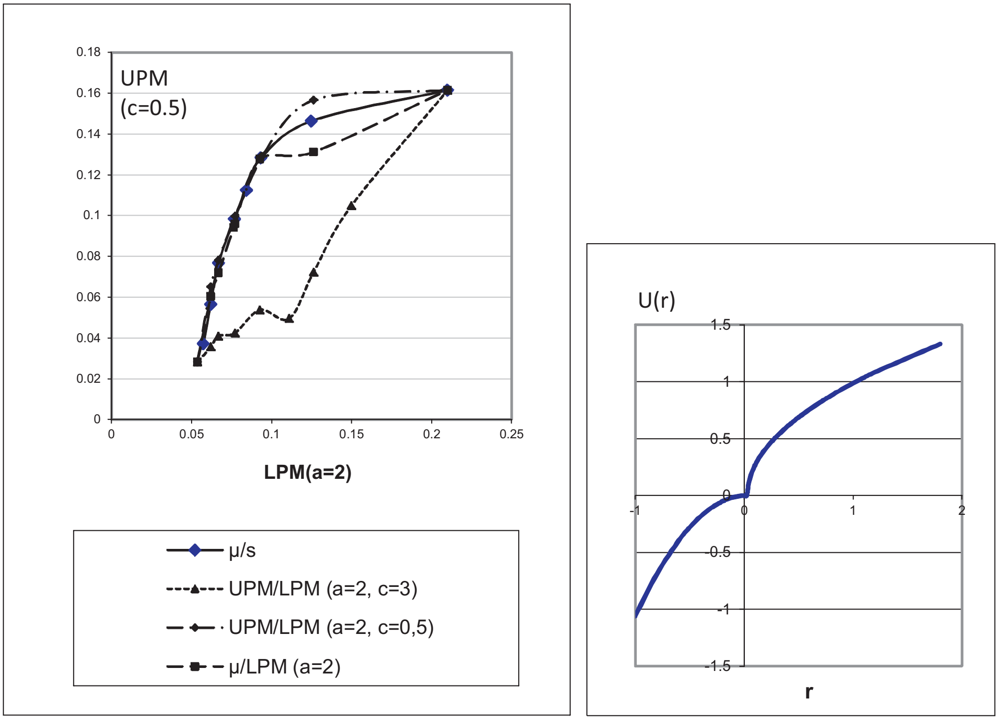

Fig. 6. Case 2 concave–convex utility function and efficient frontier graph for UPM/LPM ($a = 2$, $c = 0.5$). The Frontier Graph contains frontiers for mean-variance (μ/σ), UPM/LPM ($a = 2$, $c = 3$), and μ/LPM ($a = 2$) for comparison purposes. Graph coordinates are for UPM/LPM ($a = 2$, $c = 0.5$). This is close to the traditional quadratic utility function.

the (μ,σ) portfolios remain the same**.** *Again, the* ($\mathrm{UPM}_{0.5;3}$, $\mathrm{LPM}_{0.9;,3.}$) efficient frontier is concave in the ($\mathrm{UPM}_{0.5;3}$, $\mathrm{LPM}_{0.9;,3.}$) space. The other efficient frontiers differ mostly for high values of downside risk. However, the overall closeness of the UPM/LPM(0.9,0.5,3) and the (μ,σ) frontiers in the lower risk region provides support for the Levy and Levy (2004) finding that the (μ,σ) optimization can be employed to construct Prospect Theory-efficient portfolios. (Fig. 7)

5.6. Case 4: Downside risk seeking (a = 0.9) and upside potential seeking (c = 3)

An aggressive investment strategy is presented in this case. The investor wants to participate on the increasing markets whenever returns are above the minimum target return. Whenever returns are below-target, the main concern is not to fall short but without regard to the amount. Thus, our investor likes exposure to high returns and accepts exposure to low returns. In other words, the investor is upside potential seeking above the target return, and risk seeking below the target return. The utility function expressing these preferences is convex above the target return and close to being linear (slightly convex) below the target. The slope of convexity usually changes at the target return because that is where the investor's sensitivity to gains and losses changes. Again, we assume the degree of the risk seeking is slightly below risk neutrality *(*$a = 0.9$), and degree of upside potential seeking is $c = 3$ while the target return remains unchanged at 3%.Thus, the following utility function is generated

$$U(r) = \max[(r - 0.03); 0]^3 - h \times \max[(0.03 - r); 0]^{0.9} \tag{23}$$

For comparison purposes, the ($\mathrm{UPM}_{0.5;3}$, $\mathrm{LPM}_{0.9;3}$) efficient frontier is computed in addition to the efficient frontiers computed for this case. (Note that it does not generate a concave frontier.) *Again,*

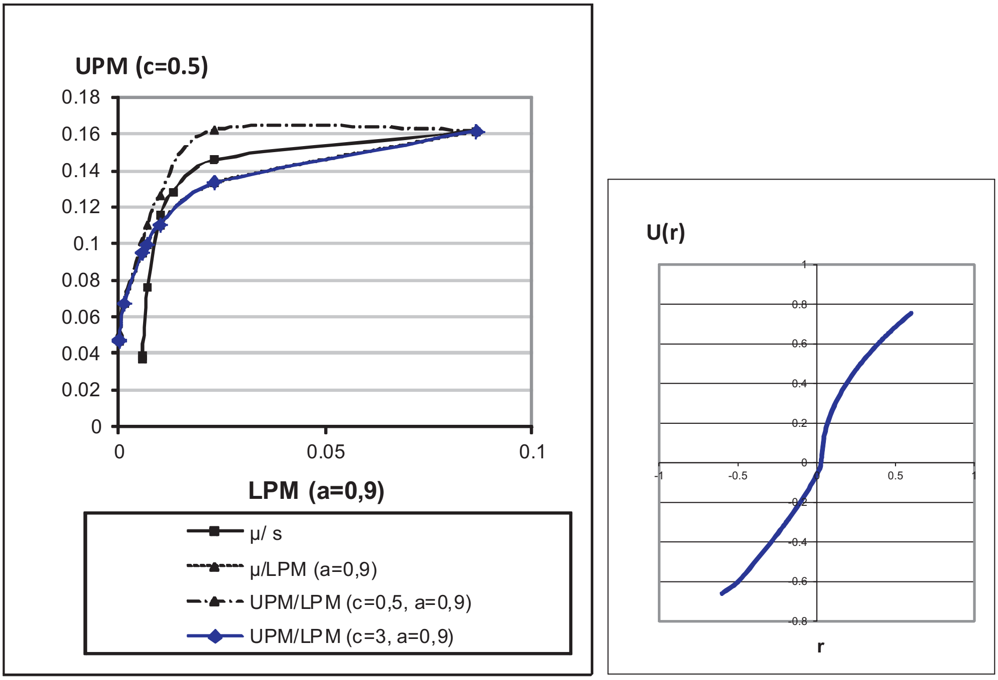

Fig. 7. Case 3 convex–concave utility function and efficient frontier graph for UPM/LPM (a = 0.9, c = 0.5). The frontier graph contains frontiers for mean-variance (μ/σ), UPM/LPM (a = 0.9, c = 3), and μ/LPM (a = 0.9) for comparison purposes. Graph coordinates are for UPM/LPM (a = 0.9, c = 0.5). This is the Kahneman and Tversky (1979) prospect theory S-shaped function.

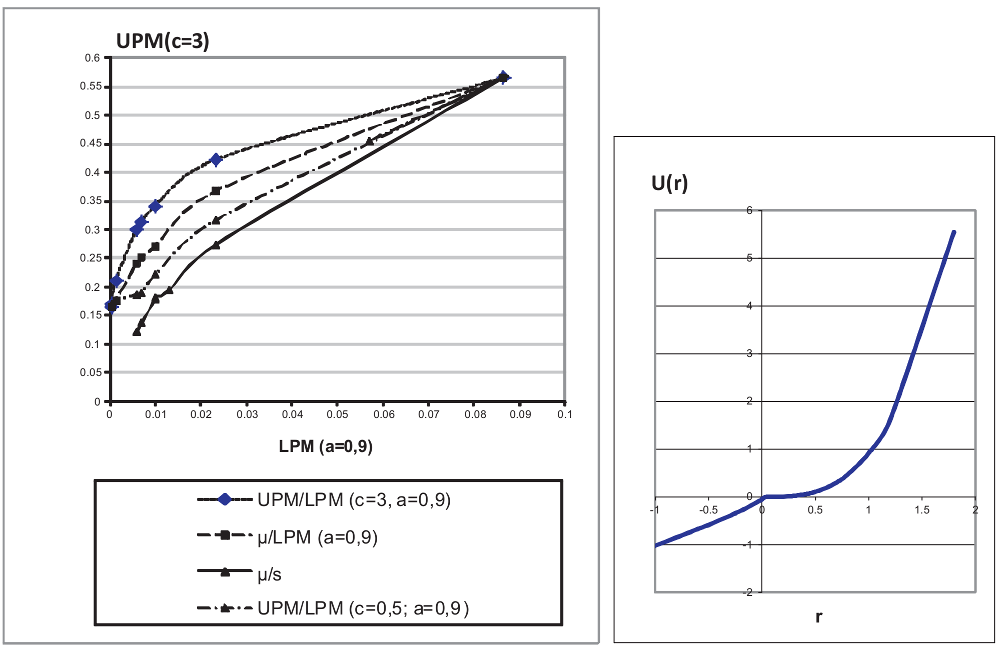

Fig. 8. Case 4 convex–concave utility function and efficient frontier graph for UPM/LPM (a = 0.9, c = 3). The frontier graph contains frontiers for mean-variance (μ/σ), UPM/LPM (a = 0.9, c = 0.5), and μ/LPM (a = 0.9) for comparison purposes. Graph coordinates are for UPM/LPM (a = 0.9, c = 3).

using the ($UPM_{3;3}$, $LPM_{0.9;3.}$) *coordinate system, the* ($UPM_{3;3}$, $LPM_{0.9;3.}$) *generates a concave efficient frontier*. Both the μ-LPM and the (μ,σ) formulations generate concave frontiers in the UPM/LPM space. With this dataset, there is a dominance of the UPM/LPM(0.9,0.5,3) frontier over the other formulations in this space. (Fig. 8)

6. Summary and conclusions

The lower partial moment (LPM) has been the downside risk measure that is most commonly used in portfolio analysis. Its major disadvantage is that its underlying utility functions are linear above some target return. As a result, the upper partial moment (UPM)/lower partial moment (LPM) analysis was suggested by Holthausen (1981), Kang et al. (1996), and Sortino et al. (1999) as a method of dealing with investor utility above the target return.[27] Unfortunately, they only provide dominance rules rather than a portfolio selection methodology that implements the intercorrelations between securities. This paper proposes a formulation of the UPM/LPM portfolio selection model which includes all intercorrelations between securities and presents four utility case studies to illustrate its ability to generate a concave efficient frontier in the appropriate UPM/LPM space.

The methods and techniques proposed in this paper are focused on the following computational issues with UPM/LPM optimization.

- Lack of positive semi-definite UPM and LPM matrices.
- Problems solving non-linear objective function subject to nonlinear constraints.
- Rank of matrix errors.
- Lack of closed form solutions in the optimization problem when using endogenous matrices.

There is still an issue with estimation error, but there is evidence in the literature that the LPM measure is a 'good' estimator of risk when there are asymmetric distributions present in the market and there is a sufficiently large sample size.[28]

The two-fold advantage of the general UPM/LPM model is that it encompasses a vast spectrum of utility theory as well as a large number of symmetric and asymmetric return distributions. Thus, it meets Markowitz's (1959, 2010) admonishment that portfolio theory has to be built on a solid expected utility theory foundation. The UPM/LPM analysis includes the reverse S-shaped utility functions of Friedman and Savage (1948) and Markowitz (1952) as well as the family of utility functions represented by stochastic dominance that are presented in Swalm (1966), Porter (1974), Bawa (1975), and Fishburn (1977). Finally, the UPM/LPM model is consistent with the prospect theory S-shaped utility functions proposed by Kahneman and Tversky (1979) as well as being congruent with Von Neumann and Morgenstern (1944) expected utility theory.

This study was focused on the expected utility and computational issues of UPM/LPM analysis and not on general equilibrium theory. If there exists an aggregate utility function as suggested by Kroll, Levy, and Markowitz (1984), Levy and Levy (2004), and Post and Levy (2005), Post and van Vliet (2002), Baltussen, Post, and van Vliet (2012), then a global optimum consistent with equilibrium asset pricing theory may possibly be obtainable using UPM/LPM analysis.[29]

Appendix A. Estimation of the amount of downside risk aversion and upside potential exposure

We can approximately estimate the investor risk behavior exponent "*a*" using the following methodology originally outlined by Fishburn (1977). We always compare two alternatives with

[27] Mausser et al. (2006) provide a UPM/LPM model that is linear above and below the target and is also a LP heurisitic that ignores the intercorrelation between securities.

[28] Nawrocki (1992), Josephy and Aczel (1993), Bond and Satchell (2002), and Cumova and Nawrocki (2011) provide this evidence.

[29] This would simply require the extension of earlier work by Hogan and Warren (1974), Bawa and Lindenberg (1977), Nantell and Price (1979), and Harlow and Rao (1989) on the LPM-CAPM model to the UPM/LPM CAPM model.

the same UPM, but with different LPM values. In the first example, we can lose A—amount with p—probability. In the second example, we can lose B—amount with the same probability p, but for two states. Using the exponent $a = 1$, i.e. risk neutrality, we would be indifferent between these two alternatives for the B-amount equal to $A/2$.

$$\begin{aligned} p \cdot A^1 &= p \cdot B^1 + p \cdot B^1 \\ p \cdot A^1 &= p \cdot \left(\frac{A}{2}\right)^1 + p \cdot \left(\frac{A}{2}\right)^1 \end{aligned} \tag{24}$$

If we prefer the second alternative, we would have a higher grade of risk aversion because this second alternative has a lower total loss for all cases of higher risk aversion, or $a > 1$. To compute the amount of risk aversion, we will have to compare the same two alternatives with a higher degree of risk aversion, for example $a = 2$.

$$\begin{aligned} p \cdot A^2 &= p \cdot B^2 + p \cdot B^2 \\ p \cdot A^2 &= p \cdot \left(\frac{A}{\sqrt[2]{2}}\right)^2 + p \cdot \left(\frac{A}{\sqrt[2]{2}}\right)^2 \end{aligned} \tag{25}$$

If we have the degree of risk aversion $a = 2$, we would be indifferent between the loss of A—amount with p—probability and the loss of $A/\sqrt{2}$ amount with the same probability p. If we prefer the second alternative, we have a higher degree of risk aversion than $a = 2$. So, we have to repeat the comparison of these two alternatives for higher and higher degrees of a. For example, for the case $a = 3$, we are indifferent between the alternatives, where the loss of B in the second alternative is equal to $A/\sqrt[3]{2}$. If we prefer the second alternative, then we have even a higher degree of risk aversion, and we have to repeat the comparison until we find indifferent alternatives.

In case of a risk seeking investor, we would prefer by the first alternative when $a = 1$. From there, we have to reduce the degree of risk seeking behavior ($a < 1$) until we reach indifference. For the degree of return exposure in the upper part of distribution "c", we proceed the same way, however, potential seeking behavior is $c > 1$ and the conservative strategy on the upside is $0 < c < 1$.

Note that Fishburn (1977), Laughhunn, Payne, and Crum (1980), and Holthausen (1981) all conclude that the a coefficient is sensitive to the total wealth of the investor, i.e. how much is at risk relative to total wealth affects the investor's a coefficient.

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