Insurance: Mathematics and Economics 51 (2012) 624–631

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Stochastic evaluation of life insurance contracts: Model point on asset trajectories and measurement of the error related to aggregation

Oberlain Nteukam T., Frédéric Planchet *

Université de Lyon, Université Claude Bernard Lyon 1, France
Institut de Science Financière et d'Assurances (ISFA), 50 avenue Tony Garnier, 69366 Lyon Cedex 07, France
WINTER & Associés, 43/47 avenue de la grande armée, 75116 Paris, France

ARTICLE INFO

Article history:
Received July 2010
Received in revised form
July 2012
Accepted 3 September 2012

Keywords:
Life Insurance contracts
Unit-linked contracts
Embedded options
TMG guarantee
ALM
Stochastic models
Monte-Carlo simulation

ABSTRACT

In this paper,[1] we are interested in the optimization of computing time when using Monte-Carlo simulations for the pricing of the embedded options in life insurance contracts. We propose a very simple method which consists in grouping the trajectories of the initial process of the asset according to a quantile. The measurement of the distance between the initial process and the discretized process is realized by the L2-norm. L2 distance decreases according to the number of trajectories of the discretized process. The discretized process is then used in the valuation of the life insurance contracts. We note that a wise choice of the discretized process enables us to correctly estimate the price of a European option. Finally, the error due to the valuation of a contract in Euro using the discretized process can be reduced to less than 5%.

1. Introduction

The implementation of an asset/liability management (ALM) model for the management and economic capital evaluation of life insurance contracts requires a very important volume of computations within the framework of Monte Carlo simulations. Indeed, for each trajectory of the asset, the whole of the liability must be simulated, because of the strong interactions between the asset and the liability through the ratchet and through the redistribution of the financial and technical results (*cf.* Planchet et al. (2011)). This leads to the well known problem of nested simulations (*cf.* Bauer et al. (2010) and Gordy and Juneja (2008)).

Various approaches were developed to overcome the practical difficulty of implementing the nested simulation approaches, among which the most used are optimizations inspired from the importance sampling (*cf.* Devineau and Loisel (2009)) and the techniques of replicating portfolio (*cf.* Revelen (2009), Schrager (2008) and Chauvigny and Devineau (2011)). More recently, Bauer et al. (2010) have used the LSMC approach initially proposed by Longstaff and Schwartz (2001) for the pricing of American options. However, optimization techniques are conceived generally for the estimation of the quantile of the excess asset/liability in the framework of the determination of the economic capital and are not always suited to compute the best estimate of the provision. Replicating portfolio approaches are wrongly adapted to the context of French insurance life contracts because of the complexity required when implementing clauses of redistribution of the financial discretionary benefit.

Therefore, practitioners sometimes use a method consist in summarizing the possible evolutions of the asset process in a limited number of characteristic trajectories. This results in proposing a limited number of scenarios of evolution for the asset process, each of these scenarios being characterized by a probability of occurrence. The difficulty is to build the scenarios in an optimal way in order to obtain a good approximation of the value of the provision.

The objective of this paper is to propose a method to build these characteristic trajectories and to provide tools to measure the impact of this simplification on the results. So we provide here a tool for best estimate computing which can be used together with other optimization techniques.

To achieve this goal in an objective manner, we propose a simple discretization of the distribution of the underlying trajectories in an L^2 Hilbert space. Many papers deal with the question of the time discretization of the path of the process (see for example the work of Gobet (2003) and the numerous references

* Corresponding author at: Institut de Science Financière et d'Assurances (ISFA), 50 avenue Tony Garnier, 69366 Lyon Cedex 07, France.
E-mail addresses: onteukam@gmail.com (O. Nteukam T.), frederic.planchet@univ-lyon1.fr, fred@planchet.net, fplanchet@winter-associes.fr (F. Planchet).

[1] Version of 2012/07/08.

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doi:10.1016/j.insmatheco.2012.09.001

therein) and the question of the bias reduction. We will adopt in this paper a different point of view and focus on the discretization of the distribution of the paths. More precisely, a stochastic process S such as those considered here can be viewed as a random variable in an L^2 space. The probability distribution of S is in practice considered as continuous. What we want to do is to find a discrete probability distribution that is "not too far" from the true one. We do not think there is many works on this topics.

2. General characteristics of the discretized process

2.1. Definition

We consider a stochastic process $S(t)$ in $\Omega = [0, +\infty[,] - \infty, +\infty[, \dots$ observed on the time interval $[0, T]$. In practice, $S(t)$ can represent a market value or a total return of assets portfolio. We replace the sample of trajectories of this process by the following simplified trajectories:

- at time t, we choose a partition of Ω, $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$;
- we then write

$$\xi_j(t) = \mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[); \tag{0.1}$$

- we define the process $\xi(t)$ by selecting one of the p trajectories of $\xi_j(t)$, each trajectory being characterized by its probability $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$.

Technically speaking, we replace the continuous distribution of the random variable S, which takes its values in a set of functions, by the discrete distribution of the variable ξ. We then make a package of trajectories according to the quantiles of the initial process $S(t)$. For example, we can choose the intervals so that $\pi_{t,j} = \frac{1}{p}$, which is the approach retained in this paper. In practice, we generally simulate trajectories of initial process S, $S_i(t)$, $1 \le i \le N$, and we usually estimate $\mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[)$ by the following estimator:

$$\tilde{\xi}_j(t) = \frac{1}{N_j} \sum_{i \in \Omega_j} S_i(t)$$

where $\Omega_j = \{i/S_i(t) \in [s_{t,j-1}, s_{t,j}[\}$ and $N_j = |\Omega_j|$.

Two errors are being made with this approximation:

- first, when replacing the trajectories of the continuous process $S(t)$ by the discretized process $\xi(t)$ obtained by selecting one of the p trajectories $\xi_j(t)$ with its probability $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$;
- second, the method of construction per simulation leads to replacing the theoretical expectation by an empirical estimation which introduces sampling fluctuations.

Generally speaking, this approximation is made within the framework of the valuation of options in life insurance contracts, and the projections are thus made under the risk neutral probability, which we shall suppose henceforward. In this context, for $r \ge 0$ interest free rate, $t \to e^{-rt}S(t)$ is a martingale under the risk neutral probability. In this paper, we are interested in the properties of the discretized process $\xi(t)$, which we call the discretized process associated with $S(t)$. We try to quantify and minimize the error generated by using this process to pricing embedded options of life insurance contracts.

First of all, we are going to study some characteristics of the discretized process $(\xi(t))_{t \le T}$. We begin by estimating distribution of this process. This distribution is established in a general context not requiring knowing characteristics of the initial process $S(t)$ (Section 2.2). The L^2-norm between the initial process and the discretized process gives a first vision of the error due to usage of the discretized process $\xi(t)$ (Section 2.3).

2.2. Distribution of the discretized process

$\xi(t)$ is a discrete process because it can take a finite number of values. Indeed, $\xi(t)$ takes p possible values $\{\xi_j(t), i = 1 \dots p\}$ with probabilities $\{\pi_{t,j}, i = 1 \dots p\}$. We note that $\xi_j(t)$ are deterministic because the partitions of Ω, $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ are not random. The process $\xi(t)$ is well defined and the mean of the random variables $\xi(t)$ is identical to the mean of the initial process $S(t)$ at the same time: $\mathbf{E}(\xi(t)) = \mathbf{E}(S(t))$.

Proof. When $\mathbf{E}(X|A) = \frac{1}{\Pr(A)}\mathbf{E}(X 1_A)$, we can write:

$$\begin{aligned}
\mathbf{E}(\xi(t)) &= \sum_{j=1}^{p} \pi_j \xi_j(t) \\
&= \sum_{j=1}^{p} \pi_j \mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[) \\
&= \sum_{j=1}^{p} \pi_j \left(\frac{1}{\Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)} \mathbf{E}\big(S(t)\mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[}\big) \right).
\end{aligned}$$

By using the definition $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$ we have

$$\begin{aligned}
\mathbf{E}(\xi(t)) &= \mathbf{E}\left(S(t) \sum_{j=1}^{p} \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} \right) \\
&= \mathbf{E}\left(S(t)\mathbf{1}_{\bigcup_{j=1}^{p} \{S(t)\in[s_{t,j-1},s_{t,j}[\}} \right) = \mathbf{E}(S(t))
\end{aligned}$$

because $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ is a partition of Ω. □

2.3. L^2-norm between $\xi(t)$ and $S(t)$

We want to estimate the L^2-norm between the initial process and the discretized process. This norm is defined by:

$$\|S - \xi\|_{L^2} = \mathbf{E}\left(\int_0^T (S(t) - \xi(t))^2 dt \right)^{\frac{1}{2}}. \tag{0.2}$$

It can be correctly calculated by using

$$\begin{aligned}
\|S - \xi\|_{L^2} &= \sqrt{\sum_{j=1}^{p} \left(\int_0^T \mathrm{Var}(X_j(t)) dt \right)} \\
&= \sqrt{\int_0^T \sum_{j=1}^{p} \mathrm{Var}(X_j(t)) dt}
\end{aligned} \tag{0.3}$$

where $X_j(t) = S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[$. The proof of this result is shown below.

Proof. We are interested in calculating $\mathbf{E}(\int_0^T (S(t) - \xi(t))^2 dt)$. Because $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ is a partition of Ω, the intersection of two distinct sets $\{S(t) \in [s_{t,j-1}, s_{t,j}[\}$ is empty. Thus, we can write

$$\begin{aligned}
&\mathbf{E}\left(\int_0^T (S(t) - \xi(t))^2 dt \right) \\
&\quad = \mathbf{E}\left(\int_0^T \left(\sum_{j=1}^{p} (S(t) - \xi_j(t)) \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} \right)^2 dt \right) \\
&\quad = \mathbf{E}\left(\sum_{j=1}^{p} \int_0^T (S(t) - \xi_j(t))^2 \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} dt \right) \\
&\quad = \sum_{j=1}^{p} \mathbf{E}\left(\int_0^T (S(t) - \xi_j(t))^2 \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} dt \right).
\end{aligned}$$

If $X_j(t) = S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[$, we have

$$\mathbf{E}\left(\int_0^T (S(t)-\xi(t))^2 dt\right) = \sum_{j=1}^{p} \mathbf{E}\left(\int_0^T (X_j(t)-\mathbf{E}(X_j(t)))^2 dt\right).$$

An application of Fubini's theorem shows that, for all j

$$\mathbf{E}\left(\int_0^T (X_j(t)-\mathbf{E}(X_j(t)))^2 dt\right) = \int_0^T \mathbf{E}((X_j(t)-\mathbf{E}(X_j(t)))^2)dt.$$

We finally obtain the result we want. In particular, we deduce the following property

$$\|S-\xi\|_{L^2} \le \sqrt{\int_0^T \mathrm{Var}(S(t))dt}. \quad \square \tag{0.4}$$

Proof. By construction, the process ξ_p converges almost surely towards S when $p \to +\infty$ and the function $f(p) = \|S-\xi\|_{L^2}$ is decreasing. Thus, $f(1) \ge f(p)$, $\forall p \ge 1$, which gives the result. □

3. The particular case of geometric Brownian motion

After having specified the main properties of the discretized process within a general framework, we are now considering the case when $S(t)$ is a geometric Brownian motion, as in the model of Black and Scholes (1973). In the first two sections of this part, we deduct some properties of the discretized process. In the third section, we show that the process $\xi(t)$ allows the correct estimation of the price of a European option.

3.1. Density distribution of the process

We suppose that $S(t)$ is a Geometric Brownian motion:

$$S(t) = S_0 \exp\left(\left(r-\frac{\sigma^2}{2}\right)t + \sigma B(t)\right). \tag{0.5}$$

In this case, $\Omega = [0,+\infty[$. $Y(t) = \frac{S(t)}{S_0}$ has a log-normal distribution with parameters $(m_t = (r-\frac{\sigma^2}{2})t, \omega_t^2 = \sigma^2 t)$. The density of the log-normal distribution is given by

$$f(y) = \frac{1}{\omega_t y\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(y)-m_t}{\omega_t}\right)^2\right]. \tag{0.6}$$

We can deduce the density of the truncated log-normal distribution $Y_j = Y(t)|Y(t) \in [y_{j-1}, y_j[$

$$f_j(y) = \frac{1}{\omega_t y \pi_j \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(y)-m_t}{\omega_t}\right)^2\right] \mathbf{1}_{[y_{j-1},y_j[}(x) \tag{0.7}$$

where $y_j = \frac{\widehat{S}_j}{S_0}$, $\pi_j = F(y_j) - F(y_{j-1})$ and $F(y) = \Phi(\frac{\ln(y)-m}{\omega})$ with Φ being the cumulative distribution function of a standard normal distribution. Note that

$$\mathbf{E}(Y_j) = \mathbf{E}(Y(t)|Y(t) \in [y_{j-1}, y_j[)$$
$$= \mathbf{E}\left(\frac{S(t)}{S_0}\middle|\frac{S(t)}{S_0} \in \left[\frac{s_{t,j-1}}{S_0}, \frac{s_{t,j}}{S_0}\right[\right)$$

and we deduce that $\mathbf{E}(X_j) = S_0 \times \mathbf{E}(Y_j)$. But we also have

$$\mathbf{E}(Y_j) = \frac{1}{\omega_t \pi_j \sqrt{2\pi}} \int_{y_{j-1}}^{y_j} \exp\left[-\frac{1}{2}\left(\frac{\ln(y)-m_t}{\omega_t}\right)^2\right] dy.$$

Let $u = \frac{\ln(y)-m_t}{\omega_t} - \omega_t$, $du = \frac{dy}{y\omega_t} = \exp(-(m_t + \omega_t \times u + \omega_t^2))\frac{dy}{\omega_t}$, we find that

$$\mathbf{E}(Y_j) = \frac{1}{\pi_j \times \sqrt{2\pi}} \int_{b_{j-1,t}}^{b_{j,t}} \exp\left[-\frac{1}{2}(u+\omega_t)^2\right] \times \exp(m_t + \omega_t u + \omega_t^2)du,$$

where $b_{j,t} = \frac{\ln(y_j)-m_t}{\omega_t} - \omega_t$ and

$$\mathbf{E}(Y_j) = \frac{\exp(m_t + \frac{\omega_t^2}{2})}{\pi_j^*} \int_{b_{j-1,t}}^{b_{j,t}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du$$
$$= \frac{\Phi(b_j,t) - \Phi(b_{j-1},t)}{\Phi(b_{j,t}+\omega_t) - \Phi(b_{j-1,t}+\omega_t)} \exp\left(m_t + \frac{\omega_t^2}{2}\right).$$

We can find, with a unique interval $b_{0,t} = -\infty$ and $b_{1,t} = +\infty$, the expectation of a log-normal distribution: $\mathbf{E}(Y_1) = \exp(m_t + \frac{\omega_t^2}{2})$. Finally

$$\mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[) = S_0 \times \mathbf{E}(Y_j)$$
$$= S_0 \times \frac{\Phi(b_{j,t}) - \Phi(b_{j-1,t})}{\Phi(b_{j,t}+\omega_t) - \Phi(b_{j-1,t}+\omega_t)} \exp\left(m_t + \frac{\omega_t^2}{2}\right)$$
$$\Rightarrow \xi_j(t) = \frac{\Phi(b_{j,t}) - \Phi(b_{j-1,t})}{\Phi(b_{j,t}+\omega_t) - \Phi(b_{j-1,t}+\omega_t)} S_0 \exp(rt) \tag{0.8}$$

where $b_{j,t} = \frac{\ln(\frac{s_{t,j}}{S_0})-m_t}{\omega_t} - \omega_t$.

To summarize, in a Black and Scholes model, the discretized process $\xi(t)$ has a discrete distribution and $\xi(t) = \xi_j(t) = \frac{\pi'_{j,t}}{\pi_j} S_0 \exp(rt)$ with probability π_j such that

$$\begin{cases} \pi'_{j,t} = \Phi(b_{j,t}) - \Phi(b_{j-1,t}) \text{ et } \pi_j = \Phi(b_{j,t}+\omega_t) - \Phi(b_{j-1,t}+\omega_t) \\ b_{j,t} = \dfrac{\ln\left(\frac{s_{t,j}}{S_0}\right) - m_t}{\omega_t} - \omega_t \\ m_t = \left(r - \dfrac{\sigma^2}{2}\right) t \text{ et } \omega_t^2 = \sigma^2 t \end{cases}$$

3.2. L^2-norm

We can establish a closed formula of the L^2-norm between the initial process $S(t)$ and the discretized process $\xi(t)$. To get it, we only need to determine the second moment of Y_j

$$\mathbf{E}(Y_j^2) = \frac{1}{\omega_t \pi_j \sqrt{2\pi}} \int_{y_{j-1}}^{y_j} y \exp\left[-\frac{1}{2}\left(\frac{\ln(y)-m_t}{\omega_t}\right)^2\right] dy.$$

Let $u = \frac{\ln(y)-m_t}{\omega_t} - \omega_t$, we have $y = \exp(u\omega_t + \omega_t^2 + m_t)$ and $du = \frac{dy}{y\omega_t} = \exp(-(m_t + \omega_t \times u + \omega_t^2))\frac{dy}{\omega_t}$, and we find that

$$\mathbf{E}(Y_j^2) = \frac{1}{\pi_j \times \sqrt{2\pi}} \int_{b_{j-1,t}}^{b_{j,t}} \exp\left[-\frac{1}{2}(u+\omega_t)^2\right] \times \exp(2m_t + 2\omega_t u + 2\omega_t^2)du,$$

where $b_{j,t} = \frac{\ln(y_j)-m_t}{\omega_t} - \omega_t$ and thus

$$\mathbf{E}(Y_j^2) = \frac{1}{\pi_j \times \sqrt{2\pi}} \int_{b_{j-1,t}}^{b_{j,t}} \exp\left[-\frac{1}{2}u^2 - \omega_t u - \frac{1}{2}\omega_t^2 + 2m_t + 2\omega_t u + 2\omega_t^2\right] du$$
$$= \frac{1}{\pi_j \times \sqrt{2\pi}} \int_{b_{j-1,t}}^{b_{j,t}} \exp\left[-\frac{1}{2}u^2 + \omega_t u\right.$$

$$- \frac{1}{2}\omega_t^2 + 2m_t + 2\omega_t^2\Big] du$$

$$= \frac{1}{\pi_j \times \sqrt{2\pi}} \int_{b_{j-1,t}}^{b_{j,t}} \exp\left[-\frac{1}{2}(u-\omega_t)^2 + 2m_t + 2\omega_t^2\right] du$$

$$= \frac{\exp(2\omega_t^2 + 2m_t)}{\pi_j} \int_{b_{j-1,t}-\omega_t}^{b_{j,t}-\omega_t} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}v^2\right] dv$$

$$= \frac{\Phi(b_{j,t}-\omega_t) - \Phi(b_{j-1,t}-\omega_t)}{\pi_j} \exp(2\omega_t^2 + 2m_t).$$

The variance is deduced from this last expression

$$(\mathbf{E}(Y_j))^2 = \left(\frac{\Phi(b_{j,t}) - \Phi(b_{j-1,t})}{\pi_j}\right)^2 \exp(2m_t + \omega_t^2).$$

$$\begin{aligned} \mathrm{Var}(Y_j) &= \mathbf{E}(Y_j^2) - (\mathbf{E}(Y_j))^2 \\ &= \frac{\Phi(b_{j,t}-\omega_t) - \Phi(b_{j-1,t}-\omega_t)}{\pi_j} \exp(2\omega_t^2 + 2m_t) \\ &\quad - \left(\frac{\Phi(b_{j,t}) - \Phi(b_{j-1,t})}{\pi_j}\right)^2 \exp(2m_t + \omega_t^2) \\ &= \exp(2m_t + \omega_t^2)\left[\frac{\Phi(b_{j,t}-\omega_t) - \Phi(b_{j-1,t}-\omega_t)}{\pi_j} \right. \\ &\quad \left. \times \exp(\omega_t^2) - \left(\frac{\Phi(b_{j,t}) - \Phi(b_{j-1,t})}{\pi_j}\right)^2\right]. \end{aligned}$$

3.3. Valuation of a European option

We are interested in the error due to the valuation of a European option when we replace the initial process $S(t)$ by discretized process $\xi_j(t) = \mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[)$. We remind that these options are for example used in unit-linked life insurance contracts with minimum death guarantee. We thus have to compare $c_\xi = \mathbf{E}([K - \xi(T)]^+)$ and $c_S = \mathbf{E}([K - S(T)]+)$.

Using the Black and Scholes formula we remind that:

$$c_S(S_0, T, K, r) = K\Phi(-d_2(T)) - S_0\Phi(-d_1(T)) \exp(rT) \tag{0.9}$$

with $d_2(T) = \frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}} = -\frac{\ln(\frac{K}{S_0})-m_T}{\omega_T}$, $d_1(T) = d_2(T) + \sigma\sqrt{T} = d_2(T) + \omega_T$.

We have

$$c_\xi(S_0, T, K, r) = \mathbf{E}([K - \xi(T)]^+) = \sum_{j=1}^{p} \pi_j \times [K - \xi_j(T)]^+. \tag{0.10}$$

We note that $\xi_j(T)$ increases with j because $\xi_j(T) \geq s_{T,j-1} \geq \xi_{j-1}(T)$. If $\xi_j(T) > K \quad \forall j$ then $c_\xi = 0$; this case is not interesting. We suppose that $\exists j_0$ so that $\xi_{j_0}(T) < K \leq \xi_{j_0+1}(T)$ thus

$$c_\xi(S_0, T, K, r) = \sum_{j=1}^{j_0} \pi_j \times \left[K - \xi_j(T)\right]. \tag{0.11}$$

Using (0.10) and the results of Section 3.1 we have

$$c_\xi(S_0, T, K, r) = \sum_{j=1}^{p} \pi_j \times \left[K - \frac{\pi'_{j,T}}{\pi_j} S_0 \exp(rT)\right]^+. \tag{0.12}$$

When we combine (0.11) and (0.12), we have:

$$\begin{aligned} c_\xi(S_0, T, K, r) &= \sum_{j=1}^{j_0} \pi_j \times \left[K - \frac{\pi'_{j,T}}{\pi_j} S_0 \exp(rT)\right] \\ &= \sum_{j=1}^{j_0} \left[\pi_j \times K - \pi'_{j,T} \times S_0 \exp(rT)\right] \\ &= (\Phi(b_{j_0,T} + \omega) - \Phi(b_{1,T} + \omega)) \\ &\quad \times K - (\Phi(b_{j_0,T}) - \Phi(b_{1,T})) \times S_0 \exp(rT). \end{aligned}$$

but $b_{1,T} = -\infty$ (because $s_{T,1} = 0$) thus

$$c_\xi(S_0, T, K, r) = \Phi(b_{j_0,T} + \omega) \times K - \Phi(b_{j_0,T}) \times S_0 \exp(rT) \tag{0.13}$$

with $b_{j_0,T} = \frac{\ln(\frac{s_{j_0}}{S_0})-m_T}{\omega_T} - \omega_T$. If we choose a partition so that $\exists j^* s_{T,j^*} = K$ then we would have $j^* = j_0$ and $s_{T,j_0} = K$ and then

$$\begin{aligned} b_{j_0,T} &= \frac{\ln(\frac{K}{S_0}) - m_T}{\omega_T} - \omega_T = -\left(-\frac{\ln(\frac{K}{S_0}) - m_T}{\omega_T} + \omega_T\right) \\ &= -d_1(T) \end{aligned}$$

and $b_{j_0} + \omega_T = -d_1(T) + \omega_T = -d_2(T)$. Finally

$$\begin{aligned} c_\xi(S_0, T, K, r) &= K \times \Phi(-d_2(T)) - S_0 \times \Phi(-d_1(T)) \times \exp(rT) \\ &= K \times \Phi(-d_2(T)) - S_0 \times \Phi(-d_1(T)) \times \exp(rT) \\ &= c_S(S_0, T, K, r). \end{aligned}$$

In the Black and Scholes model, the price of the option in the discretized process is equal to the price of the option in the initial process if the partition of $[0, +\infty[$, $\{[s_{T,j-1}, s_{T,j}[, 1 \leq j \leq p\}$ is selected like $\exists j^* s_{j^*} = K$.

To summarize, $\{[s_{T,j-1}, s_{T,j}[, 1 \leq j \leq p\}$ is a partition of $[0, +\infty[$, we can find a unique j_0 so that $s_{T,j_0} \leq K < s_{T,j_0+1}$ and the price of the European option obtained with the discretized process of process $S(t)$ is given by

$$c_\xi(S_0, T, K, r) = \begin{cases} c_S(S_0, T, s_{T,j_0}, r), & \text{when } s_{T,j_0} \leq K < \xi_{j_0+1}(T) \\ c_S(S_0, T, s_{T,j_0+1}, r), & \text{when } \xi_{j_0+1}(T) \leq K \leq s_{T,j_0+1}. \end{cases} \tag{0.14}$$

In the case of a sequence of options with different maturities (as for example in the valuation of unit-linked life insurance contracts with death minimum guarantee), the choices of the partition at time t used to construct the discretized process allows the control of the price of every option.

3.4. Illustration

We suppose that the value of asset is a Black and Scholes model with parameters $S_0 = 1$, $\mu = 8, 5\%$ and $\sigma = 25\%$ and interest free rate is $r = 5\%$. We make 100 000 simulations. The discretized process is selected by choosing $s_{t,j}$ like $\pi_{t,j} = \frac{1}{p}$. The quantiles used for the borders of the intervals of discretization are estimated empirically by using trajectories of the initial process.

The simulations are made for maturity 1 year ($T = 1$). The graphs in Fig. 1 below compare the trajectories of the initial process to those of the discretized process.

We note that the increase in the number of trajectories of the discretized process gives a better estimation. However, a discretization of 100,000 trajectories of the initial process in 100 discretization trajectories already allows to obtain a good estimate of the density distribution of the initial process (Fig. 2).

With a discretization of 100 trajectories, that is to say 1000 times less trajectories than those of the initial process, the density of the discretized process is very close to that of the initial process. This good approximation of the density of the initial process leads us to reduce the distance between the initial process and the discretized process:

We can note that the marginal profit in precision decreases very quickly as the number of trajectories of the discretized process raises, and we quickly obtain a satisfactory compromise between precision and cost in terms of computing time (Fig. 3).

4. Application in valuation portfolio of life insurance contracts

The technique of discretization of the trajectories of an asset is used to reduce the computing time while optimizing the results. In

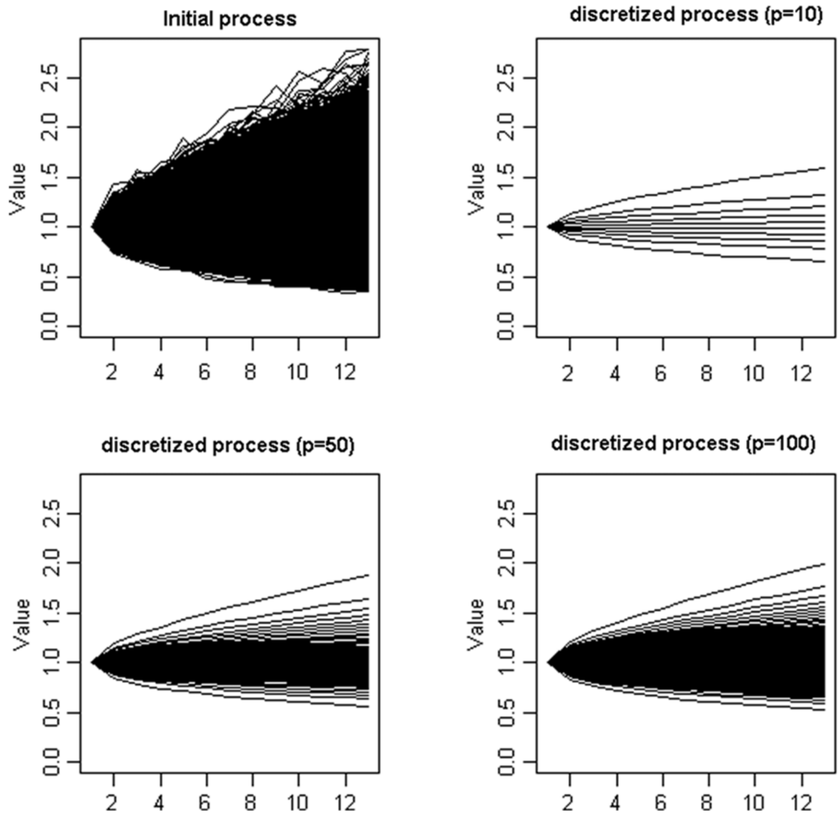

Fig. 1. Initial process vs. discretized process.

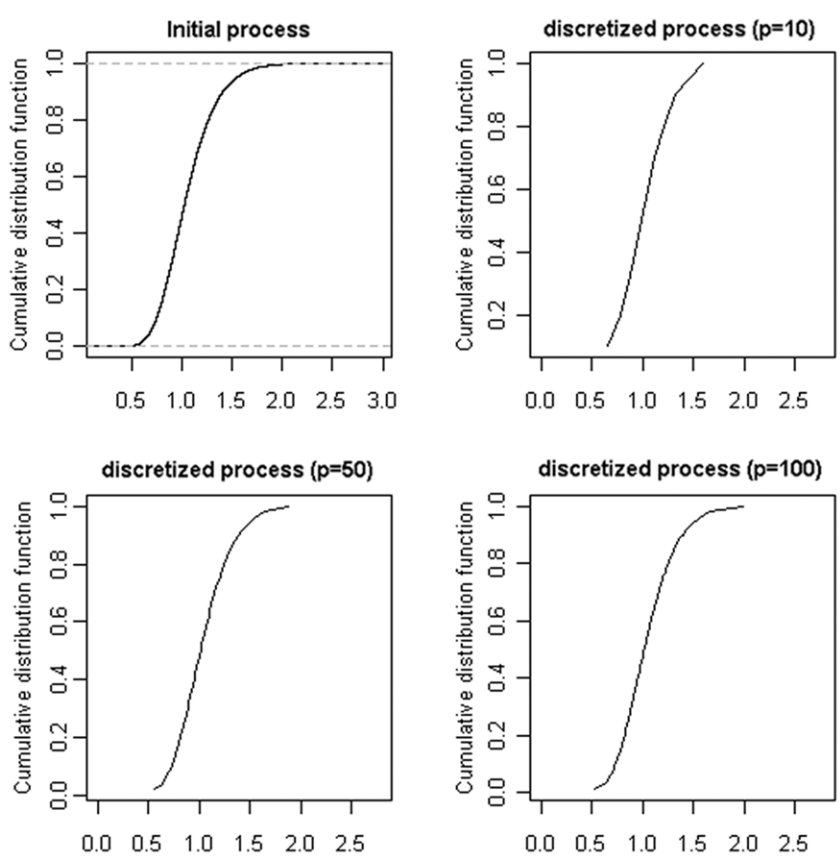

Fig. 2. Densities of probability (1 year) of the initial process and the discretized process.

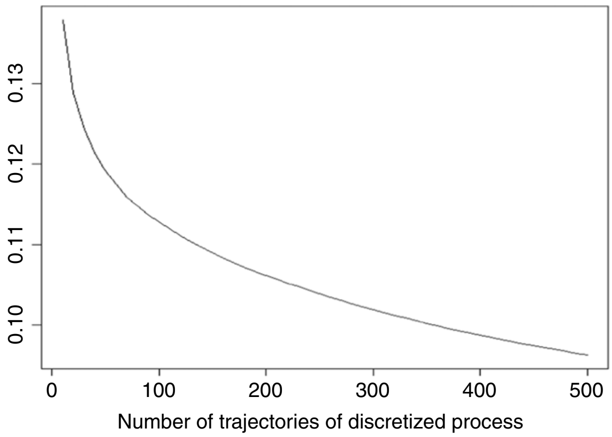

Fig. 3. L2-norm between the initial process and the discretized process.

this section, we are interested in the evaluation of a guarantee of minimum rate of return on a contract in Euros, which is classical in French life insurance contracts.

4.1. Description of the contract

We consider a life insurance contract in which the premium was revalued at a 3.50% rate (minimum rate guaranteed over one year of 60% of the TME, and profit-sharing under deduction of the guaranteed minimum rate and at least equal to 85% of the profits of the financial management and to 90% of the technical profits).

In practice, the rate of revalorization of the savings is the maximum between the guaranteed minimum rate (here 3.5%) and 85% of the profits of the financial management for which it is advisable to add 90% of the technical profits

$$\mathrm{R}_t^g = \max(TMG; 85\%R_t^f + 90\%R_t^\tau) \tag{0.15}$$

with:

- TMG: the annual minimum rate guaranteed,
- R_t^f: the financial rate of return of the portfolio of assets at year t,
- R_t^τ: a technical rate of return.

If we set $R_t^n = 85\%R_t^f + 90\%R_t^\tau$, then we can write

$$R_t^g = \max(TMG; R_t^n) = R_t^n + [TMG - R_t^n]^+ \tag{0.16}$$

$[TMG - R_t^n]^+$ can be similar to the payoff of an option of the interest rate floor type. However, the distribution of the process net return R_t^n is not explicitly known. Indeed, R_t^n depends on the evolution of the financial assets but is also impacted by the technical risks such as the mortality and the ratchet, but also by the decisions of management. The explicit form of this distribution is thus difficult to determine. The evaluation of this contract requires having recourse to some techniques of simulations.

4.1.1. Valuation of the rate guarantee

The evaluation is made contract by contract. The individual mathematical reserves are calculated according to a retrospective approach which aims at capitalizing the premiums invested by the insurants reduced of ratchets. The individual reserve is obtained by applying the following formula:

$$EA_t = [EA_{t-1} + C_t(1 - \tau_C) - R_t] \times (1 + R_t^g \times (1 - p)) \tag{0.17}$$

EA_t = savings acquired at time t;
C_t = premium at time t;
R_t = ratchet at time t;
τ_c = rate of loading on the premium ($\tau_c = 3.5\%$);
R_t^g = interest rate of capitalization at time t;
p = rate of tax and social security deduction (11.8%).

We can write the payoff related of the guaranteed TMG:

$$F_t = [EA_{t-1} + C_t(1 - \tau_C) - R_t] \times [TMG - R_t^n]^+. \tag{0.18}$$

4.1.2. Finance strategy and modeling of the portfolio of assets

We suppose that the portfolio of assets is constituted by a risk-free asset and a risky asset. The risk-free asset produces an annual return equal to 5%. We suppose that the risky asset is a Black and Scholes process, which produces an annual return on 8.5% with a 25% volatility. The target allocation of the portfolio of assets is composed of 80% of risk-free asset and 20% of risky asset. The objective is to recompose the portfolio of assets at the end of every year with respect to the target allocation.

4.2. Evaluation of a contract using simulations

4.2.1. Initial process

By using the initial process $S(t)$, the price of the guaranteed TMG can be obtained with the following formula

$$FloorLet_t = [EA_{t-1} + C_t(1 - \tau_C) - R_t] \times \frac{1}{Ns} \sum_{i=1}^{p} [TMG - 85\%R_{t,i}^f - 90\%R_t^\tau]^+ e^{-rt} \tag{0.19}$$

where

- $R_{t,i}^f = \beta \times r + \alpha \times R_i(t)$;
- $R(t) = \ln(S(t)) - \ln(S(t-1))$ is a return of risky asset between t and t -1;
- β allocation of the risk-free asset;
- α allocation of the risky asset.

4.2.2. Discretized process

In our example, we considered that the portfolio of asset was affected by a single source of risk. The TMG is a guarantee of return, the technique of discretization will apply to the process of return. Thus, the application consists of five steps:

- *Step* 1: simulation of trajectories for the risky asset $S(t)$,
- *Step* 2: calculation of the return process of the risky asset $R(t) = \ln(S(t)) - \ln(S(t-1))$,
- *Step* 3: determination of the partition of possible values of the process of return on the risky asset $\{[r_{t,j-1}, r_{t,j}[, 1 \le j \le p\}$. In our case, the sets of this partition are intervals, the borders of which are the quantiles of the process of the returns,
- *Step* 4: discretization of the process $R(t)$: $R\xi_j(t) = \mathbf{E}(R(t)|R(t) \in [r_{t,j-1}, r_{t,j}[)$,
- *Step* 5: determination of $\pi_{t,j} = \Pr(R(t) \in [r_{t,j-1}, r_{t,j}[)$, the probability of occurrence of each trajectory.

We can estimate the value of the TMG at time t

$$FloorLet_t = [EA_{t-1} + C_t(1 - \tau_C) - R_t] \times \sum_{j=1}^{p} \pi_{t,j} [TMG - 85\%R_{t,j}^f - 90\%R_t^\tau]^+ e^{-rt} \tag{0.20}$$

where

- $R_{t,j}^f = \beta \times r + \alpha \times R\xi_j(t)$;
- β allocation of risk-free asset;
- α allocation of risky asset.

4.2.3. Hypothesis of simulation

We have a 1,000 policy-holders aged 45 years which have subscribed to a contract on 8 years. The TMG is fixed to 3.5%. We

Table 1

Hypothesis	
	Parameters
Insurance portfolio	1000 insured aged 45
Maturity of insurance portfolio	8 years
Ratchet (% of contracts)	1%
Mortality	TH 00-02
Initial premium	100 €
Periodic premium	0 €
TMG	3.50%
Expenses rate of management	0.50%
Rate of tax and social security deduction	11.80%
Number of simulations	100,000

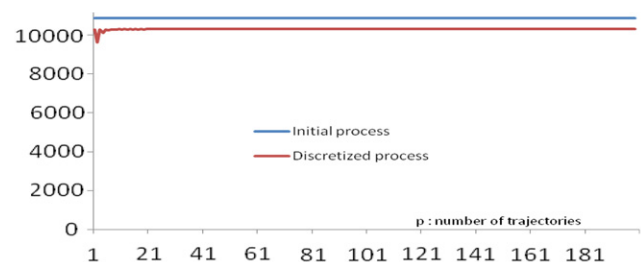

Fig. 4. Price of the TMG in the initial model and in the discretized processes.

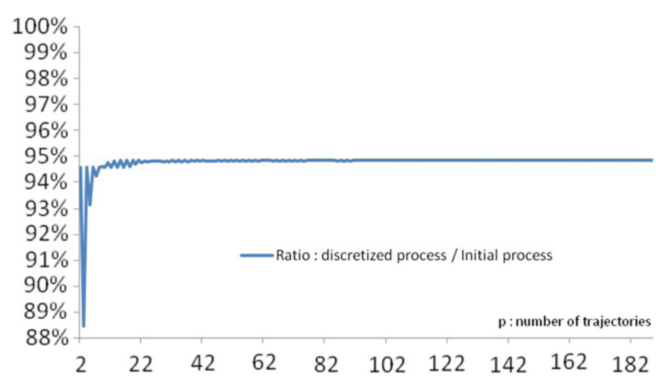

Fig. 5. Evolution of the ratio between the value of TMG guarantee in the discretized model and in the initial process.

suppose that the mortality of the portfolio is modeled by TH00-02. The annual ratchets are fixed to 1% of the number of contract. All the parameters are resumed in Table 1.

4.2.4. Results

The evaluation of the rates guarantee on the contract in Euros in both models (initial model and discretized model) shows that when the number of trajectories of the discretized process is higher than 1, the difference between the two evaluations is lower than 12%. The graph in Fig. 4 allows the visualization of the value of rates guarantee on the contract in Euros according to the number of trajectories of the discretized process.

The graph in Fig. 5 shows the evolution of the ratio between the value of rates guarantee in the discretized model and the value of rates guarantee in the initial process.

We note a very high volatility of the ratio when the number of trajectories of the discretized process is lower than 25. The ratio starts to be stabilized around 95% when the number of trajectories is higher than 100. Then, the ratio converges very slightly towards 100%, in the sense that an increase in a trajectory (p with $p + 1$) results in a marginal profit on the precision of the results obtained.

A discretized process with 100 trajectories gives a value estimated at nearly 95% of the real price of TMG guarantee. Thus, it is not necessary to indefinitely increase the number of trajectories of the discretized process; a discretization in 100 trajectories is enough to provide a correct approximation of the value of the TMG. However, it is advisable to correctly estimate the probabilities of occurrence of each of the 100 trajectories. This last point requires to know the distribution of the initial process.

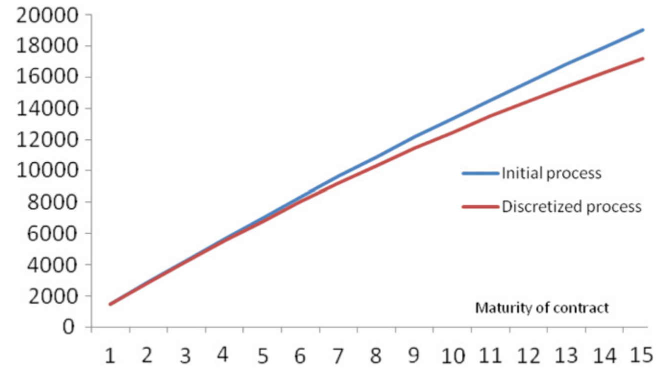

Fig. 6. Evolution of prices in both models (discretized process and initial process).

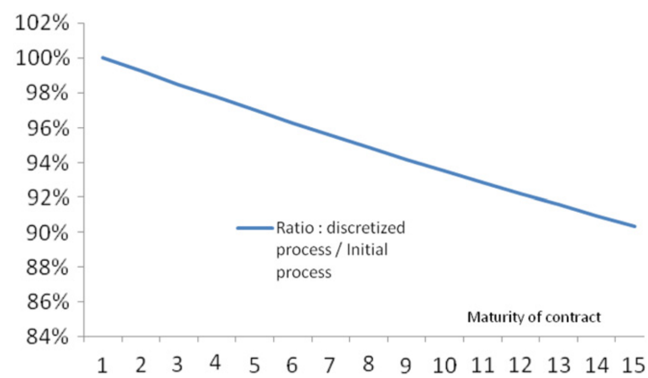

Fig. 7. Ratio of prices in both models (discretized process and initial process).

The price of TMG guarantee is determined from the values lower than the TMG; thus the convergence towards the real price is faster than the convergence of the discretized process towards the initial process.

In the following, we will work with a discretization of 100 trajectories.

4.2.4.1. Impact of the maturity of contract. The graphs in Figs. 6 and 7 show the impact of maturity on discretization.

When the maturity of the contract is equal to 1 year, the error of valorization of TMG guarantee using the discretized process is null, the ratio between the price obtained from the discretized model and the price in the initial model is 100%. Then, this ratio decreases continuously to be fixed at nearly 87.5% for a contract of maturity of 20 years.

4.2.4.2. Impact of the age of the policy-holders. The graphs in Figs. 8 and 9 show the impact of the age of the policy-holder on discretization.

The increase in the age of the policy-holder slightly impacts the error of valorization of TMG guarantee, related to the replacement of the 100,000 trajectories of the initial process by the 100 synthetic trajectories of the discretized process. For policy-holders of age ranging from 21 to 67 years, the variation of this error is in the absolute strictly lower than 0.30%. This variation is of the same scale and same width as the error related to the technique of simulation of the 100,000 trajectories. We can thus conclude that the age of the policy-holders does not have impact on the error related to the discretization of the trajectories of the initial process.

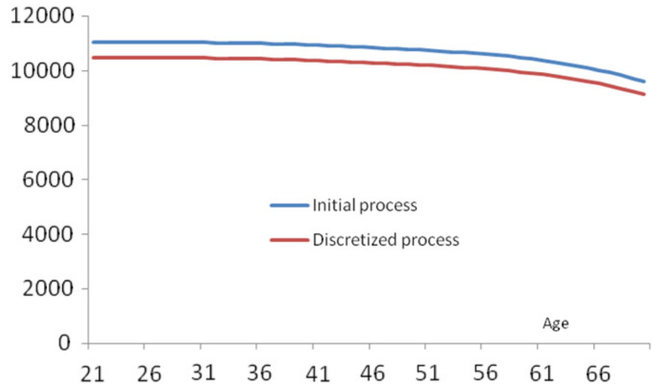

Fig. 8. Evolution of the price of the TMG in both models (discretized process and initial process).

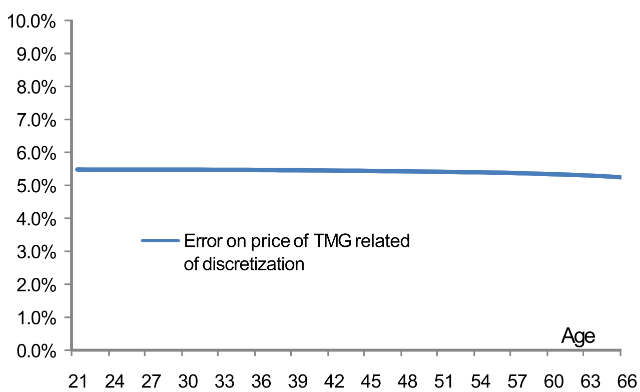

Fig. 9. Ratio of prices in the two models related to the age of policy-holders.

5. Conclusion

In this paper we are interested in a simple technique of reduction of the computing time when using Monte-Carlo simulations for the pricing of the embedded options in life insurance contracts. This technique is very easy to implement, it consists in grouping together the trajectories of the initial process according to the quantiles of the distribution all the time.

The discretized process is then used in the valuation of the life insurance contracts. We note that a wise choice of the partition of $[0, +\infty[$ allows the correct estimation of the price of a European option. These options are met in unit-linked life insurance contracts with death minimum guarantee.

We also show that the error due to the valuation of a contract in Euro using the discretized process can be reduced to less than 5% when we replace 100,000 of the trajectories of the initial process by 100 trajectories of the discretized process. This error increases with the maturity of contract but is independent of age of the policy-holder.

To use this technique, it is necessary to know the distribution of the initial process. Indeed, in addition to the constitution of the trajectories discretized, it is essential to be able to estimate the probability of occurrence of those.

The comparison of the sample of trajectories of the initial process to that of the discretized process shows clearly that the latter underestimates strongly the extreme values of the initial process. Thus, if the technique of discretization can give good results of TMG guarantee or MCEV, its use within the framework of estimating the extreme values (SCR, VAR…) can lead to biased results. However, the choice of a partition whose extreme values are strongly refined could possibly lead to reduce errors. This aspect was not treated in this article and could be the object of future developments.

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