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Interfaces with Other Disciplines

Cost efficiency measures in data envelopment analysis with data uncertainty

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ABSTRACT

This paper extends the classical cost efficiency (CE) models to include data uncertainty. We believe that many research situations are best described by the intermediate case, where some uncertain input and output data are available. In such cases, the classical cost efficiency models cannot be used, because input and output data appear in the form of ranges. When the data are imprecise in the form of ranges, the cost efficiency measure calculated from the data should be uncertain as well. So, in the current paper, we develop a method for the estimation of upper and lower bounds for the cost efficiency measure in situations of uncertain input and output data. Also, we develop the theory of efficiency measurement so as to accommodate incomplete price information by deriving upper and lower bounds for the cost efficiency measure. The practical application of these bounds is illustrated by a numerical example.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for evaluating decision making units (DMUs) based on the production possibility set. Also, traditional DEA is used for measuring the efficiency of a set of DMUs where the input and output data of the DMUs are known exactly [3]. Zhu [16,17], Wang et al. [15], Kao [10], Entani et al. [5], and Despotis and Smirlis [4] developed the theory of efficiency measurement where the data are imprecise. In these references there is no discussion concerning cost efficiency with imprecise data. In fact these references focused on the technical–physical aspects of production for use in situations where unit price and unit cost information are not available, or where their uses are limited because of variability in the prices and costs that might need to be considered. It is worthwhile to note that in many real application of DEA, the cost efficiency analysis is required, when some information on prices and costs are available. Technology and cost are the wheels that drive modern enterprises; some enterprises have advantages in terms of technology and others in cost. Hence, the management is eager to know how and to what extent their resources are being effectively and efficiently utilized, compared to other similar enterprises in the same or a similar field. There are some DEA models that deal with cost efficiency (CE) analysis when the data are known exactly. In fact, cost efficiency evaluates the ability to produce current outputs at minimal cost. See, e.g., [6,8,9,14] for more details concerning cost efficiency analysis with deterministic data. In these references, there is no discussion concerning imprecise data, whereas in most cases in industry it is usually known from experience that inputs and outputs vary over certain ranges in a short period of time. It should be mentioned that the uncertainty concerns the researchers only, who conduct efficiency analysis. Moreover, nothing is known of the distribution of the data owing to insufficient information. The only thing available to the decision maker is the two extreme points of the range. The same is true about input prices: there exist many factors in the market beyond the control of the management, which affect input prices. Factors such as interest rate, inflation, and currently exchange rate impose uncertainty of prices on the decision makers. Also, exact knowledge of prices is difficult and prices may be subject to variations in the short term. Estimation of cost efficiency is one of the vital topics in DEA. Although there are many papers for estimating cost efficiency in DEA models (see, for example, [6,8,9,14]), there are only few papers which concern the estimation of cost efficiency in the presence of imprecise data: Jahanshahloo et al. [7], Kuosmanen and Post [11–13], and Camanho and Dyson [2]. For instance Jahanshahloo et al. [7] provide some models for the treatment of ordinal data in cost efficiency analysis. In [7] the models have multiplier forms with additional weight restrictions. The main idea in constructing these models is based on the weighted enumeration of the number of inputs/outputs of each unit which are categorized on the same scale rate. Kuosmanen and Post [11,12] derived upper and lower bounds for overall cost efficiency assuming incomplete price data in the

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form of a convex polyhedral cone. Although they presented and proved a model for determining the lower bound of CE, they did not utilize the model in empirical application of their CE concepts and resorted to the Free Disposable Hull technology. For computing the lower bound, they defined set W^V. It is observed that set W^V may be non-convex, which will make it more complicated to operationalize the model they presented for computing the lower bound of CE. So, in the current paper we modify the models for obtaining the upper and lower bounds of CE, which is interesting from theory and practical point of view. In [13], Kuosmanen and Post assume that the firm analysis does not know the prices until after the production plan is fixed. Specifically, they assume that the price are random variables with domain $D \subseteq R_+^q$ and joint distribution function $F : D \to [0,1]$. They presented an approach based on first order stochastic dominance that dealt with uncertainty related to input–output prices. Anyway, we can combine the method proposed in the current paper and the method provided in [13], to obtain the economic efficiency measures. Camanho and Dyson [2] discussed the assessment of CE in complex scenarios of price uncertainty. They assumed that input prices appear in the form of ranges. The upper bound of the CE estimate is obtained with the incorporation of weight restrictions in a standard DEA model, the model which they provided (see Model (7) in [2]) is computationally expensive, because the number of constraints (7a) are $2 \times \mathscr{C}_2^m$. Let n be the number of observations. In order to obtain the lower bound of CE for n DMUs, as they mentioned, it is required to solve n^2 linear programming models. Note that the proposed model may be infeasible, and also computationally expensive.

In this paper, we propose a pair of two-level mathematical programming problems to obtain the upper and lower bounds of cost efficiency when some of the input and output data appear in the form of ranges. In turn, the resulting two-level mathematical programming problems are transformed into equivalent linear programs. Also, we consider situations in which the input and output data as well as input prices appear in the form of ranges, and we obtain the lower and upper bound for cost efficiency in these situations. We prove that when the input prices can be represent by a convex set, the upper and lower bounds of CE are obtained in extreme points of the convex set.

The rest of the paper unfolds as follows: in Section 2, some DEA CE models are reviewed and a CE model in the multiplier form is provided, which is necessary in the next sections. In Section 3, for obtaining the lower and upper bounds of CE, a pair of two-level mathematical programming problems are provided. In Section 4, the provided two-level mathematical programming problems are transformed into equivalent linear ones. Section 5, includes the main results. In fact, in this section the theory of CE is generalized to the situations in which the input prices are also imprecise in the form of ranges. In Section 6 we compare our work with other existing works. Section 7 contains an illustrative numerical example, and Section 8 gives some conclusions.

2. Preliminaries

Assume that we deal with a set of DMUs consisting of $\mathrm{DMU}_j; j = 1,\ldots,n$, with input–output vectors $(x_j, y_j); j = 1,\ldots,n$, in which $x_j = (x_{1j},\ldots,x_{mj})^T$ and $y_j = (y_{1j},\ldots,y_{sj})^T$. Define $X = [x_1, x_2,\ldots,x_n]$ and $Y = [y_1, y_2,\ldots,y_n]$ as $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively. Without the loss of generality, we assume that the input and output data x_{ij} and y_{rj} $(i = 1,\ldots,m; r = 1,\ldots,s; j = 1,\ldots,n)$ cannot be exactly obtained due to the existence of uncertainty. They are only known to lie within the upper and lower bounds represented by the ranges $\left[x_{ij}^L, x_{ij}^U\right]$ and $\left[y_{rj}^L, y_{rj}^U\right]$, where $x_{ij}^L > 0$ and $y_{rj}^L > 0$.

In order to obtain a measure of cost efficiency, when the input and output data are known exactly, Färe et al. [6] provide the following LP model:

$$\theta_o = \min\left\{\frac{w_o x}{w_o x_o} : X\lambda = x, Y\lambda \geqslant y_o, \lambda \geqslant 0\right\}. \tag{1}$$

In the above model $w_o \in \mathbb{R}_+^m$ is a user-specified row vector of the prices of the inputs of DMU_o, the unit under assessment. The variables of Model (1) are x and λ. $(\lambda_o = 1, \lambda_j = 0; j \neq o, x = x_o)$ is a feasible solution to (1) which implies that this model is feasible and bounded, and $\theta_o \in (0,1]$. Note that Model (1) has $m + s$ constraints where the RHS values of m constraints are zero, and this can lead to strong degeneracy and hence to great complexity. Regarding part (iii) of the main theorem in [8] one can use the following model instead of model (1), to determine the cost efficiency

$$\vartheta_o = \min\left\{\frac{w_o X\lambda}{w_o x_o} : Y\lambda \geqslant y_o, \lambda \geqslant 0\right\}. \tag{2}$$

Denoting the dual variables associated with the constraint sets as u, the measure of cost efficiency can be obtained by solving the dual of Model (2) as follows (see [9]):

$$\max\left\{u^T y_o : u^T Y \leqslant \frac{w_o X}{w_o x_o}, u \geqslant 0\right\}. \tag{3}$$

In Model (3) the variable is u vector. Regarding the constraints, the optimal objective value of this model is not greater than one.

3. Uncertain cost efficiency models based on uncertain data

When the input and output data are in the form of ranges, the cost efficiency measure calculated from the data should be uncertain as well. In order to deal with such uncertain situations, the following two-level mathematical program is proposed to generate the upper bound of cost efficiency range, for each DMU_o:

$$CE_o^U = \max_{\left\{\begin{array}{l} x_j^L \leqslant x_j \leqslant x_j^U, \\ y_j^L \leqslant y_j \leqslant y_j^U, \\ j = 1,\ldots,n \end{array}\right\}} \min\left\{\frac{w_o X\lambda}{w_o x_o} : Y\lambda \geqslant y_o, \lambda \geqslant 0\right\}. \tag{4}$$

The inner program, i.e., the second-level program, calculates the cost efficiency measure for each set of (x_j, y_j) defined by the outer program, i.e., the first-level program, while the outer program determines the set of (x_j, y_j) that produces the highest cost efficiency measure. The objective value of Model (4) is the upper bound of the cost efficiency measure for DMU_o. The inner program in (4) is a linear program and the dual associated with it is (3). Substituting (3) in (4) gives

$$CE_o^U = \max_{\left\{\begin{array}{l} x_j^L \leqslant x_j \leqslant x_j^U, \\ y_j^L \leqslant y_j \leqslant y_j^U, \\ j = 1,\ldots,n \end{array}\right\}} \max\left\{u^T y_o : u^T Y \leqslant \frac{w_o X}{w_o x_o}, u \geqslant 0\right\}. \tag{5}$$

In the above model, the inner program and the outer program have the same objective function of maximization. So, they can be combined into a one-level program by considering all constraints of the two programs at the same time, as follows:

$$\begin{aligned} CE_o^U = \max \quad & u^T y_o, \\ & u^T y_j \leqslant \frac{w_o x_j}{w_o x_o}, \quad j = 1,\ldots,n, \\ & x_j^L \leqslant x_j \leqslant x_j^U, \quad j = 1,\ldots,n, \\ & y_j^L \leqslant y_j \leqslant y_j^U, \quad j = 1,\ldots,n, \\ & u \geqslant 0. \end{aligned} \tag{6}$$

Notice that Model (6) is nonlinear due to nonlinear terms $w_o x_j$ and $w_o x_o$, and also fractional $\frac{w_o x_j}{w_o x_o}$. In the same way, to find the set of (x_j, y_j) that produces the lowest cost efficiency measure, a two-level mathematical program is obtained by simply replacing the outer program of Model (3) from “max” to “min”:

$$CE_o^L = \min_{\left\{\begin{array}{l} x_j^L \leqslant x_j \leqslant x_j^U, \\ y_j^L \leqslant y_j \leqslant y_j^U, \\ j = 1, \ldots, n \end{array}\right\}} \min \left\{ \frac{w_o X \lambda}{w_o x_o} : Y\lambda \geqslant y_o, \lambda \geqslant 0 \right\}. \tag{7}$$

In Model (7), the inner program calculates the cost efficiency measure for each given set of (x_j, y_j), while the outer program determines the set of (x_j, y_j) that generates the lowest cost efficiency measure. The optimal objective value of Model (7) is the lower bound of the cost efficiency measure for DMU_o.

Both the inner program and the outer program in Model (7) have the same objective of minimization. They can be combined into a one-level program by considering all constraints of the two programs at the same time. The resulting one-level program, however, is nonlinear due to uncertainty of inputs and outputs. The one-level model equivalent to (7) is as follows:

$$\begin{aligned} CE_o^L = \min \quad & \sum_{j=1}^{n} \lambda_j \left(\frac{w_o x_j}{w_o x_o} \right), \\ & \sum_{j=1}^{n} \lambda_j y_j \geqslant y_o, \\ & x_j^L \leqslant x_j \leqslant x_j^U, \quad j = 1, \ldots, n, \\ & y_j^L \leqslant y_j \leqslant y_j^U, \quad j = 1, \ldots, n, \\ & \lambda_j \geqslant 0, \quad j = 1, \ldots, n. \end{aligned} \tag{8}$$

In the next section, Models (6) and (8) are transformed into a pair of equivalent linear programs. So, the upper and lower bounds of CE for each DMU, with uncertain input and output data, can be obtained using the LP softwares.

4. Cost efficiency ranges and linear programming

Models (6) and (8), which obtain the lower and upper bounds of CE, are nonlinear due to uncertain input and output data. The following theorem establishes the point that the upper bound of cost efficiency for DMU_o is attained by setting its outputs ranges at the upper bound and the inputs ranges at the lower bounds; meanwhile, the outputs ranges of the remaining DMUs at their respective lowest levels and the inputs ranges at their respective highest levels (i.e., the DMU under evaluation in the most favorable condition and the other DMUs in the least favorable condition).

Theorem 1. *CE_o^U is equal to the optimal objective value of the following linear model:*

$$\begin{aligned} \theta_o^* = \max \quad & u^T y_o^U \\ s.t. \quad & u^T y_j^L \leqslant \frac{w_o x_j^U}{w_o x_o^L}, \quad j \neq o, \\ & u^T y_o^U \leqslant 1, \\ & u \geqslant 0. \end{aligned} \tag{9}$$

Proof. To establish the theorem it is sufficient to show that $CE_o^U = \theta_o^*$. If u^* is an optimal solution to (9), then $\left(u = u^*, x_j = x_j^U; j \neq o, y_j = y_j^L; j \neq o, x_o = x_o^L, y_o = y_o^U\right)$ is a feasible solution to (6). The objective function of Model (6), corresponding to this feasible solution is $u^{*T} y_o^U = \theta_o^*$. Since the objective function of (6) is of maximization version, so $CE_o^U \geqslant \theta_o^*$. On the other hand, if $(\widehat{u}, \widehat{x}_j, \widehat{y}_j)$ is an optimal solution to (6), then we have $\widehat{u}^T y_o^U \leqslant 1$ or $\widehat{u}^t y_o^U > 1$. If $\widehat{u}^T y_o^U \leqslant 1$, then

$$\widehat{u}^t y_j^L \leqslant \widehat{u}^T \widehat{y}_j \leqslant \frac{w_o \widehat{x}_j}{w_o \widehat{x}_o} \leqslant \frac{w_o x_j^U}{w_o x_o^L}.$$

These imply that $\widehat{u}$ is a feasible solution to (9). The value of objective function of Model (9), corresponding to this feasible solution is $\widehat{u}^T y_o^U \geqslant CE_o^U$. When $\alpha = \widehat{u}^t y_o^U > 1$, we have $(\widehat{u}^t/\alpha) y_o^U = 1$ and

$$(\widehat{u}^T/\alpha) y_j^L \leqslant (\widehat{u}^T/\alpha)\widehat{y}_j \leqslant \frac{w_o \widehat{x}_j}{\alpha w_o \widehat{x}_o} \leqslant \frac{w_o x_j^U}{\alpha w_o x_o^L} \leqslant \frac{w_o x_j^U}{w_o x_o^L}.$$

These imply that $(\widehat{u}^T/\alpha)$ is a feasible solution to (9). The objective value of Model (9), corresponding to this feasible solution is $1 \geqslant CE_o^U$. Therefore, in each case we have $\theta_o^* \geqslant CE_o^U . CE_o^U \geqslant \theta_o^*$ and $\theta_o^* \geqslant CE_o^U$ imply $CE_o^U = \theta_o^*$, and complete the proof. □

The following theorem establishes, the lower bound of cost efficiency for DMU_o is attained by setting its outputs ranges at the lower bounds and the inputs ranges at the upper bounds; meanwhile, the outputs ranges of other DMUs at their corresponding highest levels and the inputs ranges at their corresponding lowest levels (i.e., the DMU under evaluation in the worst condition and the others in the best condition).

Theorem 2. *CE^L for DMU_o is equal to optimal objective value of the following linear programming problem:*

$$\begin{aligned} \varphi_o^* = \max \quad & u^T y_o^L \\ s.t. \quad & u^T y_j^U - \frac{w_o x_j^L}{w_o x_o^U}, \quad j \neq o, \\ & u^T y_o^L \leqslant 1, \\ & u \geqslant 0. \end{aligned} \tag{10}$$

Proof. To establish the theorem, it is sufficient to show that the lower bound of cost efficiency for DMU_o is equal to optimal objective value of the following program, in which Model (10) is the dual of it

$$\begin{aligned} \min \quad & \sum_{j \neq o, j=1}^{n} \lambda_j \left(\frac{w_o x_j^L}{w_o x_o^U} \right) + \lambda_o \left(\frac{w_o x_o^U}{w_o x_o^U} \right) \\ \text{s.t.} \quad & \sum_{\substack{j \neq o, j=1 \\ \lambda_j \geqslant 0,}}^{n} \lambda_j y_j^U + \lambda_o y_o^L \geqslant y_o^L \quad j = 1, \ldots, n. \end{aligned} \tag{11}$$

If λ^* is an optimal solution to (11), then $\left(\lambda = \lambda^*, y_j = y_j^U; j \neq o, x_j = x_j^L; j \neq o, y_o = y_o^L, x_o = x_o^U\right)$ is a feasible solution to (8). The value of objective function of Model (8) corresponding to this feasible solution is φ_o^*. Since the objective function of (8) is of minimization version, hence $CE_o^L \leqslant \varphi_o^*$.

On the other hand, if $(\hat{\lambda}_j, \hat{y}_j, \hat{x}_j$, for all $j)$ is an optimal solution to (8). We have

$$\sum_{j \neq o, j=1}^{n} \hat{\lambda}_j y_j^U \geqslant \sum_{j \neq o, j=1}^{n} \hat{\lambda}_j \hat{y}_j \geqslant (1 - \hat{\lambda}_o)\hat{y}_o \geqslant (1 - \hat{\lambda}_o) y_o^L.$$

Note that $\hat{\lambda}_o \leqslant 1$, and so $(1 - \hat{\lambda}_o) \geqslant 0$. This implies that $\sum_{j \neq o, j=1}^{n} \hat{\lambda}_j y_j^U + \hat{\lambda}_o y_o^L \geqslant y_o^L$, and $\lambda = \hat{\lambda}$ is a feasible solution to (11). The value of objective function of (11) corresponding to this solution is

$$\begin{aligned} \sum_{j \neq o, j=1}^{n} \hat{\lambda}_j \left(\frac{w_o x_j^L}{w_o x_o^U} \right) + \hat{\lambda}_o \left(\frac{w_o x_o^U}{w_o x_o^U} \right) &\leqslant \sum_{j \neq o, j=1}^{n} \hat{\lambda}_j \left(\frac{w_o \hat{x}_j}{w_o \hat{x}_o} \right) + \hat{\lambda}_o \\ &= \sum_{j=1}^{n} \hat{\lambda}_j \left(\frac{w_o \widehat{x}_j}{w_o \hat{x}_o} \right) = \text{CE}_o^L. \end{aligned}$$

Since the objective function of (11) is of minimization version, hence $\varphi_o^* \leqslant CE_o^L$. Thus we get $\varphi_o^* = CE_o^L$ and the proof is completed. □

In the next section, which includes the main results the theory of the CE is generalized to the situations in which input prices are also imprecise. Since in this situations the objective function is in the form of fractional and nonlinear, so computing the bounds for CE is more difficult.

5. Extension to uncertain input prices

5.1. The upper bound

We obtained the upper and lower bounds of cost efficiency when the input and output data are in the form of ranges. In the current section, we generalize the theory of cost efficiency to situations in which input and output data as well as input prices appear in the form of ranges, due to incomplete price information, represented by ranges $[w_o^L, w_o^U]$, in which $w_o^L = (w_{1o}^L, \ldots, w_{mo}^L)$ and $w_o^U = (w_{1o}^U, \ldots, w_{mo}^U)$. In this case, we propose the following model to obtain the upper bound of cost efficiency

$$CE_o^U = \max_{\left\{\begin{array}{l} w_o^L \leqslant w_o \leqslant w_o^U, \\ x_j^L \leqslant x_j \leqslant x_j^U, \\ y_j^L \leqslant y_j \leqslant y_j^U, \\ j=1,\ldots,n \end{array}\right\}} \max\left\{u^T y_o : u^T y_j \leqslant \frac{w_o x_j}{w_o x_o};\ \text{for all } j, u \geqslant 0\right\}. \tag{12}$$

Theorem 3. *CE^U obtained from (12) is equal to the optimal objective value of the following model:*

$$\max_{w_o^L \leqslant w_o \leqslant w_o^U,} \max\left\{u^T y_o^U : u^T y_j^L \leqslant \frac{w_o x_j^U}{w_o x_o^L}; j \neq o, u^T y_o^U \leqslant 1, u \geqslant 0\right\}. \tag{13}$$

Proof. The proof is similar to that of Theorem 1 and hence omitted. □

Since the inner program and outer program have the same objective of maximization, the above program is equivalent to the following one-level program:

$$\max\left\{u^T y_o^U : w_o^L \leqslant w_o \leqslant w_o^U, u^T y_j^L \leqslant \frac{w_o x_j^U}{w_o x_o^L}; j \neq o, u^T y_o^U \leqslant 1, u \geqslant 0\right\}. \tag{14}$$

The following theorem provides a linear programming problem for obtaining the upper bound of cost efficiency in the presence of uncertainty. In fact we solve the following linear programming problem to obtain the upper bound of cost efficiency.

Theorem 4. *When the input and output data as well as input prices are in the form of ranges, the upper bound of cost efficiency is equal to the optimal objective value of the following linear program:*

$$\begin{aligned} \psi_o^* = \max \quad & \sum_{r=1}^s u_r y_{ro}^U \\ \text{s.t.} \quad & \sum_{i=1}^m \hat{w}_{io} x_{io}^L = 1, \\ & \sum_{r=1}^s u_r y_{ro}^U \leqslant 1, \\ & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m \hat{w}_{io} x_{ij}^U \leqslant 0, \quad j \neq o, j=1,2,\ldots,n, \\ & \eta w_{io}^L \leqslant \hat{w}_{io} \leqslant \eta w_{io}^U, \quad i=1,2,\ldots,m, \\ & u_r \geqslant 0, \quad r=1,2,\ldots,s, \\ & \eta \geqslant 0. \end{aligned} \tag{15}$$

Proof. First we set

$$\eta = \frac{1}{w_o x_o^L}. \tag{16}$$

We have

$$\eta w_o x_o^L = 1 \quad \text{and} \quad \eta > 0.$$

Equivalently, this can be expressed as follows:

$$\sum_{i=1}^m \eta w_{io} x_{io}^L = 1. \tag{17}$$

Now, we set $\hat{w}_{io} = \eta w_{io}, i = 1,2,\ldots,m$. Considering this variable alteration and (14), the constraints of Model (14) are transformed to the following constraints:

$$w_{io}^L \leqslant w_{io} \leqslant w_{io}^U \Longleftrightarrow \eta w_{io}^L \leqslant \eta w_{io} \leqslant \eta w_{io}^U \Longleftrightarrow \eta w_{io}^L \leqslant \hat{w}_{io} \leqslant \eta w_{io}^U, \tag{18}$$

and

$$u^T y_j^L \leqslant \frac{w_o x_j^U}{w_o x_o^L} \Longleftrightarrow u^T y_j^L - \eta w_o x_j^U \leqslant 0 \Longleftrightarrow u^T y_j^L - \hat{w}_o x_j^U \leqslant 0; \quad j \neq o, \tag{19}$$

and

$$\sum_{i=1}^m \eta w_{io} x_{io}^L = 1 \Longleftrightarrow \sum_{i=1}^m \hat{w}_{io} x_{io}^L = 1. \tag{20}$$

Constraint $u^T y_o^U \leqslant 1$ and the objective function remain unchanged. It is clear that if $(\dot{u}, \dot{w}_o)$ is an optimal solution to (14), then

$$\left(\eta = \frac{1}{\sum_{i=1}^m \dot{w}_{io} x_{io}^L}, \widehat{w}_{io} = \eta \dot{w}_{io}, u = \dot{u}\right)$$

is a feasible solution to (15) and the objective value corresponding to this solution is $\dot{u}^T y_o^U = \mathrm{CE}_o^U$. So $\psi_o^* \geqslant CE_o^U$. On the other hand if $(\ddot{u}, \ddot{\hat{w}}_o, \ddot{\eta})$ is an optimal solution to (15), then $\left(w = \frac{\hat{w}_o}{\ddot{\eta}}, u = \ddot{u}\right)$ is a feasible solution to (14) and the objective value corresponding to this feasible solution is $\ddot{u}^T y_o^U = \psi_o^*$. So $\psi_o^* \leqslant CE_o^U$. Now, regarding the above mentioned point and constraints (18)–(20), Models (14) and (15) are equivalent, and the proof is complete. □

Model (15) has $2m+1$ constraints more than Model (9), and this is the penalty that we should pay to linearize the upper bound of the cost efficiency model when the input prices are imprecise.

If an optimal solution to Model (15) is $(\hat{u}^*, \hat{w}_o^*, \hat{\eta}^*)$, then we have an optimal solution to Model (12) as: $(u^* = \hat{u}^*, w_o^* = \hat{w}_o^*/\hat{\eta}^*, x_j^* = x_j^U; j \neq o, y_j^* = y_j^L; j \neq o, x_o^* = x_o^L, y_o^* = y_o^U)$. Note that $\eta > 0$ in all of the feasible solutions of Model (15), because for all $i, \hat{w}_{io} \leqslant \eta w_{io}^U$, and $\sum_{i=1}^m \hat{w}_{io} x_{io}^L = 1$. So, in (15) we just put $\eta \geqslant 0$.

5.2. The lower bound

In this subsection we use the following notations. For vectors a^+, a^- defined by $a^+ = \max\{a, 0\}, a^- = \max\{-a, 0\}$, we have $a = a^+ - a^-, |a| = a^+ + a^-, a^+ > 0, a^- > 0$. $e = (1,1,\ldots,1)^T$ is a vector of all ones. In our description to follow, an important role is played by the set V_m of all ± 1-vectors in R^m; i.e., $V_m = \{v \in R^m : |v| = e\}$. Obviously, the cardinality of V_m is 2^m. For a given vector $v \in R^m$ we denote

$$P_v = diag(v_1, v_2, \ldots, v_m) = \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_m \end{pmatrix}.$$

An algorithm for generating V_m

It will be helpful at a later stage to generate all the ± 1-vectors forming set V_m systematically one-by-one in such a way that any two successive vectors differ in exactly one entry.

Algorithm

Set $z := 0 \in R^m$; select $v \in V_m; V_m := \{v\}$
while $z \neq e$
  $k = \min\{i : z_i = 0\}$,
  for i:= 1 to $k-1$, $z_i = 0$; end
  $z_k := 1; v_k := -v_k$;
  $V_m := V_m \cup \{v\}$;
  end,
  $V = V_m$.

Consider $W_o = \{w_o : w_o^L \leqslant w_o \leqslant w_o^U\}$, where $w_o^L, w_o^U \in R^m$. As shown later, in many cases it is more advantageous to express the data of the input prices in terms of the center vector

$$w_o^c = \frac{1}{2}(w_o^L + w_o^U)$$

and the nonnegative radius vector

$$\chi = \frac{1}{2}(w_o^U - w_o^L)$$

and we employ both forms $W_o = [w_o^L, w_o^U] = [w_o^c - \chi, w_o^c + \chi]$. For an m-dimensional range vector $W_o = [w_o^c - \chi, w_o^c + \chi]$ we define vectors $w_o^v = w_o^c + P_v \chi$ for each $v \in V_m$. Then for any such v we have

$$(w_o^v)_i = (w_o^c)_i + v_i \chi_i = \begin{cases} w_{io}^L & \text{if } v_i = -1, \\ w_{io}^U & \text{if } v_i = 1. \end{cases}$$

When the input and output data as well as the input prices are imprecise in the form of ranges, we propose the following model to obtain the lower bound of cost efficiency:

$$CE_o^L = \min_{\left\{\begin{array}{l} w_o^L \leqslant w_o \leqslant w_o^U, \\ x_j^L \leqslant x_j \leqslant x_j^U, \\ y_j^L \leqslant y_j \leqslant y_j^L, \\ j = 1, \ldots, n \end{array}\right\}} \min \left\{ \frac{w_o x}{w_o x_o} : X\lambda = x, Y\lambda \geqslant y_o, \lambda \geqslant 0 \right\}. \tag{21}$$

Theorem 5. *CE_o^L obtained by (21) is equal to the optimal objective value of the following model:*

$$\varphi_o^* = \min_{w_o^L \leqslant w_o \leqslant w_o^U} \left\{ \begin{array}{ll} \min & \dfrac{w_o x}{w_o x_o^U} \\ s.t. & \displaystyle\sum_{j \neq o, j=1}^{n} \lambda_j x_j^L + \lambda_o x_o^U = x, \\ & \displaystyle\sum_{\substack{j \neq o, j=1 \\ \lambda_j \geqslant 0,}}^{n} \lambda_j y_j^U + \lambda_o y_o^L \geqslant y_o^L, \quad j = 1, \ldots, n \end{array} \right\}. \tag{22}$$

Proof. The proof is similar to that of Theorem 2 and hence omitted. □

The following theorem gives an explicit formula for computing the lower bound of the cost efficiency measure. In fact the following model gives the lower bound of cost efficiency.

Theorem 6. *We have:*

$$\mathrm{CE}_o^L = \min_{v \in V_m} \left\{ \begin{array}{ll} \min & \dfrac{w_o^v x}{w_o^v x_o^U} \\ s.t. & \displaystyle\sum_{j \neq o, j=1}^{n} \lambda_j x_j^L + \lambda_o x_o^U = x, \\ & \displaystyle\sum_{\substack{j \neq o, j=1 \\ \lambda_j \geqslant 0,}}^{n} \lambda_j y_j^U + \lambda_o y_o^L \geqslant y_o^L, \quad j = 1, \ldots, n \end{array} \right\}. \tag{23}$$

Comment 1. By using (15), solving only one linear programming problem is needed to evaluate CE^U, whereas up to 2^m LPs are to be solved to compute CE^L by (23).

Proof. It is clear that the set $\{w_o^v : v \in V_m\}$ is the set of all extreme points of the bounded set $W_o = \{w_o : w_o^L \leqslant w_o \leqslant w_o^U\}$. To prove the theorem, it is sufficient to prove the following assertion: at least one optimal solution of Model (22) occurs at an extreme point of W_o. Assume that Model (22) has an optimal solution, say (w_o^*, λ^*, x^*), and hence, considering the representation theorem (see Theorem 2.1 on p. 69 of [1]) we have

$$w_o^* = \sum_{v \in V_m} \mu_v w_o^v, \quad \sum_{v \in V_m} \mu_v = 1.$$

If there exists a $v \in V_m$ such that

$$\frac{w_o^v x^*}{w_o^v x_o^U} = \frac{w_o^* x^*}{w_o^* x_o^U},$$

then the proof is at hand. Now, by contradiction let $\frac{w_o^v x^*}{w_o^v x_o^U} > \frac{w_o^* x^*}{w_o^* x_o^U}$, for each $v \in V_m$, then we have $(w_o^v x^*)(w_o^* x_o^U) > (w_o^v x_o^U)(w_o^* x_o^*)$. By multiplying both sides of the above inequality by μ_v; $v \in V_m$ and summation on $v \in V_m$ we have $\sum_{v \in V_m} \mu_v (w_o^v x^*)(w_o^* x_o^*) > \sum_{v \in V_m} \mu_v (w_o^v x_o^U)(w_o^* x_o^U)$. This in turn implies

$$\frac{w_o^* x^*}{w_o^* x_o^U} < \frac{\sum_{v \in V_m} \mu_v (w_o^v x^*)}{\sum_{v \in V_m} \mu_v (w_o^v x_o^U)} = \frac{\sum_{v \in V_m} \mu_v w_o^v x^*}{\sum_{v \in V_m} \mu_v w_o^v x_o^U} = \frac{w_o^* x^*}{w_o^* x_o^U}.$$

This is obviously a contradiction, and completes the proof. □

Note. Although we have obtained the upper bound of CE by solving a linear programming problem, we will also prove that the upper bound of CE can be obtained in a similar way to that for the lower bound. The only difference is that the outer program has the objective function in the form of maximization.

Theorem 7. *We have:*

$$\psi_o^* = \max_{v \in V_m} \left\{ \begin{array}{ll} \min & \dfrac{w_o^v x}{w_o^v x_o^L} \\ s.t. & \displaystyle\sum_{j \neq o, j=1}^{n} \lambda_j x_j^U + \lambda_o x_o^L = x, \\ & \displaystyle\sum_{\substack{j \neq o, j=1 \\ \lambda_j \geqslant 0,}}^{n} \lambda_j y_j^L + \lambda_o y_o^U \geqslant y_o^U, \quad j = 1, \ldots, n \end{array} \right\}. \tag{24}$$

Comment 2. When only input prices are uncertain, to obtain the bounds of CE measures only solving one of the Models (23) or (24) is required and the other is automatically obtained, since in the proof of Theorems 6 and 7, we only made use of convexity of input prices. The method proposed in this paper is applicable not only when the input prices are in the form of ranges but also when they are in the form of a convex set.

Proof. The proof is similar to that of Theorem 6 and hence omitted. □

6. Comparison with other works

6.1. Camanho and Dyson's [2] method

Camanho and Dyson [2] started with a numerical example, assuming that each DMU has two inputs and one normalized output. They further assumed that the input prices varied over the range [3,4]. So, the input prices can be expressed as follows:

$$W = \{(w_1, w_2) | 3 \leqslant w_1 \leqslant 4, 3 \leqslant w_2 \leqslant 4\}.$$

The data of their example are given in Table 1.

As can be observed in Fig. 1, there are four extreme points for the input prices: $P_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $P_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $P_3 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $P_4 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Camanho and Dyson [2] calculated CE from the optimistic and pessimistic points of view. Although they did not explicitly mention, it can be inferred from their results that cost efficiency from either of the optimistic and pessimistic points of view occurs at one of the extreme points of the set of input prices. As we proved previously, this is not by chance, and we can determine CE from the optimistic and pessimistic viewpoints by putting only the extreme points of set W in the Farrel CE model. The extreme points of input prices corresponding to optimistic and pessimistic CE are given in the third and fifth columns of Table 2, respectively. It should be mentioned that the convex combination of the extreme points in the third and fifth columns of Table 2 gives the set of all optimal input prices from the optimistic and pessimistic points of view, respectively.

The model they provided (Model (7) in [2]) to obtain the optimistic CE measure has $2 \times \mathscr{C}_2^m + n$ inequality constraints, one equality constraint and $m + s$ variables, while the model provided in this paper (Model (15)) has $2m + n$ inequality constraints, one equality constraint and $m + s + 1$ variables. In summary, Model (15) has $2 \times \mathscr{C}_2^m - 2m$ constraints fewer than the corresponding model in [2] and considering this, our model is computationally economical. In order to obtain the lower bound of CE for n DMUs, as they mentioned, it is required to solve n^2 linear programming models. Note that their model may be infeasible, and also computationally expensive.

6.2. Kuosmanen and Post's method

In [11,12], Kuosmanen and Post measured CE when the price information is incomplete. The solution they propose in Kuosmanen and Post [11,12] is based on enumerating the vertices of the FDH technology, and does not easily extend to the case of the convex technology and FDH technology with constant returns to scale. They assumed the following structure for the price domain:

$$W = \{w \in R_+^q | Aw \geqslant 0\}.$$

W represents the price domain in terms of l linear inequalities. To remove the problem in [11] for computing the lower bound of CE, in [12] they defined the set of hypothetical input vectors with a negative cost at all prices in W as:

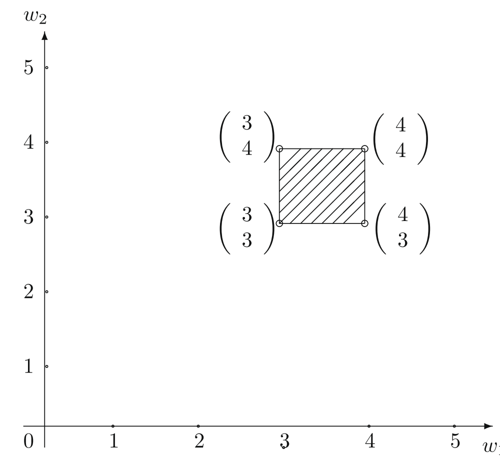

Fig. 1. Input prices.

$$C = \{x \in R^q | \exists \beta \in R_+^l : x < -\beta A\}.$$

They also defined

$$W^v = R^q - C$$

It is observed that set W^v may be non-convex, as in the example they provided in [12], which will make it more complicated to operationalize the model they presented for computing the lower bound of CE. Their proposed model is as follows:

$$\min_\theta \{\theta | \theta x \in co(L(y)) + R^q - C\}.$$

Although they presented and proved the above model for determining the lower bound of CE, they did not utilize the model in the empirical application of their CE concepts and made use of the Free Disposable Hull technology. In this paper, we will present an explicit enumerative procedure for computing the upper and lower bounds of CE. To this end, we define set $\widehat{W}$ as:

$$\hat{w} = \{w \in R_+^q | Aw \geqslant 0, \mathbf{1}w = 1\},$$

where $\mathbf{1}$ is a vector with all component equal to one. The set containing all extreme points of $\widehat{W}$ is denoted by W^{ep}. Now we modify the model for obtaining the upper and lower bounds of CE, as follows:

$$CE_o^U = \max_{w \in W^{ep}} \left(\min_{x \in L(y_o)} \frac{wx}{wx_o} \right),$$

$$CE_o^L = \min_{w \in W^{ep}} \left(\min_{x \in L(y_o)} \frac{wx}{wx_o} \right),$$

where $L(y_o) = \{x | x \text{ can produce } y_o\}$.

They provided a graphical example (Fig. 2) of a price domain, as represented by the convex cone W. By specifying matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

they obtained the price domain. By the above specification for matrix A, set $\widehat{W}$ defined above will be obtained as follows:

Table 1
Input and output data.

DMUs	DMU A	DMU B	DMU C	DMU D	DMU E	DMU F	DMU G	DMU H
x_1	2	3	5	7	3	5	9	10
x_2	7	5	3	2	7	5	2	2.5
y	1	1	1	1	1	1	1	1

Table 2
CE bounds and corresponding extreme points of input prices.

DMU	Optimistic CE	Extreme points	Pessimistic CE	Extreme points
A	0.931	P_2	0.794	P_3
B	1	P_1, P_3, P_4	0.931	P_2
C	1	P_1, P_2, P_4	0.931	P_3
D	0.931	P_2	0.794	P_3
E	0.818	P_3	0.73	P_2
F	0.8	P_1, P_4	0.771	P_2, P_3
G	0.771	P_2	0.643	P_3
H	0.675	P_2	0.568	P_3

$$\widehat{W} = \{(w_1, w_2)|2w_1 - w_2 \geqslant 0, -w_1 - 2w_2 \geqslant 0, w_1 + w_2 = 1\}.$$

Fig. 2 shows set $\widehat{W}$ graphically. As can be seen the extreme points of $\hat{w}$ are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$, one of which gives the upper bound and the other one gives the lower bound of CE.

They also proposed an enumerative LP procedure to compute the empirical estimator $\underline{CE}^{L_{FDH}}(x_o, y_o; W)$ relative to Free Disposable Hull (FDH) input set $\widehat{L}_{FDH} = \{x \in R^q_+ | y_o \geqslant y_j; x_j \leqslant x_o; j = 1, \ldots, n\}$. They also defined set $\chi(y_o) = \{x_j \in X | y_j \geqslant y_o\}$ and proposed the following model to obtain the lower bound estimator $\underline{CE}^{L_{FDH}}(x_o, y_o; W)$:

$$\underline{CE}^{L_{FDH}}(x_o, y_o; W) = \min_{x_j \in \chi(y_o)} (\min_{w \in W} \{wx_j | wx_o = 1\}).$$

Now, we propose the following model for computing the lower bound of CE, based on the evaluation of certain input ratios and extreme points of the normalized price domain and without solving any LP

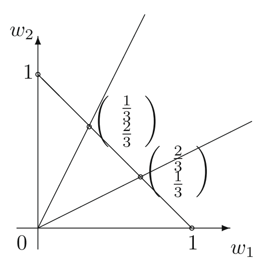

Fig. 2. Normalized input prices and their extreme points.

$$\underline{CE}^{L_{FDH}}(x_o, y_o; W) = \min_{x_j \in \chi(y_o)} \left(\min_{w \in W^{ep}} \left\{ \frac{wx_j}{wx_o} \right\} \right) = \min_{w \in W^{ep}} \left(\min_{x_j \in \chi(y_o)} \left\{ \frac{wx_j}{wx_o} \right\} \right).$$

In the same way, we propose the following model to obtain the upper bound of CE:

$$\overline{CE}^{L_{FDH}}(x_o, y_o; W) = \max_{w \in W^{ep}} \left(\min_{x_j \in \chi(y_o)} \left\{ \frac{wx_j}{wx_o} \right\} \right).$$

The models provided in the current paper can be easily extended to FDH model under constant returns to scale assumption of technology.

7. Illustrative example

In this section, we show the ability of the provided approach using a numerical example. To this end, twenty DMUs with three inputs and five outputs are considered. Without the loss of generality, in the current example we assume that all inputs and outputs are imprecise in the form of ranges. Also, the input prices are imprecise in the form of ranges. The data of the inputs ranges for these units have been listed in Table 3. The data of the input prices for these units have been listed in Table 4. The data of the outputs ranges have been listed in Table 5 and in the second, third, forth, and fifth columns of Table 6. Models (15) and (23) have been solved using GAMS software to obtain the upper and lower bounds of cost efficiency. The sixth and seventh columns of Table 6 exhibit the resulting lower bound and upper bound of cost efficiency measures.

In Table 3, for instance, the ranges for the inputs of DMU_{14}, $x_{1,14}$, $x_{2,14}$, $x_{3,14}$, are [5010.87, 5113.68], [21.62, 22.62], and [5814, 5837], respectively. And in Table 4, the ranges for the input prices of DMU_{14}, $w_{1,14}$, $w_{2,14}$, $w_{3,14}$, are [13, 18], [4, 6], [10, 18]

Table 3
The range data for the inputs of units.

	x^L_{1j}	x^U_{1j}	x^L_{2j}	x^U_{2j}	x^L_{3j}	x^U_{3j}
1	8254.56	8263.56	30.26	45.09	4847	5007
2	3600.53	38910.53	17.69	20.78	9005	10,032
3	5682.21	5697.21	17.47	19.39	15,823	17,101
4	512.76	600.76	17.8	25.18	18,319	21,305
5	12495.58	12531.58	15.39	21.35	1886	1875
6	11189.68	13193.68	19.02	34.30	14,527	14,533
7	771.61	809.64	16.34	20.12	13,977	14,056
8	4341.02	8347.33	27.75	40.34	9224	9618
9	1457.18	1958.18	19.73	20.01	9786	9961
10	9092.21	9306.25	11.89	25.89	8085	8268
11	3155.11	4195.11	20.27	22.08	1326	1345
12	8625.98	12356.03	22.5	32.45	4764	5543
13	16278.93	16679.45	24.23	30.23	9326	11,329
14	5010.87	5113.68	21.62	22.62	5814	5837
15	4011.24	4242.24	37.36	30.65	35,310	35,563
16	8702.27	8831.33	23.7	30.17	226,017	226,345
17	6927.43	8990.65	24.72	30.17	9852	10,063
18	850.67	1221.67	13.43	20.43	12,691	12,736
19	6181.84	10875.84	28.12	32.32	17,507	18,205
20	1261.57	2270.57	20.81	22.11	30,253	30,916

Table 4
The range of input prices of units.

Units	w_{1j}^L	w_{1j}^U	w_{2j}^L	w_{2j}^U	w_{3j}^L	w_{3j}^U
1	12	15	2	5	11	15
2	11	16	1	8	12	16
3	10	16	4	8	14	16
4	14	17	3	6	12	17
5	14	15	1	5	14	19
6	14	16	1	5	13	18
7	12	18	2	8	13	17
8	14	19	0.5	8	12	17
9	12	16	4	9	11	19
10	10	20	2	7	10	15
11	11	19	3	8	14	16
12	12	17	1	9	14	18
13	12	16	2	5	12	17
14	13	18	4	6	10	18
15	14	16	3	8	13	19
16	12	18	1	5	10	15
17	13	17	2	5	10	16
18	14	19	1	8	12	16
19	11	19	2	7	14	20
20	11	15	4	6	12	20

Table 5
The range output data of units (to be continued).

DMUs	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	y_{3j}^L	y_{3j}^U
1	1,262,798	1,291,506	325,071	327,038	1,092,933	1,154,312
2	302,316	332,725	38,509	41,267	66,399	66,450
3	652,583	661,236	123,230	123,580	1,517,439	1,517,687
4	737,317	737,547	261,702	26,232	301,968	302,573
5	365,134	367,007	15,612	15,786	80,153	80,893
6	537,502	567,669	51,363	51,702	229,105	435,438
7	205,122	206,143	54,177	54,196	757,565	759,043
8	243,663	247,809	264,451	264,685	728,856	734,568
9	279,091	280,974	179,083	185,632	945,771	949,551
10	383,585	386,578	13,135	13,164	1,464,666	1,465,112
11	261,142	261,829	144,716	147,218	604,120	610,986
12	401,836	402,379	61,311	61,717	151,190	151,345
13	569,375	578,903	456,902	498,437	275,812	276,361
14	261,658	262,090	220,581	221,381	735,733	737,256
15	347,687	348,762	285,715	265,945	462,277	463,478
16	433,362	455,660	80,860	82,360	304,659	332,673
17	528,743	570,965	301,168	301,464	4,146,106	4,156,223
18	396,342	425,679	177,633	177,955	32,968	33,345
19	537,025	537,327	328,473	357,623	1,662,874	1,663,364
20	876,301	877,402	104,341	109,004	1,207,702	1,218,342

Table 6
The range output data and cost efficiency bounds of units (continued).

DMUs	y_{4j}^L	y_{4j}^U	y_{5j}^L	y_{5j}^U	CE bounds	
					CE^L	CE^U
1	93128.57	93246.34	7575.97	7670.33	1.0000	1.0000
2	20179.39	20559.37	328.52	346.22	0.0178	0.3062
3	78297.51	88395.69	2409.54	2412.77	0.3947	0.5740
4	28734.36	34286.21	304.52	317.21	0.3716	0.5130
5	365,134	11995.92	279	305	0.2526	0.2909
6	21798.65	23112.45	489.53	571.73	0.1891	0.2413
7	47568.64	47989.70	431.85	448.45	0.2654	0.3977
8	55581.36	56882.25	1727.73	1745.78	0.5242	1.0000
9	40436.67	41200.90	445.82	449.06	0.4509	0.8760
10	524689.84	526284.44	90.05	92.37	1.0000	1.0000
11	12480.8	12595.35	1161.27	1201.35	1.0000	1.0000
12	16264.01	20345.15	262.08	265.18	0.2252	0.3190
13	47051.4	47906.2	848.44	849.34	0.4869	1.0000
14	19613.36	22890.38	1224.77	1235.79	0.5693	1.0000
15	131041.62	131732.56	1925.56	1937.06	0.2464	0.3595
16	186072.29	187890.37	1286.52	1311.73	0.0323	0.0602
17	11096.29	11245.62	4291.84	4330.22	1.0000	1.0000
18	9463.04	10371.8	109.15	130.79	0.3818	0.7021
19	62951.63	63045.46	1585.29	1711.12	0.3401	0.7118
20	25554.16	28095.24	1094.32	1294.32	0.2325	0.3933

respectively. And also, the output ranges data of DMU_{14}, $y_{1,14}$, $y_{2,14}$, $y_{3,14}$, $y_{4,14}$, $y_{5,14}$, in Tables 5 and 6 are [261 658, 262 090], [220 581, 221 381], [735 733, 737 256], [19613.36, 22890.38], and [1224.77, 1235.79], respectively. We solved Model (23) to obtain the lower bound of cost efficiency in the presence of uncertainty, and presented the results in the sixth column of Table 6. Finally, we solved Model (15) to obtain the upper bounds of cost efficiency in the presence of uncertainty. The resulting measures are shown in the last column of Table 6. As the sixth column of Table 6 shows, the cost efficiency measure of DMU_{14} is 0.5693, in the least favorable condition, and 1.0000 in the most favorable condition. As a result, the cost efficiency measure for DMU_{14} lies in the range of [0.5693, 1.0000]. In turn, this shows that this DMU is cost efficient in the most favorable condition, but it is cost inefficient in the least favorable condition.

As can be seen in the sixth and seventh columns of Table 6, four DMUs, 1, 10, 11, 17 are cost efficient in both the most favorable and in the least favorable conditions, but DMUs 8, 13 and 14 are cost efficient only in the most favorable condition. Consider DMU_9, for instance. The extreme points of input prices domain are $\begin{pmatrix}12\\4\\11\end{pmatrix}, \begin{pmatrix}12\\4\\19\end{pmatrix}, \begin{pmatrix}12\\9\\11\end{pmatrix}, \begin{pmatrix}12\\9\\19\end{pmatrix}, \begin{pmatrix}16\\4\\11\end{pmatrix}, \begin{pmatrix}16\\4\\19\end{pmatrix}, \begin{pmatrix}16\\9\\11\end{pmatrix}, \begin{pmatrix}16\\9\\19\end{pmatrix}$. When we put DMU_9, the DMU under analysis, in the most favorable conditions, and the other DMUs in the least favorable conditions, the CE measure corresponding to the above mentioned vertices are 0.7859, 0.5675, 0.7592, 0.5679, 0.8757, 0.6631, 0.8760, 0.6635, respectively. This shows that the upper bound of CE is 0.8760 and attained in extreme point $\begin{pmatrix}16\\9\\11\end{pmatrix}$. Also, when setting DMU_9 in the least favorable conditions, and other DMUs in the most favorable conditions, the CE measure corresponding to the above mentioned extreme points are 0.5802, 0.7203, 0.5806, 0.4509, 0.6697, 0.5124, 0.6701, 0.5127, respectively. This shows that the lower bound of CE is 0.4509 and attained in extreme point $\begin{pmatrix}12\\9\\19\end{pmatrix}$. The rank correlation between the upper and lower bounds of CE is 86.1%. Since the data are strongly uncertain, the differences between the lower bound and upper bound of CE measures, as for DMU_{13}, may be large.

8. Conclusions

This study develops a new idea for cost efficiency analysis dealing with uncertain data. In fact, when the data are imprecise in the form of ranges, the cost efficiency measure calculated from the data should be uncertain, as well. So, a pair of two-level mathematical programming problems were provided to obtain the lower bound and upper bound of CE in cases of bounded data. The provided models are very easy to understand and convenient to use. The resulting two-level mathematical programs are nonlinear and solving them is difficult. In turn, these programs are transformed into equivalent linear ones. In some cases, the input prices of all DMUs are known exactly but the prices differ from one DMU to another. In such cases, for comparing the performance of all DMUs based on their CE measure, it seems in order if we consider the minimum value of each input price, among all DMUs, as the lower bound of that respective input price; and the maximum value of each input price, among all DMUs, as the upper bound of that respective input price. Now, we can use Models (15) and (23) with exact inputs and outputs to obtain the range of CE, with the same ranges for the input prices. The method proposed in this paper is applicable not only when the input prices are in the form of a range or cone, but also when they are in the form of a convex set, provided that the extreme points of the normalized convex set can be obtained. Even when the data are exact but we have sampling error or errors-in-variables, we can utilize the model proposed in this paper for obtaining CE. Computation of economic efficiency when the data are expressed in fuzzy or qualitative terms can be considered for future research.

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