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International portfolio selection with exchange rate risk: A behavioural portfolio theory perspective

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ABSTRACT

This paper analyzes international portfolio selection with exchange rate risk based on behavioural portfolio theory (BPT). We characterize the conditions under which the BPT problem with a single foreign market has an optimal solution, and show that the optimal portfolio contains the traditional mean–variance efficient portfolio without consideration of exchange rate risk, and an uncorrelated component constructed to hedge against exchange rate risk. We illustrate that the optimal portfolio must be mean–variance efficient with exchange rate risk, while the same is not true from the perspective of local investors unless certain conditions are satisfied. We further establish that international portfolio selection in the BPT with multiple foreign markets consists of two sequential decisions. Investors first select the optimal BPT portfolio in each market, overlooking covariances among markets, and then allocate funds across markets according to a specific rule to achieve mean–variance efficiency or to minimize the loss in efficiency.

1. Introduction

According to the behavioural portfolio theory (BPT) proposed by Shefrin and Statman (2000), investors segregate total wealth into multiple mental accounts with different risk attitudes and goals. Next, the investors select the sub-portfolio in each account by attempting to achieve the account's specific investment goal, overlooking covariances among mental accounts. As a consequence, the optimal BPT portfolio is simply the combination of these sub-portfolios rather than Markowitz's (1952) optimal portfolio of all assets. In addition, in the BPT model, risk is measured by the probability (the failing probability) that the portfolio return is less than a pre-specified threshold level. While BPT investors do not follow two-fund separation, their optimal portfolios are consistent with Friedman and Savage's (1948) puzzle. Following Shefrin and Statman (2000) and Das et al. (2010) propose a new mental accounting (MA) framework, where the sub-portfolio within any given account is chosen by maximizing the account's expected return, subject to a constraint that reflects the account's motive. This constraint specifies the sub-portfolio's threshold return and the maximum probability of failing to reach that threshold in the account. Das et al. (2010) show that these sub-portfolios are actually mean–variance efficient, as is the aggregate portfolio composed of these efficient sub-portfolios.

Consider a domestic portfolio investor who wishes to diversify over foreign markets. It is important to note that the investor faces various foreign markets that have many structural and institutional distinctions, including market regulations, trading mechanisms, and trading hours. Individual foreign markets may also exhibit distinct risk-return characteristics and information processing capabilities due to their different economic and political systems as well as their particular developmental stages. Moreover, political and economic risks are distinct across foreign markets. As a result, the investor's risk attitudes may vary across markets. Therefore, instead of identifying a common goal to achieve in multiple distinct foreign markets, the investor specifies a particular investment objective in one market based on his/her risk attitude in that market, and then makes the investment decision to achieve the specific goal in the market as if there are no other portfolio risk exposures. Correspondingly, the investor views the whole portfolio as a combination of the selected portfolios in each market, rather than a combination of individual assets from all markets. This notion of international portfolio selection is supported by the empirical evidence provided by Jorion (1994), and is in line with the layered pyramid structure of portfolios described in Tversky and Kahneman (1986). It also makes practical sense, noting that investors are often recommended by professional fund managers to construct portfolios as pyramids of asset groups

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(Fisher and Statman, 1997). Thus, the investor essentially behaves in accordance with the BPT in our problem, placing assets from one market in one particular mental account with a specific goal to achieve in that account. Evidently, international portfolio selection resembles the BPT problem in the sense that portfolio optimization is divided into sub-portfolio optimizations. Two separate decisions are involved in this particular problem: portfolio selection in each individual foreign market and fund allocation across various foreign markets.

However, international portfolio investments involve not only portfolio risk but also exchange rate risk. Portfolio risk arises from movements in prices of individual assets measured in local currencies, while exchange rate risk is due to the portfolio's domestic currency return variations as a result of exchange rate fluctuations. The presence of distinct exchange rate risk in each individual market provides a further economic rationale for investors to put assets from different markets into distinct mental accounts and to follow the two-decision separation process in international portfolio selection. Given that exchange rate returns and portfolio's local currency returns are correlated (Kaplanis and Schaefer, 1991) and that domestic currency returns are the major concern, it is believed that exchange rate risk greatly impacts the portfolio selection decision in a foreign market. Thus, the selected optimal portfolio in the foreign market can deviate notably from the efficient portfolio without consideration of exchange rate risk (Jiang et al., 2010). If an investor follows the BPT strategy in foreign portfolio selection, then we must ask how exchange rate risk impacts the investor's decision, and why the optimal BPT portfolio is constructed the way it is. The BPT analysis of Das et al. (2010) considers portfolio risk only. Using the framework of Das et al. (2010) and Baptista (2012) deals with the portfolio selection problem with multiple mental accounts in the presence of background risk in each account. It is noteworthy that exchange rate risk can be considered background risk in international portfolio selection (Finkelshtain et al., 1999; Franke et al., 2006).

Motivated by Baptista (2012) and Das et al. (2010), this paper intends to provide a theoretical analysis of international portfolio selection from the perspective of BPT with consideration of exchange rate risk. Given the above-mentioned arguments, the BPT approach is of practical interest and relevance in analyzing international portfolio selection. Further, the BPT approach allows investors' risk attitudes and investment goals to vary by market. For instance, international investors may choose one market to primarily reduce risk and another market to achieve a relatively high expected return. As a result, the BPT approach allows investors to construct a sub-portfolio that meets the investment goal for any given foreign market. Additionally, in contrast with the standard deviation of returns, the failing probability in BPT is a risk measure that is closely related to the value at risk (VaR), and provides a direct application in risk management.

Our paper extends previous work in three respects. First, in contrast with Baptista (2012) and Das et al. (2010), our model includes not only risky assets but also a risk-free asset. Fund allocation between risky and risk-free assets reflects investors' precautionary motives in the present of background risk (Malevergne and Rey, 2010; Menegatti, 2009; Tzeng and Wang, 2002). Thus, our model enables us to investigate both investors' risky asset selection and their precautionary saving behaviour with exchange rate risk. Second, in our analysis the investment set differs from one market to another, whereas the investment set is the same in all mental accounts in Baptista (2012) and Das et al. (2010). Due to this difference, the general conclusions regarding aggregate portfolios in the typical BPT with multiple accounts are not true in our setting. For example, aggregate portfolios in Das et al. (2010) still lie on the mean–variance efficient frontier, while aggregate portfolios in our BPT setting with multiple foreign markets are not mean–variance efficient unless a particular condition is satisfied. Our finding is in line with those in recent work on portfolio selection with mental accounts. Alexander and Baptista (2011) develop a mental account setting with delegation where the optimal portfolios within each account and the aggregate portfolio lie generally away from the mean–variance frontier. This is because (1) investors are assumed to delegate the task of allocating wealth among assets to managers in the model, and (2) managers select portfolios that generally lie away from the mean–variance frontier. Das and Statman (forthcoming) find that optimal portfolios within accounts can noticeably deviate from the portfolios on the mean–variance frontier if asset returns have non-normal distributions. Our paper differs from both studies in that we consider background risk in each account. Baptista (2012) documents that there exist mental account settings where the aggregate portfolio is mean–variance inefficient due to aggregate background risk in mental accounts. However, the mean–variance inefficiency of the aggregate portfolio in our paper is primarily due to the fact that the investment set varies across the markets and due to the lack of integration among the investment decisions in these markets. Third, while the allocation of wealth among accounts is exogenous in Baptista (2012) and Das et al. (2010), it is endogenous in our setting, and represents an important subsequent decision in international portfolio selection.

More specifically, in this paper we explore how BPT investors choose the optimal portfolio in individual foreign markets and how exchange rate risk affects the existence of such portfolios. Our focus is not only on the impact of exchange rate risk on portfolio selection in foreign markets, but also on investors' hedging behaviour. To gain insights, we examine the properties and composition of the optimal BPT portfolio, which also has practical implications for managing exchange rate risk. Investors' precautionary saving behaviour is further analyzed by theoretical and numerical investigations of the proportion of total funds in the risk-free asset. Similar to Alexander and Baptista (2011) and Baptista (2012), we derive the condition under which the aggregate portfolio lies on the efficient frontier in our setting. Moreover, we examine the BPT investors' optimal decision on fund allocation across various markets in the case where this condition is not satisfied, and investigate the efficiency loss of the aggregate portfolio.

In the background risk literature, many portfolio selection models are proposed in an effort to provide theoretical insights into impacts of background risk on the composition of efficient portfolios or on investors' degree of risk aversion under either the utility function framework (Gollier and Pratt, 1996; Kimball, 1993; Pratt and Zeckhauser, 1987; Tsetlin and Winkler, 2005) or the mean–variance framework (Baptista, 2008; Eichner and Wagener, 2009; Jiang et al., 2010). By incorporating background risk into the framework, these models can better explain and predict investors' practical portfolio selection decisions than can traditional portfolio theory (e.g., Markowitz, 1952; Merton, 1969, 1971; Samuelson, 1969). Our work further enriches the body of literature on background risk by examining how exchange rate risk as a specific type of background risk influences international portfolio selection from a BPT perspective, which is of particular interest as argued. In addition, previous research on international portfolio selection and asset allocation is conduced primarily from the perspective of international diversification benefits, such as risk reductions and improvements in Sharpe ratios (De Roon et al., 2001; Driessen and Laeven, 2007; Eun and Resnick, 1988). Our paper complements this stream of research by analyzing the properties of the optimal international portfolio with an emphasis on investors' exchange rate risk hedging and precautionary saving behaviours.

Our contribution is as follows. We derive the conditions under which the solution to the BPT problem with exchange rate risk exists, and show that the optimal BPT portfolio contains the traditional mean–variance efficient portfolio and a component constructed to hedge against exchange rate risk. We also explore properties of the optimal BPT portfolio and explain why the hedging component can mitigate the effect of exchange rate risk. We show that the optimal BPT portfolio must be mean–variance efficient, while it is not mean–variance efficient from the perspective of local investors unless certain conditions are met. We further establish that international portfolio selection in BPT with multiple foreign markets consists of two sequential decisions. Investors first select the optimal BPT portfolio in each market overlooking covariances among markets, and then allocate funds in all markets according to a specific rule to achieve mean–variance efficiency or to minimize the loss in efficiency. We illustrate that investors with a particularly high or low degree of risk aversion experience a relatively large loss in efficiency. Our results have practical implications for the formation of global risk management and investment strategies.

The remainder of this paper is organized as follows. Section 2 presents the model and characterizes the existence and composition of the optimal portfolio. Section 3 analyzes the properties of optimal portfolios and the conditions under which optimal BPT portfolios are mean–variance efficient from a local perspective. Section 4 describes the mean–variance efficiency of the aggregate portfolio in the BPT problem with multiple foreign markets. Section 5 provides a numerical analysis, while Section 6 concludes the paper.[1]

2. Model and optimal BPT portfolio

2.1. The model

In this paper we first consider the problem with one foreign market, corresponding to one mental account in the BPT of Das et al. (2010). Then, we move onto the BPT problem with multiple foreign markets, and investigate the mean–variance efficiency of aggregate portfolios. Suppose there are n risky assets available in the foreign market with a column return vector $\boldsymbol{r}$ denominated in the local currency, and a risk-free asset with a return $\boldsymbol{r}_f$. A portfolio of these n risky assets is a vector $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)^T$, where q_i is the proportion of the portfolio invested in asset i and the superscript Trepresents the transpose operation. Accordingly, the international portfolio is the combination of $\boldsymbol{q}$ and the risk-free asset. Thus, the portfolio's local currency return is $r_p = r_f + \boldsymbol{q}^T(\boldsymbol{r} - \boldsymbol{1}_n r_f)$, where $\boldsymbol{1}_n$ is an n-column vector with all elements being equal to one. The expectation and variance of portfolio returns are given as

$$E(r_p) = r_f + \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f),$$
$$\sigma_p^2 = \boldsymbol{q}^T\boldsymbol{V}\boldsymbol{q},$$

where $\boldsymbol{V}$ stands for the covariance matrix of risky asset returns, and is non-singular.

However, the portfolio return denominated in the domestic currency is the major concern in this context. If the appreciation (or depreciation) rate of the foreign currency relative to the domestic currency is r_e, then the portfolio return denominated in the domestic currency r_D is expressed as

$$r_D = (1 + r_p)(1 + r_e) - 1 = r_p + r_e + r_p r_e. \quad (1)$$

Eq. (1) indicates that the domestic currency return on the foreign portfolio comprises three components: the foreign currency portfolio return, the foreign exchange return, and their product. As Eun and Resnick (1988) note, when the investment period is short, both r_p and r_e are small, and thus, $r_p r_e$ is negligible. To simplify the analysis, we approximate the domestic currency return on a foreign portfolio as the sum of r_p and r_e. Namely,

$$r_D \approx r_p + r_e. \quad (2)$$

Apparently, the presence of exchange rate risk, which is measured by the variability of r_e, directly impacts the domestic currency return. If the foreign currency appreciates/depreciates against the domestic currency, then the domestic currency return is increased/decreased.

The expectation of the domestic currency return and its variance are given by

$$E(r_D) = r_f + \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + E(r_e), \quad (3)$$
$$\sigma_D^2 = \boldsymbol{q}^T\boldsymbol{V}\boldsymbol{q} + 2\boldsymbol{q}^T cov(\boldsymbol{r}, r_e) + \sigma_e^2, \quad (4)$$

where σ_e^2 is the variance of r_e. Portfolio $\boldsymbol{q}$ is mean–variance efficient from the perspective of the domestic investor if its variance is minimized for a given level of expected return.

A BPT investor selects the optimal portfolio by solving the following problem:

$$\max_{\boldsymbol{q}\in\Re^n} \quad E(r_D) = r_f + \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + E(r_e), \quad (5)$$
$$\text{s.t.} \quad \text{Prob}(r_D \leqslant H) \leqslant \alpha,$$

where H is a pre-specified threshold return or aspiration level, and $\alpha \in (0, 0.5)$ is the maximum failing probability. The selection of both H and α reflects the investor's risk attitude and motive in the foreign market, which is determined by both the market's risk-return characteristics and foreign exchange risk. The constraint in model (5) defines the desire for portfolio security, stating that the probability of the portfolio return falling below the threshold level does not exceed the predetermined probability α. The goal for the BPT investor in the market is to maximize the expected return by considering the desire for security, the aspiration level, and probability of achieving the aspiration level.

It is known that if r_p and r_e have a bivariate normal distribution, the value-at-risk (VaR) for the portfolio at the confidence level $1 - \alpha$ is

$$VaR(1 - \alpha, r_D) = z_\alpha \sigma_D - E(r_D), \quad (6)$$

where $z_\alpha = -\Phi^{-1}(\alpha) > 0$ due to the fact that $\alpha \in (0, 0.5)$, and $\Phi(\cdot)$ is the cumulative standard normal distribution function. Under the normality assumption, the constraint in problem (5) is equivalent to a VaR type constraint $VaR(1 - \alpha, r_D) \leqslant -H$. The normality assumption follows Baptista (2012) and Das et al. (2010).[2] Thus, BPT problem (5) becomes

$$\max_{\boldsymbol{q}\in\Re^n} \quad E(r_D) = r_f + \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + E(r_e), \quad (7)$$
$$\text{s.t.} \quad E(r_D) \geqslant H + z_\alpha \sigma_D.$$

2.2. The model solution

We now solve problem (7) to obtain the optimal BPT portfolio. For the sake of convenience, we denote

[1] An Appendix containing proofs of the theoretical results in our paper is available at http://www.uwindsor.ca/odette/yunbi-an. It is also available from the corresponding author upon request.

[2] Optimal portfolio selection based on the mean–variance analysis typically assumes a multivariate normal distribution for asset returns or a quadratic utility function for investors. The assumption of normality allows us to derive a closed-form solution for the optimal portfolio. In addition, our normality-based results can be generalized to non-normality cases as discussed in Alexander and Baptista (2011). However, the model of Das and Statman (forthcoming) represents an illustration of a case where the normality approximation does not work well.

$$a = (E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)^T \boldsymbol{V}^{-1}(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f),$$
$$b = \boldsymbol{1}_n^T \boldsymbol{V}^{-1}(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f),$$
$$f = \boldsymbol{1}_n^T \boldsymbol{V}^{-1}\mathrm{cov}(\boldsymbol{r}, r_{\boldsymbol{e}}),$$
$$g = (E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)^T \boldsymbol{V}^{-1}\mathrm{cov}(\boldsymbol{r}, r_{\boldsymbol{e}}),$$
$$A = \mathrm{cov}(\boldsymbol{r}, r_e)^T \boldsymbol{V}^{-1}\mathrm{cov}(\boldsymbol{r}, r_e).$$

In addition, $\boldsymbol{q}_T = \frac{1}{b}\boldsymbol{V}^{-1}(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)$ is the tangency portfolio in the traditional mean–variance model with an expected excess return of a/b.

Das et al. (2010) show that portfolio optimization in the mental account framework with the VaR constraint yields an optimal portfolio that lies on the mean–variance efficient frontier. Similarly, if the solution to problem (7) exists, then the optimal portfolio is mean–variance efficient from the perspective of domestic investors. For this reason, in order to obtain the optimal BPT portfolio, we need to solve for the mean–variance efficient portfolios in the presence of exchange rate risk.

We first obtain the minimum–variance portfolio with exchange rate risk by minimizing σ_D^2. This portfolio is given as follows

$$\boldsymbol{q}_{\min} = -f\boldsymbol{q}_f, \tag{8}$$

where $\boldsymbol{q}_f = \frac{1}{f}\boldsymbol{V}^{-1}\mathrm{cov}(\boldsymbol{r}, r_e)$. It is easy to prove that the expected return on $\boldsymbol{q}_f$ in excess of the risk-free rate is g/f. This implies that in order for the portfolio to achieve the lowest possible risk in the presence of exchange rate risk, provided that $f > 0$, investors should sell f units of $\boldsymbol{q}_f$ and then invest all the proceeds together with their original funds in the risk-free asset. The reason for this will become clear in the next section. The expected value ($E_{\min}$) and variance ($\sigma_{\min}^2$) of the portfolio's domestic currency returns are

$$E_{\min} = r_f - g + E(r_e), \tag{9}$$
$$\sigma_{\min}^2 = \sigma_e^2 - A. \tag{10}$$

Theorem 1. *For any given expected domestic currency return $\pi \geqslant E_{min}$, the traditional mean–variance efficient portfolio from the perspective of domestic investors is given by*

$$\boldsymbol{q}_\pi = \boldsymbol{q}_{\min} + \frac{b}{a}(\pi - E_{\min})\boldsymbol{q}_T. \tag{11}$$

The efficient frontier is expressed as

$$\sigma_\pi^2 = \frac{1}{a}(\pi - E_{\min})^2 + \sigma_{\min}^2. \tag{12}$$

In the mean–standard deviation plane, the constraint in problem (7) represents the area on or above the straight line $E(r_D) = H + z_\alpha\sigma_D$. If the solution to problem (7) exists, then it must be the point of intersection of $E(r_D) = H + z_\alpha\sigma_D$ and the efficient frontier given by Eq. (12), as illustrated in Fig. 1 (see Fig. 2 in Baptista (2012) for a similar illustration). The following theorem characterizes the conditions under which the solution exists, and also characterizes the composition of the optimal portfolio.

Theorem 2. *If $z_\alpha > \sqrt{a}$ and $H \leqslant E_{\min} - \sigma_{\min}\sqrt{z_\alpha^2 - a}$, then problem (7) has an optimal solution, which is given by*

$$\boldsymbol{q}_{opt} = \boldsymbol{q}_{\min} + \frac{b}{a}(\pi_{opt} - E_{\min})\boldsymbol{q}_T, \tag{13}$$

where $\pi_{opt} = \frac{z_\alpha^2 E_{\min} - aH + z_\alpha\sqrt{a\left((E_{\min} - H)^2 - (z_\alpha^2 - a)\sigma_{\min}^2\right)}}{z_\alpha^2 - a}$.

This theorem is closely related to Theorems 2 and 3 in Alexander and Baptista (2011) and Theorem 1 in Baptista (2012). Alexander and Baptista (2011) assume that an investor allocates his/her wealth among various managers in each account, where each manager optimally selects a portfolio of individual assets. Theorems 2 and 3 in their paper characterize the conditions for the existence of the optimal allocations within accounts and the composition of such allocations. Note that there is no background risk involved in their problem. Baptista (2012) incorporates background risk into the mental accounting framework, and Theorem 1 in his paper characterizes the existence and composition of optimal portfolios within accounts with background risk. In both papers, the optimal portfolio is generally not mean–variance efficient due to either the portfolio delegation or background risk assumption. On the other hand, our derivation of Theorem 2 is based on the observation that from the perspective of domestic investors, the optimal BPT portfolio lies on the mean–variance efficient frontier. As a result, the optimal portfolio characterized in Theorem 2 in each foreign market is still efficient in the presence of exchange rate risk. Also, as we show shortly, this portfolio deviates notably from the mean–variance efficient portfolio without considering exchange rate risk. Moreover, the opportunity set in Alexander and Baptista (2011) and Baptista (2012) includes only risky assets, while it includes both the risky and risk-free assets in our paper. The composition of the optimal portfolio in Eq. (13) also reflects fund allocation between the risky and risk-free assets. Thus, Theorem 2 extends the results in Alexander and Baptista (2011) and Baptista (2012).

The conditions characterized in Theorem 2 as well as those in Alexander and Baptista (2011) and Baptista (2012) for the existence of the optimal portfolios within accounts reflect the trade-off between the failing probability and the threshold return, regardless of whether or not background risk is present. Specifically, for a given failing probability satisfying $z_\alpha > \sqrt{a}$, the threshold return must be lower than a certain level to ensure the existence of the optimal BPT portfolio. On the other hand, for a given threshold return satisfying $H < E_{\min}$, the failing probability must be higher than a certain level to ensure the existence of the optimal BPT portfolio. The intuition behind these conditions is clear. A high threshold return with a low failing probability is relatively hard to achieve, as investors cannot increase the individual asset and foreign exchange returns or lower their risks in a given market. However, it is noteworthy that the choice of the failing probability and threshold return may be determined by different factors in different model settings. In our paper, this selection depends not only on the risk-return characteristics of individual markets but also on the exchange rate risk, whereas it depends on the account's motive in the typical mental account setting.

3. Properties of the optimal BPT portfolio

3.1. Exchange rate risk hedging

In order to examine investors' exchange rate risk hedging behaviour, we now focus on the composition of the optimal BPT portfolio. Using Eq. (13), we have:

$$\begin{aligned}\boldsymbol{q}_{opt} &= -f\boldsymbol{q}_f + \frac{b}{a}(\pi_{opt} - r_f + g - E(r_e))\boldsymbol{q}_T \\ &= \frac{b}{a}(\pi_{opt} - E(r_e) - r_f)\boldsymbol{q}_T + f\left(\frac{g}{f}\frac{b}{a}\boldsymbol{q}_T - \boldsymbol{q}_f\right) = w\boldsymbol{q}_T + f\boldsymbol{q}_H,\end{aligned} \tag{14}$$

where $w = \frac{b}{a}(\pi_{opt} - E(r_e) - r_f)$ and $\boldsymbol{q}_H = \frac{g}{f}\frac{b}{a}\boldsymbol{q}_T - \boldsymbol{q}_f$. Note that if an investor selects his/her portfolio in the foreign market without consideration of exchange rate risk, then $\pi_{opt} - E(r_e)$ is the expected local currency return, and $w\boldsymbol{q}_T$ is the traditional efficient portfolio (Huang and Litzenberger, 1988). Thus, Eq. (14) indicates that the optimal BPT portfolio has two components. The first corresponds to the traditional efficient portfolio in the absence of exchange rate risk, which must be the same for all investors, regardless of the

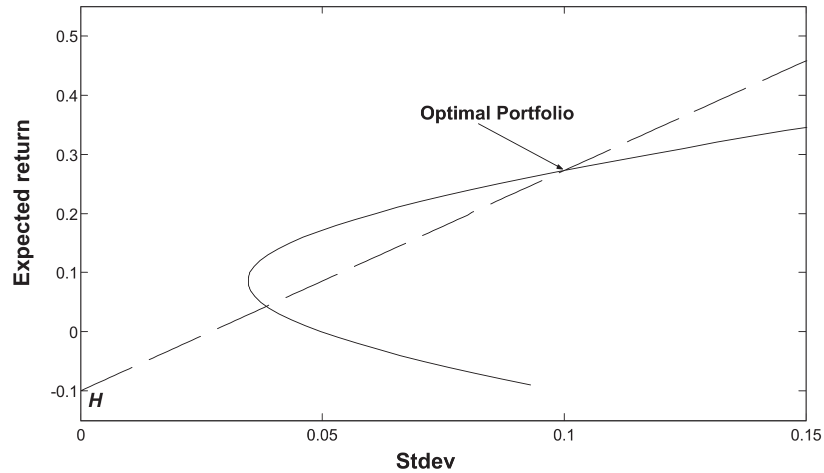

Fig. 1. The optimal BPT portfolio in a foreign market. This figure illustrates the optimal portfolio determined in the BPT model with a single foreign market. The upper segment of the curved line is the efficient frontier of risky assets with exchange rate risk, while the straight line is the plot of $E(r_D) = H + z_\alpha \sigma_D$.

country they are from. This portfolio requires investing w percent of total funds in the tangency portfolio $\boldsymbol{q}_T$ and $1 - w$ in the risk-free asset. In other words, the two fund separation theorem holds in this case. The second component represents the deviation of the optimal BPT portfolio from the traditional efficient portfolio, due to the presence of exchange rate risk. It is self-financed in the sense that if $f > 0 (f < 0)$, then investors short sell (buy) the risk-free asset in the amount of f percent of total funds and buy (short sell) the same amount of portfolio $\boldsymbol{q}_H$. Interestingly, this component varies greatly among investors who come from different countries with different exchange rate risks.

We note that the expected return on $\boldsymbol{q}_H$ in excess of the risk-free rate is zero, and it is uncorrelated with $\boldsymbol{q}_T$. This is because

$$\boldsymbol{q}_H^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) = \frac{g}{f}\frac{b}{a}\boldsymbol{q}_T^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) - \boldsymbol{q}_f^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) = \frac{g}{f}\frac{b}{a}\frac{a}{b} - \frac{g}{f} = 0. \quad (15)$$

$$\left(\frac{g}{f}\frac{b}{a}\boldsymbol{q}_T - \boldsymbol{q}_f\right)^T \boldsymbol{V}\boldsymbol{q}_T = \frac{g}{f}\frac{b}{a}\boldsymbol{q}_T^T\boldsymbol{V}\boldsymbol{q}_T - \boldsymbol{q}_f^T\boldsymbol{V}\boldsymbol{q}_T = \frac{g}{f}\frac{b}{a}\frac{a}{b^2} - \frac{g}{bf} = 0. \quad (16)$$

These results parallel those in Jiang et al. (2010). Eq. (15) implies that the expected excess return provided by the optimal BPT portfolio is completely generated by the corresponding traditional mean–variance portfolio. Eq. (16) is also intuitive, as the first component in the optimal portfolio is the mean–variance efficient portfolio constructed by local investors without considering exchange rate risk, while the second component is present due to the foreign exchange exposure. Thus, both components are uncorrelated.

We now explain the role that $\boldsymbol{q}_H$ plays in managing exchange rate risk. Since $\boldsymbol{q}_T$ and the risk-free asset are the two funds in the two-fund separation theorem in the traditional mean–variance model, we focus on $\boldsymbol{q}_f$ and see how this component helps hedge against exchange rate risk. To this end, consider the following regression

$$r_e = \theta_0 + \boldsymbol{\theta}^T(\boldsymbol{r} - \boldsymbol{1}_n r_f) + \varepsilon, \quad (17)$$

where $\boldsymbol{\theta}$ is the regression coefficient vector, and ε is the error term. Here, $E(\varepsilon) = 0$ and cov $(\boldsymbol{r}, \varepsilon) = \boldsymbol{0}$. Also, $\theta_0 + \varepsilon$ and $\boldsymbol{\theta}^T(\boldsymbol{r} - \boldsymbol{1}_n r_f)$ are the parts of foreign exchange returns that, respectively, cannot and can be explained by local currency excess asset returns. From Eq. (17), we see that

$$\boldsymbol{q}_f = \frac{1}{f}\boldsymbol{V}^{-1}\text{cov}(\boldsymbol{r}, r_e) = \frac{1}{f}\boldsymbol{V}^{-1}\text{cov}(\boldsymbol{r}, \theta_0 + \boldsymbol{\theta}^T(\boldsymbol{r} - \boldsymbol{1}_n r_f) + \varepsilon) = \frac{1}{f}\boldsymbol{\theta} = \frac{1}{\boldsymbol{\theta}^T\boldsymbol{1}_n}\boldsymbol{\theta}. \quad (18)$$

Thus, portfolio $\boldsymbol{q}_f$ is the normalized regression coefficient vector. As Eq. (17) can be expressed as

$$r_e = \theta_0 + (\boldsymbol{\theta}^T\boldsymbol{1}_n)\boldsymbol{q}_f^T(\boldsymbol{r} - \boldsymbol{1}_n r_f) + \varepsilon, \quad (19)$$

$\boldsymbol{\theta}^T\boldsymbol{1}_n$ (or f) measures the sensitivity of foreign exchange returns to changes in portfolio $\boldsymbol{q}_f$ returns in excess of the risk-free rate. This explains why the carefully constructed portfolio $\boldsymbol{q}_f$ can help mitigate the effect of exchange rate risk. Specifically, domestic investors can sell f units of $\boldsymbol{q}_f$ to offset exchange rate risk while selecting their foreign portfolio, and this is the reason why there is a term $-f\boldsymbol{q}_f$ in the optimal BPT portfolio. At the same time, the expected excess return generated by $\boldsymbol{q}_f$ is offset by the other component in $\boldsymbol{q}_H$, resulting in no expected excess returns from $\boldsymbol{q}_H$.

3.2. The efficiency of the optimal BPT portfolio from the perspective of local investors

Portfolio $\boldsymbol{q}_f$ typically lies away from the traditional mean–variance efficient frontier. As a result, the optimal BPT portfolio is generally not mean–variance efficient from the perspective of local investors unless the condition in the following theorem is satisfied. In this context local investors are those who make portfolio decisions without consideration of exchange rate risk.

Theorem 3. *The optimal BPT portfolio is mean–variance efficient from the perspective of local investors if and only if*

$$\text{cov}(\boldsymbol{r}, r_e) = \kappa(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f), \quad (20)$$

where κ is a constant.

Theorem 3 says that as long as cov $(\boldsymbol{r}, r_e)$ is proportional to $E(\boldsymbol{r}) - \boldsymbol{1}_n r_f$, the expected excess return vector, the optimal BPT portfolio is also mean–variance efficient from the perspective of local investors for the given local currency return $\pi_{opt} - E(r_e)$. In this case, the expected domestic currency return and its variance are given by:

$$E(r_D) = r_f + \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + E(r_e), \tag{21}$$

$$\sigma_D^2 = \boldsymbol{q}^T \boldsymbol{V} \boldsymbol{q} + 2\kappa \boldsymbol{q}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + \sigma_e^2. \tag{22}$$

Since both $E(r_e)$ and σ_e^2 are exogenous to the mean–variance problem based on Eqs. (21) and (22), the presence of exchange rate risk does not affect the determination of efficient portfolios. As we know, there are both local and international investors (domestic investors in our context) in any particular foreign market. Consequently, in this case the optimal portfolios selected by both local and international investors satisfy the two-fund separation theorem, where the two funds are the tangency portfolio and the risk-free asset.

Given Eq. (17), it can be shown that the expected foreign exchange return and variance are

$$E(r_e) = \boldsymbol{\theta}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f) + \theta_0, \tag{23}$$

$$\sigma_e^2 = \boldsymbol{\theta}^T \boldsymbol{V} \theta + \sigma_\varepsilon^2. \tag{24}$$

Eq. (23) indicates that the expected foreign exchange return comprises two components: $\boldsymbol{\theta}^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)$ and θ_0, and the first component is the part of returns that can be explained by foreign currency returns in excess of the risk-free rate. Similarly, Eq. (24) says that the variance includes both $\boldsymbol{\theta}^T\boldsymbol{V}\boldsymbol{\theta}$ and σ_ε^2, and $\boldsymbol{\theta}^T\boldsymbol{V}\boldsymbol{\theta}$ is the variance of the returns that can be explained by foreign currency returns in excess of the risk-free rate. In particular, if cov $(\boldsymbol{r}, r_e) = \kappa(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)$, then $\boldsymbol{\theta} = \kappa \boldsymbol{V}^{-1}(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)$. In this case, the expected return and variance of the foreign exchange rate can be re-written as

$$E(r_e) = \kappa a + \theta_0, \tag{25}$$

$$\sigma_e^2 = \kappa^2 a + \sigma_\varepsilon^2. \tag{26}$$

Under this condition, the expected return and variance of the minimum–variance portfolio are

$$E_{\min} = r_f + \theta_0, \tag{27}$$

$$\sigma_{\min}^2 = \sigma_\varepsilon^2. \tag{28}$$

Consequently, the efficient frontier generated by the foreign risky assets with exchange rate risk is

$$\sigma_\pi^2 = \frac{1}{a}(\pi - (r_f + \theta_0))^2 + \sigma_\varepsilon^2. \tag{29}$$

Eq. (29) says that the presence of exchange rate risk simply moves the efficient frontier for local investors to the right for σ_ε^2 units and moves it up or down for θ_0 units. If exchange rate risk can be completely explained by risky asset returns, then $\sigma_\varepsilon^2 = \theta_0 = 0$. In this special case, the efficient frontiers for both domestic and local investors are identical.

3.3. Proportion of total funds in the risk-free asset

As described in the model, investments in the risk-free asset represent investors' precautionary savings in the presence of background risk. Therefore, investors' hedging behaviour is not only reflected by including portfolio $\boldsymbol{q}_f$ in $\boldsymbol{q}_{opt}$, but also reflected by investing in the risk-free asset. Eq. (13) implies that the proportion of the optimal portfolio invested in the risk-free asset w_f is given by

$$\begin{aligned} w_f &= 1 - \boldsymbol{q}_{opt}^T \boldsymbol{1}_n = \left[1 - \frac{b}{a}(\pi_{opt} - E(r_e) - r_f)\right] + f\left(1 - \frac{g/f}{a/b}\right) \\ &= \left[1 - \frac{b}{a}(\pi_{opt} - E(r_e) - r_f)\right] + f\left(1 - \frac{\boldsymbol{q}_f^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)}{\boldsymbol{q}_T^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)}\right). \end{aligned} \tag{30}$$

For the given expected local currency return $\pi_{opt} - E(r_e)$, the traditional efficient portfolio weight in the risk-free asset is the first term on the right-hand side of Eq. (30). Thus, the difference in the portfolio weights in the risk-free asset between the efficient portfolios with and without exchange rate risk is equal to $f\left(1 - \frac{\boldsymbol{q}_f^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)}{\boldsymbol{q}_T^T(E(\boldsymbol{r}) - \boldsymbol{1}_n r_f)}\right)$. In the case where $f > 0$ and also the expected excess return on $\boldsymbol{q}_f$ is lower than the expected excess return on $\boldsymbol{q}_T$, domestic investors should place more funds in the risk-free asset in the portfolio than local investors.

4. Asset allocation across foreign markets

Let us move onto the case in which investors allocate total wealth among multiple foreign markets. In contrast with BPT with multiple accounts (Baptista, 2012; Das et al., 2010) where the investment opportunity set is identical across mental accounts, in our analysis the available set of assets differs from market to market. Given that each individual market exhibits distinct risk-return characteristics and exchange rate risk, investors may have different risk preferences in various foreign markets. Accordingly, investors set different maximum failing probabilities and threshold returns in individual markets to achieve different goals. One distinguishing feature in our framework is that BPT investors do not consider their portfolio as a whole of individual assets from all markets. Instead, they regard their portfolio as a combination of optimal BPT portfolios in individual markets. Thus, an important question in our problem is how total funds should be allocated across these optimal BPT sub-portfolios.

In this section we focus on asset allocation across foreign markets, and characterize the mean–variance efficiency of aggregate portfolios. As argued in Shefrin and Statman (2000), BPT investors with multiple accounts segregate their portfolios into distinct mental accounts, overlooking covariances among these mental accounts. Consistent with this important assumption in BPT, our investors select their optimal BPT portfolio in a particular foreign market as if this market were uncorrelated with other markets. This is still the case in the subsequent asset allocation decision. As noted by Tversky and Kahneman (1986), the presence of covariances among mental accounts imposes great difficulties on the analysis; this is why people simply divide joint distributions into the assumed uncorrelated mental accounts. Nevertheless, this model setup makes practical sense, as there is strong experimental and practical evidence showing that investors ignore covariances when constructing their portfolios (Kroll et al., 1988). Hence, BPT investors generally end up holding aggregate portfolios that differ from those of mean–variance investors. Moreover, BPT portfolios are inherently suboptimal.

Without loss of generality, we consider an international investor who chooses to invest in m foreign markets. It is important to note that all assets in a foreign market are risky from the perspective of domestic investors, as even the risk-free asset in the market generates variable domestic currency returns due to exchange rate risk. If the vector for proportions of total wealth in the ith market is $\boldsymbol{q}_i$ with a domestic currency return vector $\boldsymbol{r}_i$, then the return on the aggregate portfolio is

$$r_A = \sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{r}_i, \tag{31}$$

where $\sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{1}_{(i)} = 1$, and $\boldsymbol{1}_{(i)}$ is a column vector with all elements being equal to one and the number of elements being equal to the number of assets in the ith market. Thus, the expected return and variance of the aggregate portfolio are given as follows

$$E(r_A) = \sum_{i=1}^{m} \boldsymbol{q}_i^T E(\boldsymbol{r}_i), \tag{32}$$

$$\sigma_A^2 = \sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{V}_i \boldsymbol{q}_i + \sum_{i=1, i\neq j}^{m} \sum_{j=1}^{m} \boldsymbol{q}_i^T \boldsymbol{V}_{ij} \boldsymbol{q}_j, \tag{33}$$

where $\boldsymbol{V}_i$ is the covariance matrix of domestic currency returns on assets in the ith market, and $\boldsymbol{V}_{ij} = \text{cov}(\boldsymbol{r}_i, \boldsymbol{r}_j)$. Thus, for an expected return of v, the efficient portfolio of all assets in these m markets solves the following optimization problem:

$$\min_{\boldsymbol{q}_i \in \Re^{(i)}, i=1,\ldots,m} \quad \sigma_A^2 = \sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{V}_i \boldsymbol{q}_i + \sum_{i=1, i\neq j}^{m} \sum_{j=1}^{m} \boldsymbol{q}_i^T \boldsymbol{V}_{ij} \boldsymbol{q}_j, \tag{34}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} \boldsymbol{q}_i^T E(\boldsymbol{r}_i) = v,$$

$$\sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{1}_{(i)} = 1.$$

One interesting question is whether an efficient portfolio of all assets is simply a portfolio of efficient portfolios in individual markets.

Lemma 1. *If $\boldsymbol{V}_{ij} = \boldsymbol{0}$ for any $i \neq j$, then any portfolio on the aggregate efficient frontier is a combination of efficient portfolios on the sub-frontiers generated by assets in individual foreign markets, and the percentage of funds placed in the ith market is given by*

$$x_i = \frac{C_m v - B_m}{A_m C_m - B_m^2} b_i + \frac{A_m - B_m v}{A_m C_m - B_m^2} c_i, \tag{35}$$

where v is the expected return on the aggregate portfolio. In addition, the portfolio in the ith market is q_i/x_i, with

$$\boldsymbol{q}_i = \frac{C_m v - B_m}{A_m C - B_m^2} \boldsymbol{V}_i^{-1} E(\boldsymbol{r}_i) + \frac{A_m - B_m v}{A_m C_m - B_m^2} \boldsymbol{V}_i^{-1} \boldsymbol{1}_{(i)}, \tag{36}$$

where $a_i = E(\boldsymbol{r}_i)^T \boldsymbol{V}_i^{-1} E(\boldsymbol{r}_i)$, $b_i = \boldsymbol{1}_{(i)}^T \boldsymbol{V}_i^{-1} E(\boldsymbol{r}_i)$, $c_i = \boldsymbol{1}_{(i)}^T \boldsymbol{V}_i^{-1} \boldsymbol{1}_{(i)}$, $A_m = \sum_{i=1}^{m} a_i$, $B_m = \sum_{i=1}^{m} b_i$, *and* $C_m = \sum_{i=1}^{m} c_i$.

If the condition in Lemma 1 holds, then any aggregate efficient portfolio can be decomposed into sub-portfolios $\boldsymbol{q}_i/x_i (i = 1, 2, \ldots, m)$, which are mean–variance efficient in their respective markets. The expected return on the efficient portfolio $\boldsymbol{q}_i/x_i$ is as follows:

$$\frac{1}{x_i} \boldsymbol{q}_i^T E(\boldsymbol{r}_i) = \frac{v(C_m a_i - B_m b_i) + A_m b_i - B_m a_i}{v(C_m b_i - B_m c_i) + A_m c_i - B_m b_i}. \tag{37}$$

Eq. (37) indicates that once the expected return on the aggregate efficient portfolio is given, the expected return on each sub-portfolio is determined. Consequently, the decomposition of any aggregate efficient portfolio is unique.

Using Lemma 1, we can derive the aggregate efficient frontier for BPT investors, which is generated by the assets in all foreign markets. In particular, the variance of aggregate portfolio returns for a given expected return v is given by

$$\sigma_A^2 = \sum_{i=1}^{m} \boldsymbol{q}_i^T \boldsymbol{V}_i \boldsymbol{q}_i = \frac{C_m}{A_m C_m - B_m^2} \left(v - \frac{B_m}{C_m} \right)^2 + \frac{1}{C_m}. \tag{38}$$

Varying v in Eq. (38) results in a set of variances, tracing out graphically the aggregate mean–variance efficient frontier in the mean–variance space.

To illustrate Lemma 1, we consider the case in which there are only two markets. Fig. 2 plots the aggregate efficient frontier, the efficient frontier for each market, and the efficient frontier generated by two efficient portfolios that lie on the two sub-frontiers. Lemma 1 suggests that any aggregate efficient portfolio C can be decomposed into sub-portfolios A and B. In this case, the efficient frontier generated by A and B is inner-tangent to the aggregate efficient frontier at point C.

Now we examine the same issue from the opposite direction, and consider whether the aggregate portfolio of optimal BPT sub-portfolios still resides on the aggregate efficient frontier. It turns out that the aggregate portfolio is generally not mean–variance efficient unless a specific condition is satisfied.

Theorem 4. *Suppose that the optimal BPT portfolio in the ith foreign market has an expected return of π_i, which is a function of the threshold return and failing probability in that market. There exists an aggregate portfolio that lies on the aggregate efficient frontier if and only if*

$$\frac{A_m b_i - B_m a_i - (A_m c_i - B_m b_i)\pi_i}{\pi_i (C_m b_i - B_m c_i) - (C_m a_i - B_m b_i)} = \Pi, \quad \forall i. \tag{39}$$

Π is a constant, and the weight of the portfolio in the ith market x_i is given by

$$x_i = \frac{C_m \Pi - B_m}{A_m C_m - B_m^2} b_i + \frac{A_m - B_m \Pi}{A_m C_m - B_m^2} c_i. \tag{40}$$

The intuition behind Theorem 4 is clear. We note that the optimal BPT portfolio in each market corresponds to the predetermined threshold return and failing probability. This investment objective is different from one market to another. On the other hand, an aggregate portfolio decision simply implies one particular objective in all markets. Condition (39) requires that the predetermined BPT parameters in individual markets satisfy a precise relationship in order for the aggregate portfolio to be efficient. However, these BPT parameters in each market are determined based on the market's unique risk-return characteristics without considering risk exposures in other markets, and thereby are not connected in various markets. As a consequence, the aggregate portfolio of these BPT sub-portfolios with multiple investment objectives is typically inefficient because of the lack of integration among the motive parameters in various markets. In addition, while any aggregate efficient portfolio can be decomposed into efficient portfolios on sub-frontiers, these sub-portfolios are typically not exactly the optimal BPT sub-portfolios associated with the predetermined threshold returns and failing probabilities in individual markets.

Theorem 4 has two important practical implications. First, if Equation (39) is true, a BPT investor starts with the optimal portfolio selection in individual markets, and then combines them according to Eq. (40). This aggregate portfolio is mean–variance efficient, and the decision of the BPT investor coincides with and the decision of the mean–variance investor with the assumption of zero covariances among markets.

Second, since the BPT approach allows investors to specify different BPT investment objectives and to select the optimal portfolio based on risk and return characteristics in each individual foreign market, the aggregate BPT portfolio is typically not on the aggregate efficient frontier. Unless Eq. (39) holds, the aggregate portfolio of optimal BPT portfolios results in a loss in efficiency even if investors can accurately specify their risk attitudes in different markets. This observation is in contrast with Das et al.'s (2010) conclusion that in their model framework the aggregate portfolio is mean–variance efficient with short-selling. In this case, BPT investors should allocate total wealth across these optimal BPT portfolios to minimize the loss in efficiency.

To obtain the weight in each optimal BPT portfolio in the second case, we denote the BPT portfolio return in the ith market as r_{Di}, the variance as σ_{Di}^2, and the expected return as π_i. Since the optimal BPT portfolio is efficient in the ith market, we have

$$\sigma_{Di}^2 = \frac{c_i}{a_i c_i - b_i^2} \left(\pi_i - \frac{b_i}{c_i} \right)^2 + \frac{1}{c_i}. \tag{41}$$

The expected return vector on these optimal BPT portfolios is $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)^T$. Given the assumption cov $(\boldsymbol{r}_i, \boldsymbol{r}_j) = \boldsymbol{0}$, the covari-

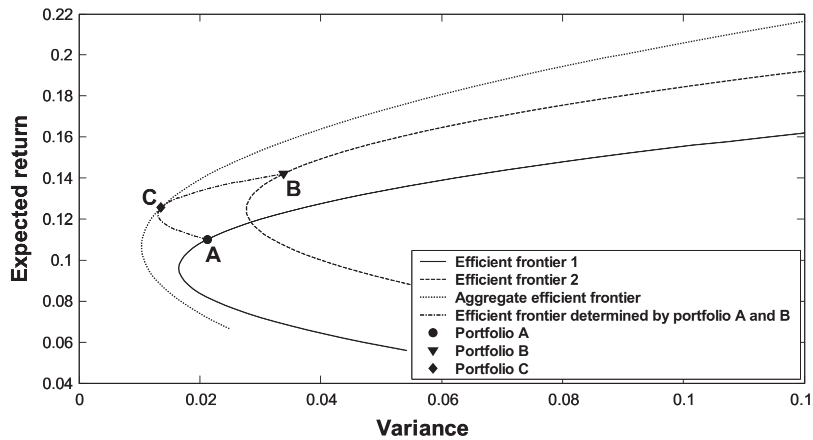

Fig. 2. Decomposition of aggregate efficient portfolios. This figure illustrates the decomposition of aggregate efficient portfolios described in Lemma 1 in the case of two foreign markets. Efficient frontiers 1 and 2 are the frontiers generated by assets in markets 1 and 2, respectively. The aggregate efficient frontier is the frontier generated by all available assets in both markets.

ance matrix of these portfolio returns $\mathbf{\Sigma}$ is a diagonal matrix, i.e., $\mathbf{\Sigma} = diag(\sigma^2_{D1}, \sigma^2_{D2}, \ldots, \sigma^2_{Dm})$. Let $\mathbf{1}_m$ be an m-column vector with all elements being equal to one, $A_\pi = \boldsymbol{\pi}^T\mathbf{\Sigma}^{-1}\boldsymbol{\pi}, B_\pi = \mathbf{1}_m^T\mathbf{\Sigma}^{-1}\boldsymbol{\pi}$, and $C_\pi = \mathbf{1}_m^T\mathbf{\Sigma}^{-1}\mathbf{1}_m$, then the efficient frontier generated by these m optimal portfolios is

$$\sigma^2_{A\pi} = \frac{C_\pi}{A_\pi C_\pi - B_\pi^2}\left(v - \frac{B_\pi}{C_\pi}\right)^2 + \frac{1}{C_\pi}, \tag{42}$$

where v is the expected return on the aggregate portfolio.[3]

Theorem 5. *If none of the combinations of optimal BPT portfolios lies on the aggregate efficient frontier, then BPT investors choose the following aggregate portfolio:*

$$\boldsymbol{x}^* = \frac{C_\pi v^* - B_\pi}{A_\pi C_\pi - B_\pi^2}\mathbf{\Sigma}^{-1}\boldsymbol{\pi} + \frac{A_\pi - B_\pi v^*}{A_\pi C_\pi - B_\pi^2}\mathbf{\Sigma}^{-1}\mathbf{1}_m, \tag{43}$$

where v^* is the expected return on the optimal aggregate portfolio, and

$$v^* = \left(\frac{B_m}{A_m C_m - B_m^2} - \frac{B_\pi}{A_\pi C_\pi - B_\pi^2}\right) \Bigg/ \left(\frac{C_m}{A_m C_m - B_m^2} - \frac{C_\pi}{A_\pi C_\pi - B_\pi^2}\right). \tag{44}$$

The optimal aggregate portfolio is the portfolio that minimizes the loss in efficiency. For the aggregate portfolio with an expected return of v, the loss in efficiency is defined as the difference between Eqs. (42) and (38). Namely,

$$\delta = \frac{C_\pi}{A_\pi C_\pi - B_\pi^2}\left(v - \frac{B_\pi}{C_\pi}\right)^2 + \frac{1}{C_\pi} - \frac{C_m}{A_m C_m - B_m^2}\left(v - \frac{B_m}{C_m}\right)^2 - \frac{1}{C_m}. \tag{45}$$

It is easy to show that δ is minimized at point v^*. Accordingly, the optimal aggregate portfolio weights are given by Eq. (43).

Theorem 5 is of particular interest to international portfolio investors, as it provides a strategy for fund allocation among optimal BPT portfolios. Investors may use alternative methods to choose the optimal aggregate portfolio. For example, given the overall degree of risk aversion to foreign markets, an investor may apply the BPT method again or use the mean–variance utility framework. However, the optimal aggregate portfolios based on these alternative methods do not typically minimize the loss in efficiency.

Given the optimal BPT portfolios in each market, Eq. (45) shows that a too high or too low expected return leads to a relatively higher loss in efficiency. Suppose, instead, that investors select the aggregate portfolio based on the mean–variance utility function $v - \frac{\Gamma}{2}\sigma^2_{A\pi}$, where Γ represents the degree of risk aversion. In this case, the optimal mean–variance aggregate portfolio minimizes the loss in efficiency if

$$\Gamma^* = \frac{1}{\frac{C_\pi}{A_\pi C_\pi - B_\pi^2}\left(v^* - \frac{B_\pi}{C_\pi}\right)}.$$

Das et al. (2010) find that the efficiency loss may arise either from misspecification of investors' risk aversion or from imposition of the short-selling constraint. Using a numerical example, they demonstrate that the losses are higher for investors who are less risk-averse in both cases. In our setting, the efficiency loss of aggregate portfolios is primarily due to the lack of integration among the objectives in various markets where the opportunity sets are different. When the efficiency loss is measured by δ defined in Eq. (45), we find that while investors with a very low degree of risk aversion ($\Gamma < \Gamma^*$) to foreign assets may experience a large loss in efficiency, those who are very risk-averse ($\Gamma > \Gamma^*$) may also suffer a relatively large loss in efficiency.

Overall, Theorems 4 and 5 establish that international portfolio selection in the BPT with multiple foreign markets consists of two sequential decisions: First, investors choose the optimal BPT portfolio in each market for a given threshold return and a maximum failing probability, ignoring covariances among markets. Then, they allocate funds across these optimal portfolios according to Eq. (40) to obtain an aggregate mean–variance efficient portfolio if Eq. (39) is true. If Eq. (39) does not hold, funds must be allocated according to Eq. (43) to minimize the loss in efficiency.

5. A numerical analysis

In order to illustrate the main results in our paper, we now present a numerical example. This example involves a Chinese investor who wishes to invest in the US market and the euro zone. For this purpose, we choose four US industry portfolios as the risky assets

[3] The efficient frontier is derived from the optimal BPT portfolios assuming zero covariance among each pair of such portfolios. Consequently, it is similar to the characterization of the mean–variance frontier of available assets in the traditional analysis of Huang and Litzenberger (1988).

Table 1
Mean, standard deviation, and correlations of the industry/market portfolios in the US and euro zone.

US market						Euro zone market					
	Cnsmr	Manuf	HiTec	Hlth	Exchange		France	Germany	Netherlands	Spain	Exchange
Mean	0.1013	0.1365	0.1228	0.1219	−0.0408		0.0135	0.0598	0.0162	0.0268	−0.0037
Stdev	0.2644	0.2573	0.2400	0.2397	0.0159		0.1774	0.1934	0.1934	0.2156	0.1155
Correlation											
Cnsmr	1.0000	0.8976	0.9308	0.8459	0.0118	France	1.0000	0.9361	0.8985	0.8379	0.3458
Manuf	0.8976	1.0000	0.9386	0.8512	−0.0961	Germany	0.9361	1.0000	0.8399	0.7706	0.3149
HiTec	0.9308	0.9386	1.0000	0.9138	0.0006	Netherlands	0.8985	0.8399	1.0000	0.7415	0.2163
Hlth	0.8459	0.8512	0.9138	1.0000	−0.0155	Spain	0.8379	0.7706	0.7415	1.0000	0.4851
Exchange	0.0118	−0.0961	0.0006	−0.0155	1.0000	Exchange	0.3458	0.3149	0.2163	0.4851	1.0000

This table reports the mean and standard deviation of the industry portfolios in the US and market indices in the euro zone considered in the paper. The correlations are also presented. The sample period is from July 2005 to June 2011.

in the US market, namely Cnsmr, Manuf, HiTec, and Hlth.[4] Monthly data for these industry portfolios as well as the risk-free rates for the period from July 2005 to June 2011 are obtained from Kenneth R. French's personal website. For the euro zone, we use stock market indices in France, Germany, Netherlands, and Spain as the risky assets, while the risk-free rate is set to zero. The data for these market indices denominated in Euros from July 2005 to June 2011 are obtained from the Morgan Stanley Capital International (MSCI) website. The exchange rates between Chinese RMB and USD, as well as those between Chinese RMB and Euros for the same time period, are obtained from the People's Bank of China website.

Table 1 reports the data's summary statistics. The four industry portfolios from the US market all generate an annualized return of more than 10% with standard deviations ranging from 23.9% to 26.5%. In addition, they are highly correlated, with a correlation coefficient between any pair of portfolio returns higher than 0.84. The risk-free rate is approximately 0.1775%, and Chinese RMB appreciates by 4.1% every year against US dollars during this sample period. The mean returns on these four market indices from the euro zone range from 1.35% to 5.98%, with standard deviations from 17.74% to 21.56%. Chinese RMB appreciates by 0.37% every year against the Euro during this sample period. However, Chinese investors face a higher exchange rate risk when investing in these euro zone markets than when investing in the US. From the data we obtain the values of a number of model parameters and present these in Table 2.

Table 2
Model variable values.

Parameters	US market	Euro zone market
$E_{\min}$	−0.0379	−0.0103
$\sigma^2_{\min}$	0.0002	0.0095
a	0.3678	0.4915
a_i	6.5788	0.5026
b_i	−163.8541	−1.0788
c_i	4322.6420	105.0906

This table presents values of some model parameters in the US and euro zone markets. $E_{\min}$ and $\sigma^2_{\min}$ are, respectively, the mean and variance of the portfolio's domestic currency returns. $a = (E(\mathbf{r}) - \mathbf{1}_n r_f)^T \mathbf{V}^{-1}(E(\mathbf{r}) - \mathbf{1}_n r_f)$, where returns are expressed in the foreign currency. $a_i = E(\mathbf{r}_i)^T \mathbf{V}_i^{-1} E(\mathbf{r}_i)$, $b_i = \mathbf{1}_{(i)}^T \mathbf{V}_i^{-1} E(\mathbf{r}_i)$, and $c_i = \mathbf{1}_{(i)}^T \mathbf{V}_i^{-1} \mathbf{1}_{(i)}$, where returns are expressed in the domestic currency.

5.1. The trade-off between the failing probability and threshold return

It follows from Theorem 2 that, provided that the failing probability $\alpha < 1 - \Phi\left(\sqrt{0.3678}\right) = 0.2721$ in the US market and $\alpha < 1 - \Phi\left(\sqrt{0.4915}\right) = 0.2416$ in the euro zone, the optimal portfolio exists in each market if and only if $H \leqslant E_{\min} - \sigma_{\min}\sqrt{z_\alpha^2 - a}$. Fig. 3 displays the maximum threshold return for a given failing probability that ensures the existence of an optimal portfolio (see Fig. 4 in Baptista (2012) for a similar illustration). Investors can find an optimal portfolio if and only if they choose a pair of (α, H) on or below the curved line. This figure shows clearly that there is a trade-off between the failing probability and the maximum threshold return level. In other words, a low/high threshold return corresponds to a low/high failing probability for both markets, and vice versa. In addition, the curve for the euro zone is below the curve for the US market, and is steeper for a sensible range of failing probabilities (e.g., when the failing probability is lower than 20%), indicating that in order to ensure the existence of an optimal portfolio we need to choose a lower threshold return in the euro zone than in the US. This is particularly pronounced for a relatively low failing probability.[5]

5.2. The proportion in the risk-free asset, selection of (α, H), and exchange rate risk

To examine the impact of the failing probability and threshold return on the proportion of total funds in the risk-free asset, Fig. 4 plots the weight of the risk-free asset as a function of the threshold return in the cases that the failing probability $\alpha = 0.05$, 0.10, and 0.15. Note that the lower the failing probability or the higher the threshold return, the more funds are placed in the risk-free asset. This is not surprising, as a lower failing probability or higher threshold return means a more stringent control over total risk, and more investments in the risk-free asset help achieve this investment objective. However, this limits investors' capital gain potential. The plots of expected returns in Fig. 4 clearly demonstrate this point by showing that the expected return decreases as the failing probability falls or the threshold return rises.

The proportion of the optimal portfolio in the risk-free asset reflects not only the investment objective but also investors' precautionary saving behaviour in the presence of exchange rate risk. To investigate how exchange rate risk can through a precautionary motive influence holdings of the risk-free asset for a given set of (α, H), we consider two scenarios where $(\alpha, H) = (0.05, -0.10)$ and $(0.10, -0.05)$ in the US market. In both scenarios, we set $\sigma_e = 2\%$ or 6% and allow $E(r_e)$ to change between −4% and 4%, while keeping

[4] The following are the definitions of these industry portfolios. Cnsmr: Consumer Durables, NonDurables, Wholesale, Retail, and Some Services; Manuf: Manufacturing; HiTec: Business Equipment, Telephone and Television Transmission; Hlth: Health Care, Medical Equipment and Drugs.

[5] There are two reasons for this observation. First, the US market generates a higher average return than does the euro zone, and this is still true even if the foreign exchange return is considered. Second, the Euro exchange rate risk is higher than the US dollar exchange rate risk from the perspective of Chinese investors. This results in a higher risk-return ratio for investments in the euro zone than in the US.

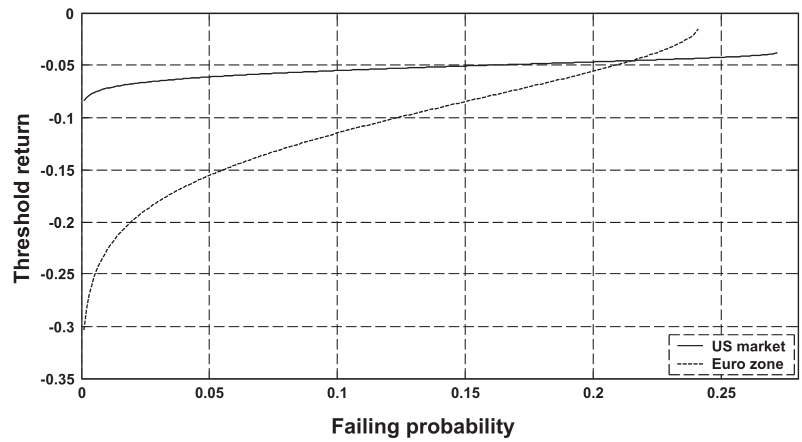

Fig. 3. Threshold return and failing probability. This figure plots the maximum threshold return as a function of the failing probability in the US market and euro zone. An optimal solution to the BPT model exists in the area on or below the curve.

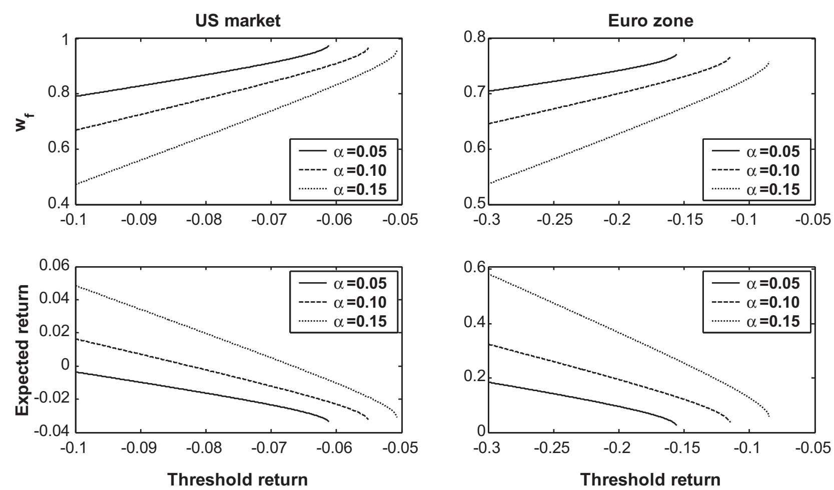

Fig. 4. Proportions in the risk-free asset, expected return of the optimal portfolio, and (α, H). This figure displays the proportions of the optimal portfolio in the risk-free asset (top) as well as its expected returns (bottom) as a function of the threshold return for various failing probabilities in the US market (left) and euro zone (right). w_f is the proportion of total funds in the risk-free asset, while (α, H) represents the set of the failing probability and threshold return.

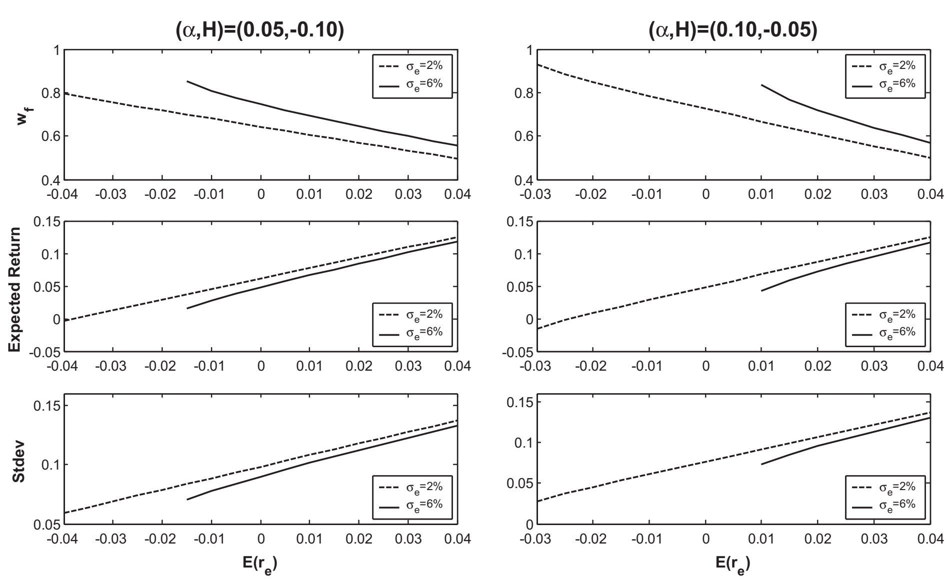

Fig. 5. Proportion in the risk-free asset, expected return, and standard deviation of the optimal portfolio and exchange rate risk. This figure illustrates how the proportion in the risk-free asset, expected return, and standard deviation of the optimal portfolio changes with the expected foreign exchange return and exchange rate risk. The left figures are for the case in which $(\alpha, H) = (0.05, -0.1)$, and the right figures are for the case in which $(\alpha, H) = (0.10, -0.05)\cdot(\alpha, H)$ represents the set of the failing probability and threshold return.

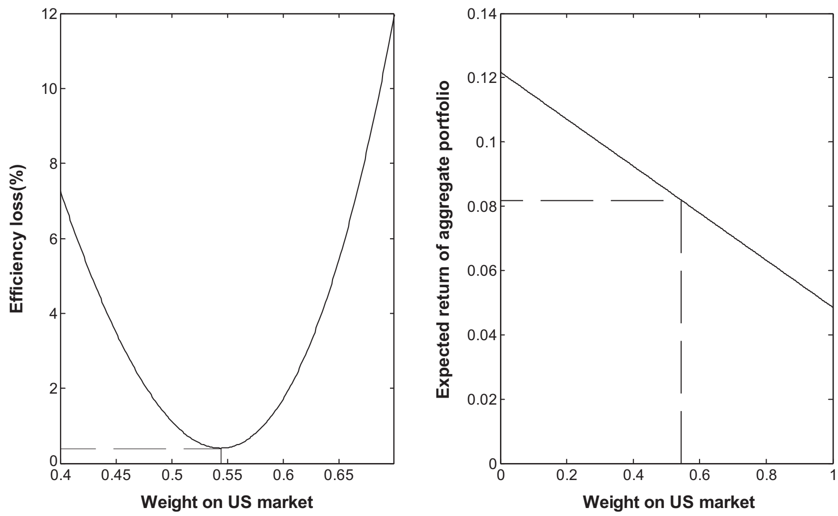

Fig. 6. Efficiency loss, expected return of aggregate portfolio, and asset allocation. This figure plots the efficiency loss (left) as well as the expected return (right) of aggregate portfolios against the weight of total funds in the US market.

the correlations between exchange rate returns and asset returns constant.[6] Fig. 5 plots the proportion in the risk-free asset, as well as the optimal portfolio's return and standard deviation against $E(r_e)$. Evidently, as the expected exchange rate return increases, the fraction of the risk-free asset in the optimal portfolio decreases for a given value of σ_e, regardless of the investment objective (α, H). The intuition is that as the foreign currency appreciates against the domestic currency, more funds should be allocated to the risky asset to better capture the foreign currency return. In this case, both the optimal portfolio's expected return and risk rise. On the other hand, a higher exchange rate risk results in a higher proportion in the risk-free asset for a given $E(r_e)$ to meet the risk control constraint in the model. This is consistent with findings in the literature that higher background risk enhances investors' demand for protective savings (Courbage and Rey, 2007; Fei and Schlesinger, 2008; Malevergne and Rey, 2010; Menegatti, 2009; Tzeng and Wang, 2002).

5.3. Assets allocation and efficiency loss

To illustrate the efficiency loss of aggregate portfolios in our example, we set (α, H) equal to (0.15, −0.10) and (0.10, −0.15) in the US and euro zone, respectively. Using the parameters in Table 2, we find that the optimal BPT portfolio in the US has a lower expected return and risk than the optimal portfolio in the euro zone, and Eq. (39) does not hold. Thus, none of combinations of both optimal BPT portfolios lies on the aggregate efficient frontier. In this case, when 54.44% of total funds are allocated to the US market, the efficiency loss is minimized, and the loss is 0.4% of the variance of the corresponding aggregate efficient portfolio. Fig. 6 displays the percentage efficiency loss and the expected return of the aggregate portfolio as a function of the portfolio weight in the US market.

6. Conclusions

This paper examines international portfolio selection with exchange rate risk based on BPT. We derive conditions under which the BPT problem with one foreign market has an optimal solution, and characterize the composition of the optimal portfolio. We document that the optimal BPT portfolio contains the traditional mean–variance efficient portfolio without consideration of exchange rate risk, and an uncorrelated component constructed to hedge against exchange rate risk. We find that the optimal BPT portfolio is typically not mean–variance efficient from the perspective of local investors unless certain conditions are satisfied. We further demonstrate that the aggregate portfolio in the BPT problem with multiple foreign markets does not typically reside on the aggregate efficient frontier, and therefore funds should be allocated across optimal BPT portfolios to minimize the loss in efficiency. We show that the loss in efficiency depends on the BPT investors' degree of risk aversion. These results provide insights into international portfolio selection that complement those in previous papers.

Using a numerical example, we illustrate that as the failing probability increases, investors can set a relatively high threshold return for a given market. However, for a market with a higher level of exchange rate risk, the threshold return must be lower to ensure that the optimal portfolio is available, and this is especially true for a relatively low failing probability. In addition, the fraction of the risk-free asset in the optimal international portfolio is affected by investment objectives, expected foreign exchange return, and exchange rate risk. In particular, the higher the exchange rate risk, the higher the proportion of total funds placed in the risk-free asset, given the failing probability and threshold return level.

[6] In the scenario of $(\alpha, H) = (0.05, -0.10)$, if $\sigma_e = 2\%$ and $E(r_e)$ changes between −4% and 4%, the optimal portfolio always exists. However, if $\sigma_e = 6\%$, then the optimal portfolio exists only if $E(r_e)$ is higher than a specific level. On the other hand, in the scenario of $(\alpha, H) = (0.10, -0.05)$, if $\sigma_e = 2\%$ and $E(r_e)$ changes between −3% and 4%, the optimal portfolio always exists. However, if $\sigma_e = 6\%$, then the optimal portfolio exists only if $E(r_e)$ is higher than a specific level.

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