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Locating emergency facilities with random demand for risk minimization

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ABSTRACT

Locating emergency service facilities is a challenging problem. Planners do not know specifically where emergencies will occur and, therefore, struggle to find a location that effectively ensures that the risk of poor service to any specific emergency is minimized. In this paper, we study the problem where locations of each demand point (emergency occurence) are random. Our objective is to minimize the expected maximum rectilinear distance from the facility to the demand points. This problem has practical importance in public sector as it aims to minimize the expected maximum risk when locating an emergency response facility. We start with a one dimensional problem and extend the results to the more complex two dimensional case. We present some properties of the problem along with examples for special cases. We propose a simulation approach to solving complex two dimensional cases and present simulation results for general cases to illustrate the problem and provide insight into solutions. We show that the simulation approach provides solutions very close to optimal for the linear case and suggest that it may provide valuable insight into the location selection system.

1. Introduction

The location of emergency service facilities has received considerable research attention given its societal importance. Improving the decision process for locating these facilities could be extremely valuable. These problems have historically been primarily modelled as a minimax facility location problem. The objective in the minimax format has been to locate a new facility such that the maximum distance to the areas requiring emergency services is minimized. The objective function is based on the objective that even the farthest and/or the least important client should get a reasonable level of service by attempting to sensure that any emergency is addressed in a reasonable time. There have been a number of different variations of the objective function applied to the problem. Research has also addressed the location of an emergency service facility on networks, on the plane, location on the restricted plane and others. Another variation is the case where demand points have different weights which can reflect different risk (either frequency or severity of an emergency) or different priority for response.

We propose an approach that explicitly addresses the uncertainty with respect to where an emergency will occur by minimizing the maximum risk through the minimization of expected maximum distance to emergencies. Uncertainty is a real issue for many facilities problems generally and for emergency facility location problems specifically. There has been some research into the case where some of the parameters and/or variables are uncertain. It is clear that the specific location of a crisis requiring emergency service cannot be known in advance. Explicitly accounting for that uncertainty may improve the performance of a practical application of a problem and, therefore, improve response outcomes when a facility is located.

Consider a simple example of locating a fire hall which will serve a number of residential neighborhoods and industrial and commercial areas. In some cases there may be overlap between neighborhoods and commercial areas. A fire or other emergency could occur anywhere within these regions. In the context of the model we introduce, the boundaries of the neighborhoods are the parameters of the probability distribution of where the emergency occurs in the neighborhood. We use a bivariate uniform for our work. Weights may reflect both the severity of a potential problem (higher risk industrial sites, for example) or the probability of a potential problem which can vary due to factors such as population density. This real life variability could be more robustly addressed by incorporating uncertainty into the model. We propose an approach to this problem that minimizes the expected maximum distance rather than minimizing the maximum expected distance. This distinction is important. Our approach minimizes the expected worst case in terms of emergency service. This minimizes the maximum risk. Previous approaches have considered the expected distances. This contribution provides an alternative approach to this real-life problem. This work is an extension of the work of Carbone and Mehrez (1980) and Mehrez and Stulman (1984). They introduce the simplest case in which a number of demand points with equal weights occupy the same probabilistic

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location. We consider locating a server facility on the plane to serve n demand regions or neighborhoods whose locations follow bivariate uniform distributions. We want to locate this facility minimax to this probabilistic demand points in the area to minimize the expected maximum distance.

The paper is laid out as follows. In Section 2, we provide an overview of the literature into facility location problems that incorporate uncertainty. Then we define our minimization problem on the line in Section 2.1. We explore two different cases of the problem and present detailed examples for illustration. The problem on the line is already very difficult to solve analytically. Therefore, in Section 3, we tackle the planar version of the problem using a simulation approach. The simulation approach, while not proven analytically optimal, provides excellent results. We also evaluate and report approximate Expected Value of Perfect Information (EVPI) values for different numbers. The EVPI approach gives us an upper bound on the improvement we can make as we better understand the risk in individual demand locations. Finally, Section 4 discusses key conclusions and outlines a number of future research directions.

2. Previous research on facility location under uncertainty

2.1. Planar facility location problems under uncertainty

Planar facility location problems under uncertainty have been studied under two categories. The first category deals with problems that contain random parameters which follow certain probabilistic distributions. For example the weights attached to demand points could be associated with a known probability distribution. Various objective functions are considered. Facility layout problems with random parameters are also included in this area although the research on this topic is limited. The second category, on the other hand, deals with so called robust facility location problems where distributions of the random parameters are unknown. This type of parameters are either represented by interval values or by parameter estimators. Objective functions are designed in such a way that a minimal change (robustness) in the objective function for a given change in the parameters is sought. In Table 1 we provide the cited research on planar facility location problems under uncertainty, classified by their main characteristics.

Wesolowsky (1977a) was one of the earliest papers that considered a facility location problem under uncertainty. The author considered a problem of locating a facility on a line in the presence of n demand points associated with probabilistic weights that follow the multivariate normal distribution. The Weber objective was considered. Because the problem is one-dimensional, the solution method is the same for both rectangular and euclidean distances. The probability of a facility being located on any point on the line is found. As in the result of the Hakimi property for the p-median problems, it is shown that only the locations at demand points have nonzero probabilities for the optimal location of the facility. The author also determined the EVPI (Expected Value of Perfect Information) for the problem. EVPI is defined as the difference in costs between the best location and the location found by using the expected values of the weights. Later, Drezner and Wesolowsky (1981) extended the work of Wesolowsky (1977a) by considering a similar problem on the plane with p-norm distances. The property found in Wesolowsky (1977a) is no longer valid for this general case. Which means that any point on the plane may have nonzero probability of a facility being located there.

In another study, Wesolowsky (1977b) proposed a solution to the single facility location problem with rectangular distances in which the locations of demand points have random coordinates that follow a bivariate normal distribution. It is shown that the objective function, which is the expected sum of the weighted rectilinear distances in x and y coordinates, is separable, and is thus not affected by correlation of demand point coordinates. Because the objective function is unimodal along each axis, the author proposed a rather easy method in which one can take the derivative of the objective function for each axis and apply an interval bisection method to find the values of coordinates that make the derivative zero.

There is a number of facility location papers that are using the minimax criterion. Carbone and Mehrez (1980) was the first that studied the problem of minimizing the expected maximum distances where the coordinates of the demand points $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ are identical, pairwise independent, and normally distributed random variables with mean 0 and variance 1. The authors showed that in this problem, the optimal location of the single facility is at the (0,0) point. Later, for the same problem, (Mehrez and Stulman, 1984) proposed a general statement that if the distance distribution between any demand point and a facility placed at coordinate (x,y) dominates the distribution of the

Table 1
Planar facility location problems under uncertainty.

Uncertain parameter	Underlying distribution	Objective function	Distance norm	Space	Study
Demand weights	Multivariate normal	Expected minisum	Rectilinear	One dimensional	Wesolowsky (1977a)
Location of demand points	Bivariate normal	Expected minisum	Rectilinear	Planar	Wesolowsky (1977b)
Location of demand points	Standard normal	Min expected maximum	Rectilinear	Planar	Carbone and Mehrez (1980)
Demand weights	Multivariate normal	Expected minisum	p-norm	Planar	Drezner and Wesolowsky (1981)
Location of demand points	Bivariate uniform	Max expected minimum	Rectilinear	One dimensional	Mehrez et al. (1983)
Location of demand points	Arbitrary	Min expected maximum	Arbitrary	Planar	Mehrez and Stulman (1984)
Demand weights	Triangular fuzzy	Maximin aspiration level	Euclidean	Planar	Bhattacharya and Tiwari (1994)
Existence of demand points	Binomial	Min expected maximum	Euclidean	Planar	Berman et al. (2003a)
Demand weights	Uniform	Minimax probability with threshold	Euclidean	Planar	Berman et al. (2003b)
Location of demand points	Uniform	Expected minimax	Euclidean	Planar	Foul (2006)
Product mix and product demand	Discrete	Expected minisum	Euclidean	Planar layout	Benjafaar and Sheikhzadeh (2000)
Demand weights	Unknown	Minimax regret (Robust)	Rectilinear	Planar	Carrizosa and Nickel (2003)
Demand weights and locations	Unknown	Minimax regret (Robust)	Rectilinear	Planar	Averbakh and Bereg (2005)
Demand weights	Arbitrary	Minimax probability with threshold	Arbitrary	Planar	Pelegrin et al. (2008)

distance between the same point and the facility placed at any other feasible coordinate, then (x,y) will be the dominating point hence provides an optimal solution to the problem. A necessary and sufficient condition for distribution $F(x)$ to dominate distribution $G(x)$ is that $F(x) \leqslant G(x)$ for all x. Only certain types of problems can be solved using this method under stringent assumptions, and actual values of objective functions may require extensive calculations. Berman, Drezner, and Wesolowsky (2003a) approached the same expected maximum objective from a different perspective. In their model, the problem is designed to minimize the expected maximum distances where for each demand point, there is a probability associated with its existence. In their words, the discussed model aims for minimizing expected 'undesirability'. The model also separates itself from the minimax model by using expectation to balance 'damage equity' (using information from all demand points in the optimum solution).

Berman, Wang, Drezner, and Wesolowsky (2003b) studied extensively a probabilistic version of the weighted minimax location problem on the plane where the weights of the demand points are uniformly distributed. The objective of their problem is to minimize the probability that the maximum distance to all demand points is greater than or equal to some pre-specified threshold value T. The authors proved that the problem is convex for certain parameters of the uniform distributions therefore it can be solved using standard optimization methods.

Foul (2006) studied a similar problem where the demand points have probabilistic locations that follow a bivariate uniform distribution. The best location for a facility is determined under the objective such that the maximum expected weighted distance to all probabilistic demand points is minimized.

And lately, Pelegrin, Fernandez, and Toth (2008) argued that the 1-center problem on the plane with probabilistic weights has only been studied for a number of specific probability distributions and distance measures. The authors proposed a general framework where weights are associated with arbitrary probability distributions and distances are measured by any distance norm. Two objective functions are evaluated. The first maximizes the covering probability for all demand points within a given threshold, while the second satisfies a minimum allowed coverage probability. Two algorithms that provide global optimal solutions are tested with different values of parameters and both found to be highly efficient.

Mehrez, Sinuanystern, and Stulman (1983) analyzed the problem of locating a facility on a line, in the presence of n hazardous points that have probabilistic locations. The objective is to maximize the expected minimum distance from these hazardous points. The authors showed that even for $n = 2$, acquiring an analytical result is cumbersome, therefore suggested a simulation model for solving the problem.

As a different approach to handle uncertainty, Bhattacharya and Tiwari (1994) presented a cost minimization model to locate multiple facilities on the plane where the cost per unit distances are not known precisely. Uncertainty in the cost is handled through the use of fuzzy numbers. The fuzzy model is tranformed into a crisp model by generating some aspiration levels (goals) by using different levels of the fuzzy numbers. A suitable solution is determined through finding a compromise solution which maximizes the minimum aspiration level.

When it comes to the probabilistic facility layout problems, the research is very limited. One of the important papers published in this area is Benjafaar and Sheikhzadeh (2000). The authors proposed a model for the design of plant layouts under uncertainty. It is argued in the paper that in manufacturing environments where product variety is high, using functional layouts causes inefficiency. Thus there is a need for probabilistic models that make the process more flexible and more efficient. The authors presented a probabilistic layout model for the design of plant layouts which considers random product mix and product demand. Demand for each product is represented by a finite discrete distribution where demands can be correlated or independent from each other. It is also considered that there might be a duplicate or duplicates of the same department in the same facility which is not possible in a job shop layout. Based on a combination of different products and demands, a set of scenarios with a probability of occurrence is considered. The authors used a heuristic approach first to find a minimum cost flow allocation between departments in a fixed layout, then to find a minimum cost layout with fixed flow allocation.

Robust models are used when uncertainty can not be defined by known probability distributions. Robust facility location problems differ from probabilistic location problems where the latter have uncertainty associated with some distribution functions with known parameters, but the former have uncertainty associated with no known distribution functions hence no known parameters. Because decisions are made in the presence of unknown parameters, and estimation of parameters need to be used, the researchers, in general, aim to find a minimax regret location in order to minimize the maximum loss. Research in the area is recent and mostly on the discrete facility location problems.

Carrizosa and Nickel (2003) considered the robust planar facility location problem when uncertainty is high in demand weights and only estimations of the weights are provided. The authors defined the robustness of the new facility location as the minimum increase in the weights needed to exceed a given threshold on cost. The objective is then to find a location that maximizes the robustness to get the most robust location.

In the case of rectilinear distances and uncertain weights and coordinates of demand points, the planar minimax regret location problems for both median and centre objectives are investigated recently by Averbakh and Bereg (2005). The authors proposed polynomial algorithms for 1-median and 1-centre problems. For a state-of-the-art literature review of facility location problems in general or under uncertainty, the reader is referred to ReVelle and Eiselt (2005) or Snyder (2006), respectively.

3. The problem on the line

We first consider the problem on the line. The problem can be represented as;

$$V(x) = \min_{x} E\left[\max_{1 \leqslant j \leqslant n}\{w_j|X_j - x|\}\right] \tag{1}$$

where $X_j j = 1,\ldots,n$ is a uniformly distributed random variable with parameters (a_j,b_j) and w_j is the weight for demand point j where $w_j > 0$. Clearly, since X_j is a random variable, $w_j|X_j - x|$ will also be a random variable. Let $f_{Z_j}(z)$ be the probability distribution function (pdf) and $F_{Z_j}(z)$ be the cumulative distribution function (cdf) of $Z_j = w_j|X_j - x|$. $f_{Z_j}(z)$ will be one of the followings depending upon the value of x.

If $x < a_j$, $f_{Z_j}(z)$ is uniformly distributed with parameters $(w_j(a_j - x), w_j(b_j - x))$. Then,

$$F_{Z_j}(z) = \left\{\begin{array}{ll} 0; & z < w_j(a_j - x) \\ \frac{z - w_j(a_j - x)}{w_j(b_j - a_j)}; & w_j(a_j - x) \leqslant z \leqslant w_j(b_j - x) \\ 1; & z > w_j(b_j - x) \end{array}\right\} \tag{2}$$

If $x > b_j$, $f_{Z_j}(z)$ is uniformly distributed with parameters $(w_j(x - b_j), w_j(x - a_j))$. Then,

$$F_{Z_j}(z) = \left\{\begin{array}{ll} 0; & z < w_j(x - b_j) \\ \frac{z - w_j(x - b_j)}{w_j(b_j - a_j)}; & w_j(x - b_j) \leqslant z \leqslant w_j(x - a_j) \\ 1; & z > w_j(x - a_j) \end{array}\right\} \tag{3}$$

If $a_j \leqslant x \leqslant b_j$ then,

$$F_{Z_j}(z) = P(w_j|X_j - x| \leqslant z) = \frac{\min\left(b_j - x, \frac{z}{w_j}\right) + \min\left(x - a_j, \frac{z}{w_j}\right)}{b_j - a_j} \quad (4)$$

which can also be represented as;

$$F_{Z_j}(z) = \begin{cases} \frac{2z}{w_j(b_j-a_j)}; & z < w_j \min\{b_j - x, x - a_j\} \\ \frac{z+\min\{b_j-x,x-a_j\}}{w_j(b_j-a_j)}; & w_j \min\{b_j - x, x - a_j\} \leqslant z \leqslant w_j \max\{b_j - x, x - a_j\} \\ 1; & z > w_j \max\{b_j - x, x - a_j\} \end{cases} \quad (5)$$

Therefore,

$$f_{Z_j}(z) = \begin{cases} \frac{2}{w_j(b_j-a_j)}; & z < w_j \min\{b_j - x, x - a_j\} \\ \frac{1}{w_j(b_j-a_j)}; & w_j \min\{b_j - x, x - a_j\} \leqslant z \leqslant w_j \max\{b_j - x, x - a_j\} \\ 0; & z > w_j \max\{b_j - x, x - a_j\} \end{cases} \quad (6)$$

which is a general distribution with a histogram that looks like a combination of two uniform distributions. Without loss of generality, we call this distribution as 'two-bin uniform distribution'.

We illustrate this distribution using following example. Consider a demand point with a unit weight located on a line whose location follows a uniform distribution with the parameters $U(2,5)$. For a given $x = 4$, the expected value for the two-bin uniform distribution (See Fig. 1) will be;

$$E_{Z_1}[Z] = \int_0^1 \frac{2}{3} z dz + \int_1^2 \frac{1}{3} z dz = \frac{5}{6} \quad (7)$$

This is also equal to the expected maximum distance for a given x, because we consider only a single demand point. If there is more than one point, we need to first find its cdf and then pdf in order to find the expected value of the maximum distribution. Let $F_{\max}(z)$ be the cdf for the maximum distribution. Assuming that the distributions of X_j's are independent, we can write the following;

$$F_{\max}(z) = P(Max \leqslant z) = P(Z_1 \leqslant z, Z_2 \leqslant z, \ldots, Z_n \leqslant z) = \prod_{j=1}^{n} P(w_j|X_j - x| \leqslant z) \quad (8)$$

and then the pdf will be:

$$f_{\max}(z) = \frac{\partial}{\partial z} F_{\max}(z) \quad (9)$$

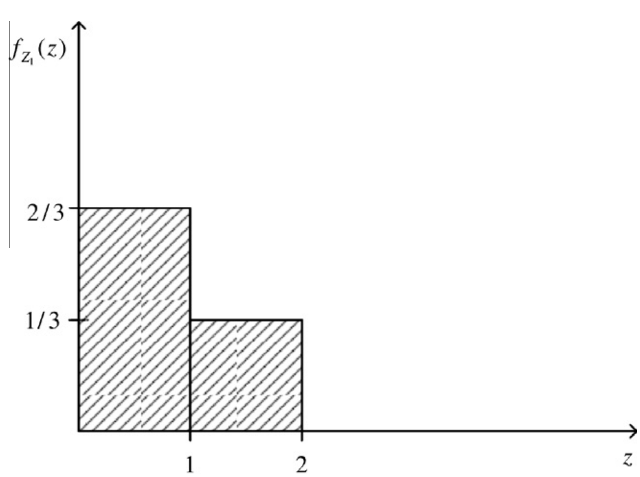

Fig. 1. Probability density function of Z_1 for a given x.

3.1. Case 1: Demand points location distributed uniformly over same range and weights equal 1

If there are n demand points located on a line with uniformly distributed x-coordinates with the same parameters $X_j \sim U(a,b)$ and $w_j = 1, j = 1, \ldots, n$, then the optimal $x = x^*$ value for minimizing the expected maximum weighted distance will be in the range $a \leqslant x \leqslant b$. And we can write the cdf and pdf of the maximum distribution as;

$$F_{Max}(z) = \frac{(\min(b - x, z) + \min(x - a, z))^n}{(b - a)^n} \quad (10)$$

When $x < (b - a)/2$ then $b - x > x - a$ which means that the pdf for the maximum distribution will be;

$$f_{Max}(z) = \begin{cases} \frac{n2^n z^{n-1}}{(b-a)^n}; & z < x - a \\ \frac{n(x-a+z)^{n-1}}{(b-a)^n}; & x - a \leqslant z \leqslant b - x \\ 0; & z > b - x \end{cases} \quad (11)$$

Otherwise;

$$f_{Max}(z) = \begin{cases} \frac{n2^n z^{n-1}}{(b-a)^n}; & z < b - x \\ \frac{n(b-x+z)^{n-1}}{(b-a)^n}; & b - x \leqslant z \leqslant x - a \\ 0; & z > x - a \end{cases} \quad (12)$$

And then the Expected Value of this maximum distribution will be as follows.

$$E_{Max}[Z] = \begin{cases} \frac{2^n(x-a)^{n+1} + (-(n+1)x + a + bn)(b-a)^n}{(n+1)(b-a)^n}; & x < (b-a)/2 \\ \frac{2^n(b-x)^{n+1} + ((n+1)x - an - b)(b-a)^n}{(n+1)(b-a)^n}; & x \geqslant (b-a)/2 \end{cases} \quad (13)$$

We need to find the x that minimizes this expected value. The following lemma will help finding the x.

Lemma 1. *The Expected Value function is a piecewise non-linear convex function of x when the underlying distribution is uniform and a < x < b. Also the function has a minimum (not necessarily unique) at (b − a)/2.*

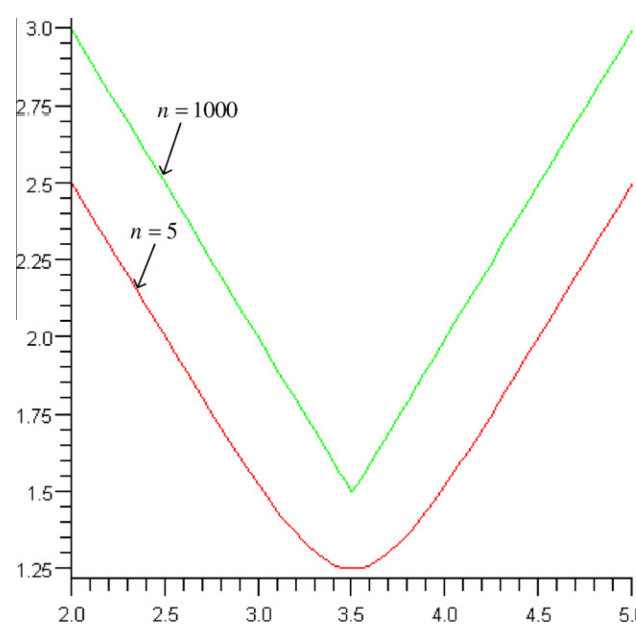

Fig. 2. Expected maximum function with $n = 5$ and $n = 1000$.

Proof. When $a \leqslant x \leqslant (b-a)/2$, $E_{Max}[Z]$ is a non-increasing function of x. And when $(b-a)/2 \leqslant x \leqslant b$, $E_{Max}[Z]$ is a non-decreasing function of x. Therefore, there exists a minimum at their intersection where $x^* = (b-a)/2$. Also the second derivatives for the functions are positive in this range which proves that the function is convex. □

We demonstrate this result by considering another example with 5 demand points located on a line with random x coordinates following a uniform distribution with the same parameters $U(2,5)$ and $w_j = 1$. Then the expected value function will take following form as in the Fig. 2. If we increase the number of demand points to 1000 then the piecewise nonlinear function of x for the expected value of maximum distance gets a form close to a piecewise linear function with the same minimizer x^*.

3.2. Case 2: Demand points location distributed uniformly over individual ranges and weights equal 1

We now assume now that the probabilistic demand points on the line have equal weights but different location parameters. In order to find an optimal point that minimizes the expected maximum distance, we make use of the property from the deterministic minimax problem in the case that none of the extreme demand points overlap with the other demand points. In other words, if there exist extreme demand points that dominate the other demand points, it is enough to consider only these demand points. This is due to the fact that for the deterministic minimax problem, when all demand point weights are equal to each other, we know that only the extreme points play role in finding the minimax point. This dominance result is proven by (Mehrez and Stulman, 1984) for general distributions. However, unfortunately, when overlapping occurs, we cannot consider the non-overlapping extreme parts as truncated extreme distributions that dominate other demand points. Therefore, we need to determine the maximum distribution and its expected maximum distance function of x for a given problem which is a tedious job even for a small n. This expected maximum distance function overall in not convex and, therefore, line search techniques will only provide a local optimum value. We suggest a line partitioning procedure that divides the x-axis into smaller ranges denoted by Ω_k, in which the distribution of each Z_j will be the same hence the distribution of the maximum. In the worst case, the number of ranges is bounded by $O(2n(2n-1)(n-1)) = O(n^3)$ where n is the number of demand points.

As in the deterministic minimax location problem, optimal location of the new facility will be in the convex hull (a line in this one dimensional case) of the demand points. We illlustrate the solution procedure in another example.

We now consider 2 demand points located on a line with random x coordinates following a uniform distribution with the parameters provided in Table 2. We want to find a point that minimizes the expected maximum distance to these two points.

In order to determine the distribution of each distance for a given x value, we divide the x-axis into three main ranges divided by dark lines as illustrated in the Fig. 3. For example, in the first range $(0 \leqslant x < 2)$, f_{Z_2} will follow the 'two-bin distribution' whereas f_{Z_1} will follow the uniform distribution. In order to find unique a $F_{Max}(z)$ which does not change with the value of x, we need to further divide this range into two sub-ranges (divided by light lines). Total ranges add up to seven for this small example.

Table 2
Parameters of the demand points.

j	a_j	b_j	w_j
1	2	8	1
2	0	4	1

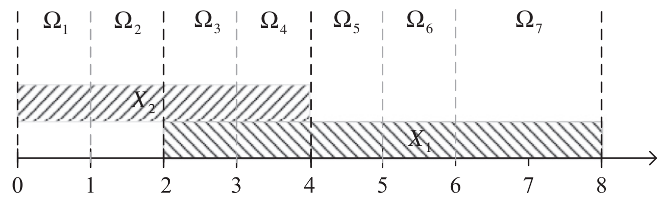

Fig. 3. Partitioning of x-axis.

To illustrate the idea, let us consider the first range that contains two sub-ranges Ω_1, and Ω_2. F_{Z_1} and F_{Z_2} for the first range are as follows;

$$F_{Z_1}(z) = \begin{Bmatrix} \frac{z}{3}; & z < (2-x) \\ \frac{z+x-2}{6}; & (2-x) \leqslant z < (8-x) \\ 1; & z \geqslant (8-x) \end{Bmatrix} \tag{14}$$

$$F_{Z_2}(z) = \begin{Bmatrix} \frac{z}{2}; & z < (4-x) \\ \frac{z-x+4}{4}; & (4-x) \leqslant z \leqslant x \\ 1; & z \geqslant x \end{Bmatrix} \tag{15}$$

Then the cdf for the maximum distribution will be,

$$F_{Max}(z) = F_{Z_1}(z)F_{Z_2}(z). \tag{16}$$

To determine $F_{Max}(z)$ in this range, the functions x, $(4-x)$, $(2-x)$, $(8-x)$ have to be sorted from the smallest to the largest when x takes values between 0 and 2. Unfortunately, these functions do not have the same ordering in the whole range $0 \leqslant x \leqslant 2$ (See Fig. 4).

Therefore, we further divide this range into two as Ω_1 $(0 \leqslant x < 1)$, and Ω_2 $(1 \leqslant x < 2)$.

After finding the cdf of maximum distributions for each range, we find the expected value for each range by,

$$\underset{x\in\Omega_k}{E}[Z_{\max}] = \int_0^\infty z\left(\frac{\partial}{\partial z}F_{\max}(z)\right)dz \tag{17}$$

The calculation of expected values is an intensive process that is done using Maple 10. Table 3 provides the expected values for the all ranges.

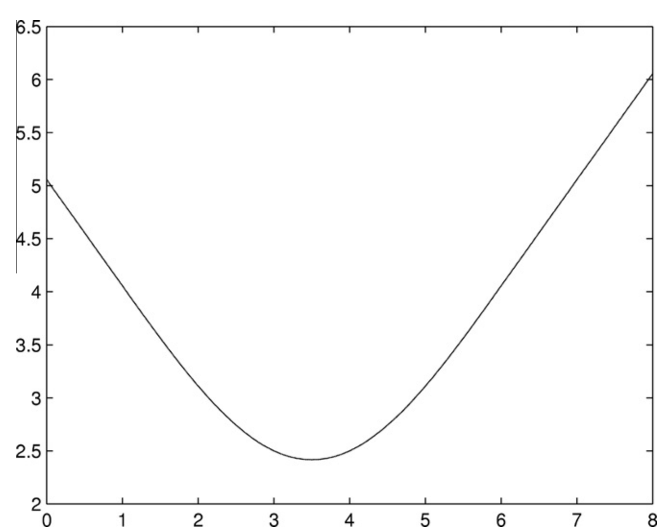

Fig. 4. Comparison of break point functions.

Table 3
Expected maximum functions for example 3.

Range	Expected maximum function
Ω_1	$5.05 - x$
Ω_2	$5.00 - .83x - .17x^2 + 0.056x^3$
Ω_3	$5.00 - .83x - .17x^2 + 0.056x^3$
Ω_4	$6.50 - 2.33x + .33x^2$
Ω_5	$9.17 - 4.33x + .83x^2 - 0.04x^3$
Ω_6	$10.05 - 5x + x^2 - 0.056x^3$
Ω_7	$-1.94 + x$

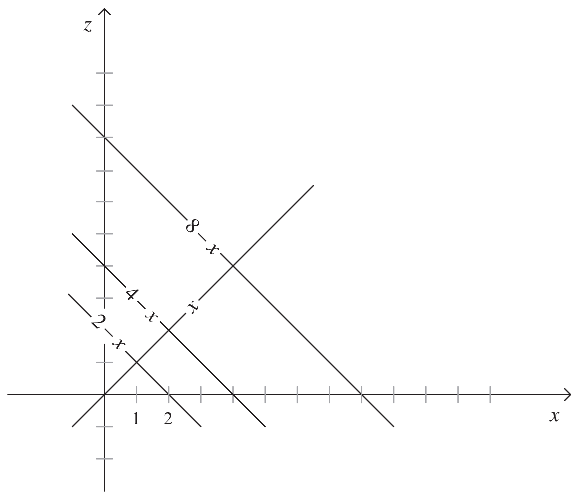

Fig. 5. The Objective function for example 3.

This piecewise nonlinear function has a unique minimum at $x^* = 3.5$. The objective value at this point is 2.417. Fig. 5 illustrates the expected value function where $0 \leqslant x \leqslant 8$.

Now we consider the question of how much we would gain if we follow this wait-and-see approach. That is, we wait to observe the actual location of the demand points and then decide on the facility location. That value lets us calculate EVPI which is the difference in the expected cost of the best location (ECBL) without knowing the exact locations of the demand points minus the expected cost with perfect information (ECPI) about the location of the demand points. Therefore,

$$EVPI = ECBL - ECPI \tag{18}$$

where,

$$ECBL = \min_x E\left[\max_j(|x - x_j|)\right], \tag{19}$$

and

$$ECPI = E\left[\min_x\left(\max_j(|x - x_j|)\right)\right]. \tag{20}$$

Clearly, $ECBL = 2.417$ which is the objective value of the expected maximum function. In order to find the value of ECPI we will use the closed form expression given by Elzinga and Hearn (1972) for the objective of the deterministic minimax location problem, S^*:

$$S^* = \left(\max_{1\leqslant j\leqslant n} x_j - \min_{1\leqslant j\leqslant n} x_j\right)/2.$$

Then, we can write $ECPI$ as,

$$ECPI = E[S^*] = \left(E\left[\max_{1\leqslant j\leqslant n} x_j\right] - E\left[\min_{1\leqslant j\leqslant n} x_j\right]\right)/2.$$

The exact value of $ECPI$ can not be easily found even for a small number demand points since it involves in determining difference of the expected values of the extreme (minimum and maximum) distributions of a number of uniformly distributed random variables. For this small example, we have found it using Maple 10 as $(91/18 - 35/18)/2 = 1.55$.

Therefore $EVPI = 2.417 - 1.55 = 0.912$.

4. An approach to the problem on the plane

It is clear that, even for a small number of demand points, there are significant difficulties in finding a solution for the problem on the line analytically. The problem on the plane will be significantly more difficult. Therefore,we employ a simulation approach to find a good solution for the problems on the plane. Our simulation approach is twofold. First, we used a limited version of MS Excel add-in called RISKOptimizer[1] developed by Palisade decision tools company to determine the optimal value of ECBL. RISKOptimizer has been used to solve a variety of optimization problems in a wide range of industries including financial institutions, airlines and manufacturing companies. The tool combines the Monte Carlo simulation techniques with genetic algorithm heuristic for approximate optimization of mathematical models with random variables. It generates random values in adjustable cells for random variables, runs a Monte Carlo simulation, and finds the combination of values that provides the optimal simulation results. Second we determine the value of ECPI through another Palisade tool @Risk.[2] @Risk performs risk analysis using Monte Carlo simulation by generating possible outcomes on a spreadsheet and gives details about how likely they are going to occur by providing confidence intervals (CI) on them. The main difference between RISKOptimizer and @ Risk is that the former contains an optimization process but the later is a straight simulation which simply generates random values according to their distribution parameters.

We first validated the performance of RISKOptimizer and @Risk by comparing their results with the analytical results of Example 3. A screenshot showing the model to find the ECBL using RISKOptimizer along with the model settings is shown in Fig. 6. Fig. 7 is a screenshot providing the @Risk result to determine the ECPI.

It can be seen from the figures, the best value found for ECBL (2.4178) is very close to the true value (2.417) as the difference is almost zero. Also the best value found for ECPI (1.5541) is almost the same as the true value (1.5555).

We can conclude that, for practical purposes, simulation approach for emergency facility location problems is a valid heuristic method that does not require heavy computations and can provide very good results. Therefore, we now can perform simulation runs for larger size planar problems using RISKOptimizer and compare their results with the wait and see approach done again by simulation using @Risk. The distribution parameters given in Table 4 are used to generate the demand point locations used in the computational work. We first generate random numbers a_i, b_i, c_i, and d_i and then use these random numbers to generate demand point locations (x_i, y_i). We also choose to have equal weights for the demand points for the sake of simplicity, but this is not necessary and having different demand weights does not affect the simulation approach at all.

The results of the analysis of the problem on the plane are presented in Table 5. It can be seen from Table 5 that there is a decrease in $EVPI$ with the increase in the number of demand points. It is also clear that in the case of an emergency facility, once the specific location of a 'demand' is known, the demand has passed.

1 http://www.palisade.com/RISKoptimizer/.
2 http://www.palisade.com/risk/.

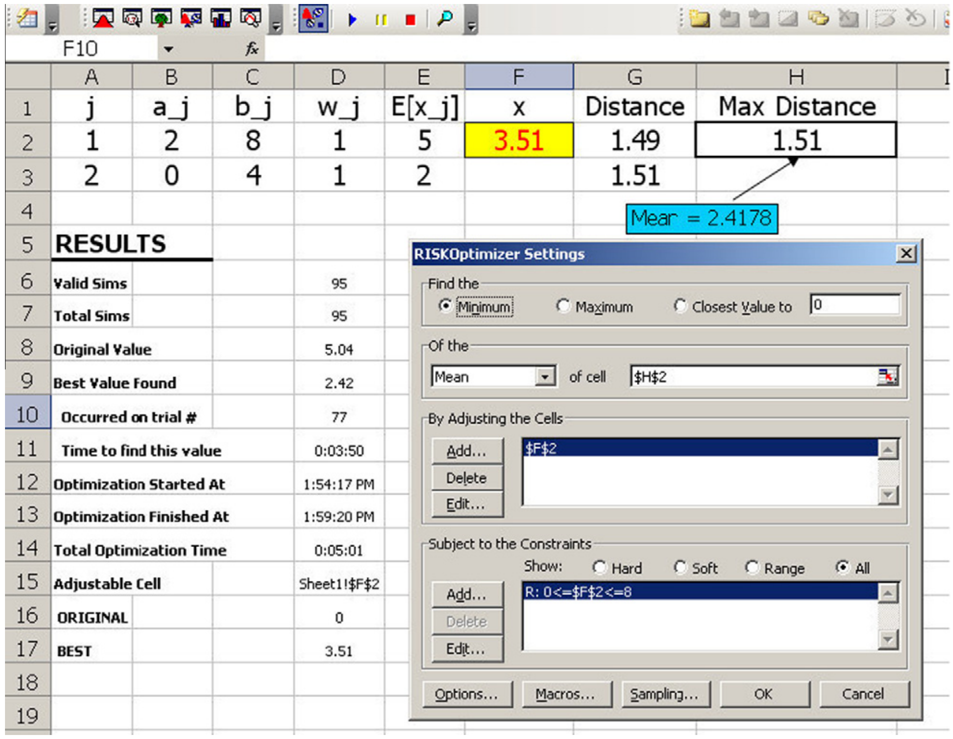

Fig. 6. A Screenshot for RISKOptimizer for Finding ECBL.

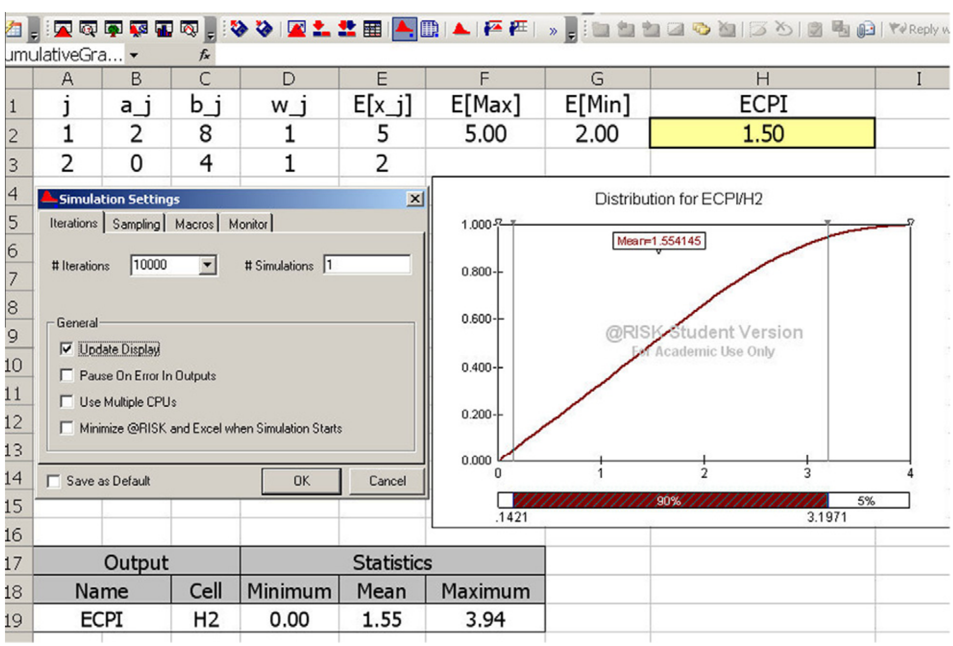

Fig. 7. A screenshot for @Risk for finding ECPI.

Table 4
Parameters for demand points.

a_i	b_i	c_i	d_i	x_i	y_i	w_i
$U(0,0.5)$	$U(0.5,1)$	$U(0,0.5)$	$U(0.5,1)$	$U(a_i,b_i)$	$U(c_i,d_i)$	1

Notwithstanding the political difficulty in waiting to see where emergencies will occur, the wait and see approach is clearly not practical. It may make more sense, in the context of this problem, to consider the parameters of the uniform distribution as the boundaries of neighborhoods to be served by the emergency facility rather than a distribution that a single emergency will be drawn from. That approach moots the 'wait and see' approach as it really only evaluates a single occurrence (one specific emergency) rather than how best to minimize the risk for the specific neighborhood, particularly as the neighborhoods are bigger and, as a result, the n is smaller. This is highlighted in the larger gaps between the simulation solution and the wait and see solution when n is smaller.

Table 5
Simulation Results.

n	Probabilistic objective	Wait and see objective	Approximate EVPI	% Approximate EVPI
5	0.512	0.391	0.121	30.95
10	0.537	0.455	0.082	18.02
15	0.613	0.531	0.082	15.44
20	0.621	0.546	0.075	13.74
25	0.629	0.554	0.075	13.54
30	0.647	0.571	0.076	13.31
35	0.656	0.581	0.075	12.91
40	0.669	0.595	0.074	12.44
45	0.681	0.61	0.071	11.64
50	0.682	0.616	0.066	10.71

In this case, the simulation approach, in the absence of a manageable analytical approach could provide solutions with less risk.

5. Conclusions

We present a risk minimization model as an approach to facility location under uncertainty. This could be implemented in such applications as the location of emergency response facilities. We first present the problem on the line generally and in some specific cases and outline the characteristics of the problem and approaches to solving the problem. The problem is very difficult to solve analytically except for the smallest cases on a line. The problem on the plane is, therefore, extremely difficult to solve analytically. We suggest a simulation approach to solving the problem on the plane. We also evaluate the expected value of perfect information by solving the problem after a 'wait and see' approach which solves the problem once the exact location of the random points are known. Once a single demand point is manifest, the next one is most likely to be somewhere else. Given the nature of the problem on the plane, and that each emergency or demand point is not known in advance, using the simulation approach which explicitly allows for the emergency or demand to appear anywhere in the defined 'neighborhood' will provide a better and lower risk solution to the problem.

We believe there are a number of opportunities to extend this research. It would be interesting to undertake one or more case studies to evaluate other distributions for specific neighborhoods. It could also be interest to evaluate a combination of neighborhoods and specific demand points. As an example, a large high risk industrial facility could be modeled as a fixed demand point with a risk weight while a residential or lower risk commercial or industrial neighborhood could be represented by a distribution. It may also be interesting to treat the weights as a random variable. We recall that the weights are a measure of the relative risk of emergency within different neighborhoods. That relative risk could reflect either the frequency or severity of the emergencies within a neighborhood, both of which clearly could have some uncertainty.

This problem may also be well suited to either nonparametric analysis or the application of fuzzy analysis. The nonparametric analysis could provide for sensitivity analysis and a more robust location decision given uncertainty relative to the specifics of the risk associated with different neighborhoods. The application of fuzzy parameters instead of random variables has the potential to simplify the problem and allow for the consideration of extensions such as constraints on the location of the facility (forbidden regions).

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