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Short Communication

# Identifying the anchor points in DEA using sensitivity analysis in linear programming

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ABSTRACT

In this paper, the anchor points in DEA, as an important subset of the set of extreme efficient points of the production possibility set (PPS), are studied. A basic definition, utilizing the multiplier DEA models, is given. Then, two theorems are proved which provide necessary and sufficient conditions for characterization of these points. The main results of the paper lead to a new interesting connection between DEA and sensitivity analysis in linear programming theory. By utilizing the established theoretical results, a successful procedure for identification of the anchor points is presented.



## 1. Introduction

Nowadays, Data Envelopment Analysis (DEA) is one of the most popular tools for efficiency analysis in multiple input-multiple output framework of the production theory (see Cooper, Seiford, & Tone, 2007; Emrouznejad, Parker, & Tavares, 2008). As can be seen in the DEA literature, the anchor points play a vital role in DEA theory and applications. Thanassoulis and Allen (1998) used the concept of these points, at first, for the generation of unobserved DMUs and extending the DEA frontier. Bougnol and Dulá (2009) defined these points formally as production possibilities which give the transition from the Pareto-efficient frontier to the free-disposability portion of the boundary of the production possibility set (PPS). Rouse (2004) utilized this notion for identifying prices for health care services. Bougnol and Dulá (2009) used the geometrical properties of the anchor points to design and test an algorithm for their identification under variable returns to scale (VRS) assumption of the production technology. In a recently published paper, Thanassoulis, Kortelainen, and Allen (2011) provided another method for identifying the anchor points based upon the radial efficiency scores and slack variables at the optimal solution of envelopment models. They have used this concept for improving envelopment under multiple inputs and outputs in a VRS technology. See also Bougnol (2001) and Allen and Thanassoulis (2004) for more details about the notion and applications of the anchor points.

Since the set of anchor points is a subset of the set of extreme efficient points, the first step for obtaining the anchor points is obtaining the extreme efficient points. There are different algorithms to do this. See Charnes, Cooper, and Thrall (1991) and Dulá and López (2006) among others.

In this paper, a basic definition of anchor points, based upon the optimal solutions of the multiplier DEA models and the supporting hyperplanes of the PPS, is given. This definition is close to that given by Bougnol and Dulá (2009). Afterwards, it is proved that a DMU under consideration is an anchor point if and only if by increasing some input or decreasing some output the new unobserved point is located on the boundary of the production possibility set. Due to this result, we obtain a characterization of the anchor points using input-oriented and output-oriented models. Then, utilizing the established theoretical results, an approach for identification of the anchor points is introduced. The presented approach follows the problem from a different standpoint, and determines the anchor points using some sensitivity analysis techniques, while the existing methods do this by obtaining all extreme efficient points of some polyhedrals or by resolving LP problems. It is interesting from a theoretical point of view too, because it provides a new connection between the DEA and sensitivity analysis in linear programming.

The rest of the paper unfolds as follows: Section 2 contains some preliminaries. In Section 3, some basic theoretical results are addressed. In Section 4, sensitivity analysis in linear programming theory is utilized to provide a new procedure for identification of the anchor points. Section 5 is devoted to conclusions; and the proofs of the main results are given in Appendix A of the

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paper. Appendix B provides two initial simplex tableaus for two studied LP problems.

## 2. Preliminaries

Suppose that we have a set of $n$ peer DMUs, $\{DMU_j,\ j = 1,\ldots,n\}$, such that each $DMU_j$ produces multiple outputs $y_{rj}$ $(r = 1,\ldots,s)$ utilizing multiple inputs $x_{ij}$ $(i = 1,\ldots,m)$. We assume that $x_{ij} > 0$ (for all $i,j$) and $y_{rj} > 0$ (for all $r,j$). Also, we assume that there is not any duplicated DMU. Furthermore, let $x_j = (x_{1j},\ldots,x_{mj})^T$ and $y_j = (y_{1j},\ldots,y_{sj})^T$.

Considering $DMU_o$ as the unit under assessment, the BCC efficiency measure of $DMU_o$, in input orientation is obtained by solving the following model. Note that, $DMU_o$ is an observed DMU, i.e., $o \in \{1,\ldots,n\}$.

$$\begin{aligned} \theta_o^{BCC} = \min \quad & \theta, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j \leqslant \theta x_o, \\ & \sum_{j=1}^{n} \lambda_j y_j \geqslant y_o, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geqslant 0, \quad \forall j. \end{aligned} \tag{1}$$

The above model is called the envelopment form of the BCC model. In the above model, $(\lambda,\theta)$ is the decision variables vector. The vector $\lambda$ is named the intensity vector and the optimal value of $\theta$, denoted by $\theta_o^{BCC}$, exhibits the “input-oriented BCC-efficiency score” of the unit under assessment.

Considering $(u, v, u_0)$ as the vector of dual variables, the dual of the above model, which is called the multiplier form of the BCC model, is as follows:

$$\begin{aligned} \theta_o^{BCC} = \max \quad & u y_o + u_o, \\ \text{s.t.} \quad & v x_o = 1, \\ & u y_j - v x_j + u_o \leqslant 0, \quad \forall j, \\ & (u, v) \geqslant 0, \\ & u_o \text{ free in sign.} \end{aligned} \tag{2}$$

The $DMU_o$ is called input-oriented BCC-weakly efficient (radially efficient), if $\theta_o^{BCC} = 1$. It is worth mentioning that $\theta_o^{BCC} \in (0, 1]$.

The BCC output-oriented models in envelopment and multiplier forms can be expressed as follows, respectively:

$$\begin{aligned} \varphi_o^{BCC} = \max \quad & \varphi_o, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j \leqslant x_o, \\ & \sum_{j=1}^{n} \lambda_j y_j \geqslant \varphi_o y_o, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geqslant 0, \quad \forall j, \end{aligned} \tag{3}$$

$$\begin{aligned} \varphi_o^{BCC} = \min \quad & v x_o - u_o, \\ \text{s.t.} \quad & u y_o = 1, \\ & u y_j - v x_j + u_o \leqslant 0, \quad \forall j, \\ & u \geqslant 0, \\ & v \geqslant 0, \\ & u_o \text{ free in sign.} \end{aligned} \tag{4}$$

In the above two output-oriented models, $\frac{1}{\varphi_o^{BCC}}$ is called the “output-oriented BCC-efficiency score” of the unit under assessment. Here, $\varphi_o^{BCC} \geqslant 1$, and so the output-oriented BCC-efficiency score belongs to $(0, 1]$. Developing an example demonstrating $\frac{1}{\varphi_o^{BCC}} \neq \theta_o^{BCC}$ is not difficult.

The $DMU_o$ is called output-oriented BCC-weakly efficient (radially efficient), if $\varphi_o^{BCC} = 1$.

The definition below introduces the notion of extreme efficient DMUs. Notice that in the present paper all input–output values of the observed DMUs are positive.

**Definition 1.** The $DMU_o(x_o, y_o)$, is called an extreme efficient DMU, if it is an extreme point of the production possibility set (PPS). Here, we consider the PPS under variable returns to scale assumption of technology, denoted by $T_v$, which is expressed as follows:

$$T_v = \left\{ (x,y) : \sum_{j=1}^{n} \lambda_j x_j \leqslant x, \quad \sum_{j=1}^{n} \lambda_j y_j \geqslant y \geqslant 0, \quad \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geqslant 0, \quad j = 1,\ldots,n \right\}.$$

Hereafter, we denote the set of extreme efficient points of $T_v$ by $E$.

## 3. Some basic results

As can be seen in the DEA literature (see Allen & Thanassoulis, 2004; Bougnol, 2001; Bougnol & Dulá, 2009; Rouse, 2004; Thanassoulis et al., 2011), the anchor points play a vital role in DEA theory and application. These points delineate the Pareto-efficient frontier of the PPS ($T_v$) from the free-disposability portion of the boundary. The anchor points are usually production points with small or big size of input–output factors. These points are far from the central part of the efficiency frontier. As can be seen from Theorem 1 of the present paper and the results given by Bougnol and Dulá (2009), the role of some input–output factors in efficiency situation of these units is not considerable.

Anchor points were first used by Thanassoulis and Allen (1998). More developments about the anchor points from both conceptual and applied points of view have been done by Bougnol (2001), Rouse (2004), Bougnol and Dulá (2009), and Thanassoulis et al. (2011). They used the properties of the anchor points to design and test some algorithms for their identification in $T_v$. In this section, we address some basic results for identifying the anchor points of $T_v$. The first method, which results from some notions and corollaries given by Bougnol and Dulá (2009), works based upon the multiplier models.

Since $T_v$ is a polyhedral in $\mathbb{R}^{m+s}$, the hyperplanes supporting to this set can be expressed as $H_{(u,v,u_o)} = \{(x,y) \mid uy - vx + u_o = 0\}$, in which $(u, v) \neq 0$ is the normal vector of the hyperplane, and $-u_o$ is its level value. Recall that, the hyperplane $H_{(u,v,u_o)}$ supports $T_v$ at $(\bar{x},\bar{y}) \in T_v$ if $u\bar{y} - v\bar{x} + u_o = 0$ and $uy - vx + u_o \leqslant 0$ for each $(x,y) \in T_v$.

The following definition introduces the concept of anchor points in $T_v$. In this definition, $E$ denotes the set of extreme efficient DMUs. In fact, this property comes from Result 1 in Bougnol and Dulá (2009).

**Definition 2.** (Bougnol & Dulá, 2009). Let $DMU_o = (x_o, y_o)$ be the unit under consideration. $DMU_o$ is called an anchor point if $(x_o, y_o) \in E$ and it is located on a supporting hyperplane of $T_v$, say $H_{(u,v,u_o)}$, such that at least one component of the normal vector $(u, v)$ is zero.

It is known that $H_{(u,v,u_0)}$ is a hyperplane supporting to $T_v$ at $(x_o, y_o) \in E$ if and only if a positive multiplier of $(u, v, u_0)$ is an optimal solution to Model (2), see Cooper et al. (2007). Based upon this fact and the above definition (Result 1 in Bougnol & Dulá, 2009), the following optimization problems can be used for testing whether $DMU_o = (x_o, y_o) \in E$ is an anchor point or not:

$$u_k^* = \min \quad u_k, \tag{5}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leqslant 0, \quad \forall j \neq o, \tag{5.1}$$

$$\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o = 0, \tag{5.2}$$

$$\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, \tag{5.3}$$

$$u_r \geqslant 0, \quad r = 1, \ldots, s, \tag{5.4}$$

$$v_i \geqslant 0, \quad i = 1, \ldots, m, \tag{5.5}$$

for $k \in \{1, \ldots, s\}$; and

$$v_t^* = \min \quad v_t, \tag{6}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leqslant 0, \quad \forall j \neq o, \tag{6.1}$$

$$\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o = 0, \tag{6.2}$$

$$\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, \tag{6.3}$$

$$u_r \geqslant 0, \quad r = 1, \ldots, s, \tag{6.4}$$

$$v_i \geqslant 0, \quad i = 1, \ldots, m, \tag{6.5}$$

for $t \in \{1, \ldots, m\}$. Now, defining $\chi_o = \min\{u_1^*, \ldots, u_s^*, v_1^*, \ldots, v_m^*\}$, we have the following central result. The proof of this theorem is given in Appendix A.

**Theorem 1.** *$DMU_o \in E$ is an anchor point if and only if $\chi_o = 0$.*

For obtaining $\chi_o$, solving $m + s$ LP problems is required, which can be computationally expensive. But, it is worthwhile to note that when the optimal objective value corresponding to one of the above LP models becomes zero, then $DMU_o \in E$ is an anchor point and it is not necessary to solve other linear programs. Although the above mentioned approach may not be suitable computationally, this method gives a (weak) hyperplane which makes $DMU_o$ anchor point. In the next section, a new approach is given which works invoking the sensitivity analysis techniques.

## 4. Identifying anchor points utilizing sensitivity analysis

In this section, we utilize the sensitivity analysis techniques in linear programming theory to identify the anchor points. It has advantages from an applied point of view. Furthermore, it is interesting theoretically, because provides a new link between linear programming theory and performance analysis (DEA).

The following theorem provides a characterization of anchor points based upon the weak-efficiency of $(x_o + e_i, y_o)$ or $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ in output-orientation or input-orientation, respectively. This result helps us in providing the sensitivity analysis techniques in sequel. The proof of Theorem 2 is given in Appendix A.

**Theorem 2.** *$DMU_o = (x_o, y_o) \in E$ is an anchor point if and only if one of the following conditions holds:*

*(i) $(x_o + e_i, y_o)$ is output-oriented BCC-weakly efficient, for some $i \in \{1, \ldots, m\}$,*
*(ii) $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ is input-oriented BCC-weakly efficient, for some $r \in \{1, \ldots, s\}$.*

Regarding the above theorem, for checking that "whether $DMU_o = (x_o, y_o)$ is an anchor point or not" we should check two conditions (i) and (ii). For surveying conditions (i) and (ii), evaluating $(x_o + e_i, y_o)$ and $(x_o, y_o - \frac{y_{rto}}{2} e_r)$ for all $i, r$ may be required. To do this, Models (1) and (3) assessing $(x_o + e_i, y_o)$ and $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ should be solved. But, for reducing the computational requirements, we do not solve Models (1) and (3) from scratch. We use sensitivity analysis techniques in linear programming theory to do this.

Consider LP problem (3) evaluating $(x_o, y_o)$. Suppose that the Simplex method has produced an optimal basis $B$ by solving this LP. We shall describe how to make use of optimality conditions (primal–dual relationships) for finding the output-oriented efficiency measure of $(x_o + e_k, y_o)$ without solving the following LP problem from scratch:

$$\varphi_k^+ = \max \quad \varphi, \tag{7}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^+ = x_{io}, \quad \forall i \neq k, \tag{7.1}$$

$$\sum_{j=1}^{n} \lambda_j x_{kj} + s_k^+ = x_{ko} + 1, \tag{7.2}$$

$$-\sum_{j=1}^{n} \lambda_j y_{rj} + s_r^- + \varphi y_o = 0, \quad \forall r, \tag{7.3}$$

$$\sum_{j=1}^{n} \lambda_j = 1, \tag{7.4}$$

$$s_i^+, s_r^-, \lambda_j \geqslant 0, \quad \forall i, j, r. \tag{7.5}$$

The initial simplex tableau for Problem (7) can be seen in Appendix B (Table 1).

Comparing Models (3) and (7), it is seen that Model (7) is gotten by replacing the right-hand-side vector $\begin{pmatrix} x_o \\ 0 \\ 1 \end{pmatrix}$ in Model (3) by $\begin{pmatrix} x_o + e_k \\ 0 \\ 1 \end{pmatrix}$. Therefore, denoting the simplex right-hand-side of models (3) and (7) by $\bar{b}$ and $\bar{b}^{new}$, respectively, $\bar{b} = B^{-1} \begin{pmatrix} x_o \\ 0 \\ 1 \end{pmatrix}$ will be replaced by $\bar{b}^{new} = B^{-1} \begin{pmatrix} x_o + e_k \\ 0 \\ 1 \end{pmatrix}$. The new simplex right-hand-side can be calculated without explicitly evaluating $B^{-1} \begin{pmatrix} x_o + e_k \\ 0 \\ 1 \end{pmatrix}$. This is evident by noting that

$$\bar{b}^{new} = B^{-1} \begin{pmatrix} x_o + e_k \\ 0 \\ 1 \end{pmatrix} = B^{-1} \begin{pmatrix} x_o \\ 0 \\ 1 \end{pmatrix} + B^{-1} \begin{pmatrix} e_k \\ 0 \\ 0 \end{pmatrix} = B^{-1} \begin{pmatrix} x_o \\ 0 \\ 1 \end{pmatrix} + B^{-1} a_{s_k^+} = \bar{b} + B^{-1} a_{s_k^+},$$

in which $a_{s_k^+}$ denotes the column corresponding to $s_k^+$ in the technological matrix of Problem (3). Also, $\bar{a}_{s_k^+} := B^{-1} a_{s_k^+} = B^{-1} \begin{pmatrix} e_k \\ 0 \\ 0 \end{pmatrix}$ is the column corresponding to variable $s_k^+$ in the optimal simplex tableau. So, $\bar{b}^{new} = \bar{b} + \bar{a}_{s_k^+}$. Since in the optimal simplex tableau the objective row is nonnegative ($z_j - c_j \geqslant 0$) for all variables, the only possible violation of optimality is that the new vector $\bar{b}^{new}$ may have some negative components. The variable $\varphi$ is a basic variable in $B$, because the optimal value of $\varphi$ is always greater than or equal to one. Two different cases may occur.

**Case (A)** : $\bar{b}^{new} \geqslant 0$: In this case, $B$ remains optimal for Problem (7), and the values of the basic variables are $\bar{b}^{new} = \bar{b} + \bar{a}_{s_k^+}$. This case is further divided into two subcases. (Notice that $\varphi$ is a basic variable.)

**Subcase A1** : The component of $\bar{a}_{s_k^+} = B^{-1}\begin{pmatrix} e_k \\ 0 \\ 0 \end{pmatrix}$ corresponding to $\varphi$, denoted by $\bar{a}_{\varphi,s_k^+}$, is 0.

**Subcase A2** : $\bar{a}_{\varphi,s_k^+} > 0$.

Note that $\bar{a}_{\varphi,s_k^+}$ is not negative because the optimal value of LP (7) is greater than or equal to one.

In subcase A1, $B$ remains optimal basis for Problem (7) and the optimal value of $\varphi$ in Model (7) remains one. So, the unobserved DMU $(x_o + e_k, y_o)$ is output-oriented BCC-weakly efficient, i.e., condition (i) is happened. Hence, in this subcase $DMU_o$ is an anchor point.

In subcase A2, $B$ remains optimal basis for Problem (7) and the optimal value of $\varphi$ in Model (7) is greater than one. So, the unobserved DMU $(x_o + e_k, y_o)$ is not output-oriented BCC-weakly efficient, i.e., condition (i) is not happened for considered $k$. Notice that, in this subcase we cannot say whether $DMU_o$ is anchor point or not.

**Case (b)**: $\bar{b}^{new} \ngeqslant 0$. In this case, the dual simplex algorithm (Bazaraa, Sherali, & Shetty, 1993) is used to obtain a new optimal solution for Model (7) by getting feasibility. After implementing the dual simplex algorithm, if the new optimal value of $\varphi$ is equal to one, then $(x_o + e_k, y_o)$ is output-oriented BCC-weakly efficient (and hence, $DMU_o$ is an anchor point); otherwise (i.e., if the new optimal value of $\varphi$ is greater than one), $(x_o + e_k, y_o)$ is not output-oriented BCC-weakly efficient, i.e., condition (i) does not happen for considered $k$ (and hence, here too we cannot say whether $DMU_o$ is anchor point or not).

If condition (i) holds, then the $DMU_o$ is an anchor point; otherwise we check condition (ii). For checking condition (ii), we use the sensitivity analysis techniques similar to the above discussion. Here, the optimality conditions for the following LP are examined:

$$
\begin{aligned}
\theta_t^+ = \min \quad & \theta, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^+ - \theta x_{io} = 0, \quad \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^- = y_{ro}, \quad \forall r \neq t, \\
& \sum_{j=1}^{n} \lambda_j y_{tj} - s_t^- = \frac{y_{to}}{2}, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^+, s_r^-, \lambda_j \geqslant 0, \quad \forall i, j, r.
\end{aligned}
\tag{8}
$$

The initial simplex tableau for Problem (8) can be seen in Appendix B (Table 2).

The right-hand-side of the above model is obtained by adding the vector $\begin{pmatrix} 0 \\ -\frac{y_{to}}{2} e_t \\ 0 \end{pmatrix}$ to that in Model (1), and $\bar{b}^{new} = \bar{b} + \frac{y_{to}}{2}\bar{a}_{s_t^-}$. Note that here in subcase A2, the component of $\bar{a}_{s_t^-}$ corresponding to $\theta$ is negative.

If condition (ii) holds, then $DMU_o$ is an anchor point; otherwise $DMU_o$ is not an anchor point (note that we checked conditions (i) and (ii) so far), and the procedure terminates.

The following remark decreases the computational requirements of the above procedure in some situations strongly.

**Remark 1.** In checking condition (i), if in optimal tableau of Model (3) one of the input-slack variables, say $s_k^+$, is a basic variable, then $\bar{b}^{new} = \bar{b} + \bar{a}_{s_k^+} \geqslant 0$ and $\bar{a}_{\varphi,s_k^+} = 0$. Hence subcase A1 is happened. Thus, we stop with decision that $(x_o, y_o)$ is an anchor point. In checking condition (ii), this remark is also valid and helpful.

To sum up, in this section we provided a new method for checking that whether a DMU is anchor point or not. The presented procedure gives a new connection between the sensitivity analysis techniques in linear programming theory and DEA. It is interesting theoretically. From a computational point of view, in the method given in this section, we do not solve the DEA models for checking the efficiency position of $(x_o + e_k, y_o)$ and $(x_o, y_o - \frac{y_{to}}{2} e_t)$ from scratch. Also, in this method finding all extreme points of any polyhedral is not required. Furthermore, from an applied point of view, the introduced procedure automatically gives the weak supporting hyperplane which makes $DMU_o$ anchor point.

## 5. Conclusions

Anchor points build an important class of extreme efficient points in DEA. These points define the transition from the Pareto-efficient frontier to the free-disposability portion of the frontier of the PPS, as introduced by Bougnol and Dulá (2009). On the other hand, these points define the transmittance from the Pareto-efficiency to weak efficiency. The properties, applications, and identification of these points have been studied by some authors, including Bougnol (2001), Allen and Thanassoulis (2004), Rouse (2004), Bougnol and Dulá (2009), and Thanassoulis et al. (2011). In this paper, we dealt with these points from a different point of view, and provided some main theorems for characterization of these points. Utilizing these theoretical results, a procedure has been introduced for identification of the anchor points. In this method, solving the LP problems from scratch is not required and the sensitivity analysis tools from linear programming theory are utilized. In fact, it provides a new connection between the DEA and sensitivity analysis in linear programming theory.

Reduction in the computational requirements of the methods given for identification of the anchor points can be worth studying in future. Also, it seems that the connection between the anchor points and super-efficiency models (see Andersen & Petersen, 1993; Banker & Chang, 2000; Chen, 2005; Soleimani-damaneh, Jahanshahloo, & Foroughi, 2006; Tone, 2002) can be a worthwhile direction for future research.

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## Appendix A. The proofs

**Theorem 1.** *$DMU_o \in E$ is an anchor point if and only if $\chi_o = 0$.*

**Proof.** Let $(x_o, y_o)$ be an anchor point. Then, by Definition 2, it is located on a supporting hyperplane of $T_v$, say $H_{(u,v,u_0)}$, such that at least one component of the normal vector $(u, v)$ is zero (notice that $(u, v) \neq 0$). Since

$$H_{(u,v,u_0)} = \{(x,y) : \ uy - vx + u_0 = 0\}$$

supports $T_v$ at $(x_o, y_o)$, we have

$$uy_o - vx_o + u_0 = 0,$$

$$uy - vx + u_0 \leqslant 0, \quad \forall (x,y) \in T_v. \tag{9}$$

Setting

$$(\bar{u}, \bar{v}, \bar{u}_0) = \left(\frac{u}{\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i}, \frac{v}{\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i}, \frac{u_0}{\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i}\right),$$

we get

$$\bar{u}y_o - \bar{v}x_o + \bar{u}_0 = 0, \tag{10}$$

and

$$\sum_{r=1}^{s} \bar{u}_r + \sum_{i=1}^{m} \bar{v}_i = 1. \tag{11}$$

Furthermore, since $(x_j, y_j) \in T_v$ for each $j$, inequality (9) leads to

$$\bar{u}y_j - \bar{v}x_j + \bar{u}_0 \leqslant 0, \quad \forall j = 1, 2, \ldots, n. \tag{12}$$

Since $(x_o, y_o) \in T_v$, by presentation of $T_v$ (possibility axiom), $(x_o + e_i, y_o) \in T_v$ for each $i$. Therefore, according to Eqs. (9) and (10),

$$0 \geqslant \bar{u}y_o - \bar{v}(x_o + e_i) + \bar{u}_0 = -\bar{v}_i, \quad \forall i = 1, 2, \ldots, m.$$

Hence $\bar{v} \geqslant 0$. Since $(x_o, y_o) \in T_v$ and $y_o > 0$, by presentation of $T_v$ (possibility axiom), $(x_o, y_o - \frac{y_{ro}}{2} e_r) \in T_v$ for each $r$. Therefore, according to Eqs. (9) and (10),

$$0 \geqslant \bar{u}\left(y_o - \frac{y_{ro}}{2} e_r\right) - \bar{v}x_o + \bar{u}_0 = -\frac{y_{ro}}{2}\bar{u}_r, \quad \forall r = 1, 2, \ldots, s.$$

Hence $\bar{u} \geqslant 0$. Since $(\bar{u}, \bar{v}) \geqslant 0$, because of Eqs. (10)–(12), the vector $(\bar{u}, \bar{v}, \bar{u}_0)$ is a feasible solution to both LPs (5) and (6) in which at least one component of the vector $(\bar{u}, \bar{v})$ is zero. Without loss of generality, assume that $\bar{u}_1 = 0$. According to the constraint $u \geqslant 0$ in LP (5) and since its objective function is minimization, $\bar{u}_1 = 0$ implies $u_1^* = 0$. Therefore, $\chi_o = 0$.

To prove the converse, let $\chi_o = 0$. Without loss of generality, assume that $\chi_o = u_1^*$. Therefore, there exists a vector $(u^*, v^*, u_0^*)$ satisfying

$$u_1^* = 0, \tag{13}$$

$$u^*y_j - v^*x_j + u_o^* \leqslant 0, \quad \forall j = 1, 2, \ldots, n, \tag{14}$$

$$u^*y_o - v^*x_o + u_o^* = 0, \tag{15}$$

$$\sum_{r=1}^{s} u_r^* + \sum_{i=1}^{m} v_i^* = 1, \tag{16}$$

$$u^* \geqslant 0, \tag{17}$$

$$v^* \geqslant 0. \tag{18}$$

Considering arbitrary $(x, y) \in T_v$, there exists $\lambda \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} \lambda_j x_j \leqslant x, \quad \sum_{j=1}^{n} \lambda_j y_j \geqslant y \geqslant 0, \quad \sum_{j=1}^{n} \lambda_j = 1, \lambda \geqslant 0. \tag{19}$$

Multiplying (14) by $\lambda_j$ and summing the obtained inequalities over $j$, implies

$$u^* \sum_{j=1}^{n} \lambda_j y_j - v^* \sum_{j=1}^{n} \lambda_j x_j + u_o^* \leqslant 0. \tag{20}$$

By (19) and (20), we have $u^*y - v^*x + u_o^* \leqslant 0$. Also, by (16), $(u^*, v^*) \neq 0$. Since $(x, y) \in T_v$ is arbitrary and due to (15), the hyperplane $H_{(u^*,v^*,u_o^*)}$ supports $T_v$ at $(x_o, y_o)$, while $u_1^* = 0$. Therefore, $(x_o, y_o)$ is an anchor point because of Definition 2. □

**Theorem 2.** *$DMU_o = (x_o, y_o) \in E$ is an anchor point if and only if one of the following conditions holds:*

(i) $(x_o + e_i, y_o)$ is output-oriented BCC-weakly efficient, for some $i \in \{1, \ldots, m\}$,
(ii) $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ is input-oriented BCC-weakly efficient, for some $r \in \{1, \ldots, s\}$.

**Proof.** Let $(x_o, y_o)$ be an anchor point. Then there exits a vector $(u^*, v^*, u_o^*)$ satisfying constraints (6.1)–(6.5) in which $v_k^* = 0$ for some $k \in \{1, \ldots, m\}$ or $u_t^* = 0$ for some $t \in \{1, \ldots, s\}$.

*Case 1.* $v_k^* = 0$ for some $k \in \{1, \ldots, m\}$. We have two possible subcases:

*Case 1.1:*

$u^* \neq 0$: In this subcase, defining $\beta = u^*y_o$, we have $\beta > 0$ and

$$\frac{u^*}{\beta} y_j - \frac{v^*}{\beta} x_j + \frac{u_o^*}{\beta} \le 0, \quad \forall j,$$

$$\frac{u^*}{\beta} y_o = 1, \quad \text{and} \quad \frac{v^*}{\beta}(x_o + e_k) - \frac{u_o^*}{\beta} = 1.$$

Hence $\left(\frac{u^*}{\beta}, \frac{v^*}{\beta}, \frac{u_o^*}{\beta}\right)$ is a feasible solution to Model (4) when assessing $(x_o + e_k, y_o)$ and the value of the objective function at this feasible solution equals 1. This implies that $(x_o + e_k, y_o)$ is output-oriented BCC-weakly efficient. Note that, by possibility axiom, we have $(x_o + e_k, y_o) \in T_v$.

*Case 1.2:*

$u^* = 0$: In this case, $v^* \neq 0$. Defining $\alpha = v^*x_o$, we have $\alpha > 0$ and similar to the above, it can be shown that $\left(u^* = 0, \frac{v^*}{\alpha}, \frac{u_o^*}{\alpha}\right)$ is a feasible solution to Model (2) when assessing $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ and the value of the objective function at this feasible solution equals 1. This implies that $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ is input-oriented BCC-weakly efficient.

*Case 2.* $u_t^* = 0$ for some $t \in \{1, \ldots, s\}$. In this case, similar to the previous case, it can be shown that (i) or (ii) occurs.

Conversely, if case (i) or case (ii) happens, then $(x_o + e_k, y_o)$ or $(x_o, y_o - \frac{y_{ro}}{2} e_r)$ is located on the boundary of $T_v$ (see Lemma 8 in Soleimani-damaneh, Jahanshahloo, Mehrabian, & Hasannasab, 2009). Therefore, $(x_o, y_o)$ is an anchor point because of Corollary 1 in Bougnol and Dulá (2009). □

## Appendix B. The initial simplex tableaus

In the simplex tableaus given in this section, RHS means "Right Hand Side". $R_i$ variables are artificial variables which are required to obtain the initial BFS. Here we have used the Big-M method to find the initial BFS, and so in these tables $M$ is a positive sufficiently large number (see Bazaraa, Jarvis, & Sherali, 1990; Soleimani-damaneh et al., 2006) (see Tables 1 and 2).

**Table 1**
The initial simplex tableau related to Problem (8).

| | $Z$ | $\lambda_1$ | $\cdots$ | $\lambda_n$ | $s_1^+$ | $\cdots$ | $s_k^+$ | $\cdots$ | $s_m^+$ | $s_1^-$ | $\cdots$ | $s_s^-$ | $\varphi$ | $R$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | $-M$ | $\cdots$ | $-M$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | $-1$ | 0 | $-M$ |
| $s_1^+$ | 0 | $x_{11}$ | $\cdots$ | $x_{1n}$ | 1 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | 0 | 0 | $x_{1o}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_k^+$ | 0 | $x_{k1}$ | $\cdots$ | $x_{kn}$ | 0 | $\cdots$ | 1 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | 0 | 0 | $x_{ko}+1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_m^+$ | 0 | $x_{m1}$ | $\cdots$ | $x_{mn}$ | 0 | $\cdots$ | 0 | $\cdots$ | 1 | 0 | $\cdots$ | 0 | 0 | 0 | $x_{mo}$ |
| $s_1^-$ | 0 | $-y_{11}$ | $\cdots$ | $-y_{1n}$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 1 | $\cdots$ | 0 | $y_{1o}$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_s^-$ | 0 | $-y_{s1}$ | $\cdots$ | $-y_{sn}$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 1 | $y_{so}$ | 0 | 0 |
| $R$ | 0 | 1 | $\cdots$ | 1 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | 0 | 1 | 1 |

**Table 2**
The initial simplex tableau related to Problem (9).

| | $Z$ | $\lambda_1$ | $\cdots$ | $\lambda_n$ | $s_1^+$ | $\cdots$ | $s_m^+$ | $s_1^-$ | $\cdots$ | $s_t^-$ | $\cdots$ | $s_s^-$ | $\theta$ | $R_1$ | $\cdots$ | $R_t$ | $\cdots$ | $R_s$ | $R_{s+1}$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | $m_1$ | $\cdots$ | $m_n$ | 0 | $\cdots$ | 0 | $-M$ | $\cdots$ | $-M$ | $\cdots$ | $-M$ | $-1$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $Z$ |
| $s_1^+$ | 0 | $x_{11}$ | $\cdots$ | $x_{1n}$ | 1 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | $-x_{1o}$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_m^+$ | 0 | $x_{m1}$ | $\cdots$ | $x_{mn}$ | 0 | $\cdots$ | 1 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | $-x_{mo}$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 |
| $R_1$ | 0 | $y_{11}$ | $\cdots$ | $y_{1n}$ | 0 | $\cdots$ | 0 | $-1$ | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 1 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | $y_{1o}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $R_t$ | 0 | $y_{t1}$ | $\cdots$ | $y_{tn}$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | $-1$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 1 | $\cdots$ | 0 | 0 | $\frac{y_{1o}}{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $R_s$ | 0 | $y_{s1}$ | $\cdots$ | $y_{sn}$ | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | $\cdots$ | $-1$ | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 1 | 0 | $y_{so}$ |
| $R_{s+1}$ | 0 | 1 | $\cdots$ | 1 | 0 | $\cdots$ | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 1 | 1 |

In this table, $m_j$ denotes the reduced cost of $\lambda_j$, equal to $m_j = M\left(\sum_{r=1}^{s} y_{rj} + 1\right)$. Also, $Z$ stands for the value of the objective function at the current BFS, equal to $Z = M\left(\sum_{r=1,r\neq t}^{s} y_{ro} + 1\right) + \frac{y_{to}}{2}$.

## References


Allen, R., & Thanassoulis, E. (2004). Improving envelopment in data envelopment analysis. *European Journal of Operational Research, 154*, 363–379.

Andersen, P., & Petersen, N. C. (1993). A procedure for ranking efficient units in data envelopment analysis. *Management Science, 39*, 1261–1264.

Banker, R. D., & Chang, H. (2000). *Evaluating the super-efficiency procedure in data envelopment analysis for outlier identification and for ranking efficient units*. Richardson, TX: School of Management, The University of Texas at Dallas.

Bazaraa, M. S., Jarvis, J. J., & Sherali, H. D. (1990). *Linear programming and network flows* (2nd ed.). New York: John Wiley & Sons.

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (1993). *Nonlinear programming theory and algorithms* (2nd ed.). New York: Wiley.

Bougnol, M. L. (2001). *Nonparametric frontier analysis with multiple constituencies*. Ph.D. dissertation, The University of Mississippi.

Bougnol, M. L., & Dulá, J. H. (2009). Anchor points in DEA. *European Journal of Operations Research, 192*, 668–676.

Charnes, A., Cooper, W. W., & Thrall, R. M. (1991). A structure for classifying and characterizing efficiency and inefficiency in data envelopment analysis. *Journal of Productivity Analysis, 2*, 197–237.

Chen, Y. (2005). Measuring super-efficiency in DEA in the presence of infeasibility. *European Journal of Operational Research, 161*, 545–551.

Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software* (2nd ed.). Springer.

Dulá, J. H., & López, F. J. (2006). Algorithms for the frame of a finitely generated unbounded polyhedron. *INFORMS Journal on Computing, 18*, 97–110.

Emrouznejad, A., Parker, B., & Tavares, G. (2008). Evaluation of research in efficiency and productivity: A thirty years survey of the scholarly literature in DEA. *Socio-Economic Planning Sciences, 42*(3), 151–157.

Rouse, P. (2004). *Using DEA to set prices for health care delivery in New Zealand hospitals*. Working paper, The University of Auckland, New Zealand.

Soleimani-damaneh, M., Jahanshahloo, G. R., & Foroughi, A. A. (2006). A comment on "Measuring super-efficiency in DEA in the presence of infeasibility". *European Journal of Operational Research, 170*, 323–325.

Soleimani-damaneh, M., Jahanshahloo, G. R., Mehrabian, S., & Hasannasab, M. (2009). Scale elasticity and returns to scale in the presence of alternative solutions. *Journal of Computational and Applied Mathematics, 233*, 127–136.

Thanassoulis, E., & Allen, R. (1998). Simulating weights restrictions in data envelopment analysis by means of unobserved DMUs. *Management Science, 44*, 586–594.

Thanassoulis, E., Kortelainen, M., & Allen, R. (2011). Improving envelopment in data envelopment analysis under variable returns to scale. *European Journal of Operational Research*. http://dx.doi.org/10.1016/j.ejor.2011.10.009.

Tone, K. (2002). A slack-based measure of super-efficiency in data envelopment analysis. *European Journal of Operational Research, 30*, 498–509.