Constitutive models of linear viscoelasticity using Laplace transform

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January 2009

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1 Introduction

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Viscous materials, like honey, resist shear flow and strain linearly with time when a stress is applied. Elastic materials strain instantaneously when stretched and just as quickly return to their original state once the stress is removed. Viscoelastic materials have elements of both of these properties and, as such, exhibit time dependent strain [5].

This work deals with one-dimensional linear viscoelastic models using Laplace transform that is used for solving differential equations with the great benefit of producing easily solvable algebraic equations.

2 Theory

2.1 Serial viscoelastic models

Viscoelasticity is the result of the diffusion of atoms or molecules inside a material but the viscoelasticity models are symbolized with macroscopic object like string and damper while it is still describing the microscopic material behaviour. One of the models can be found on Figure 1 describing the series of elastic string and viscous damper.

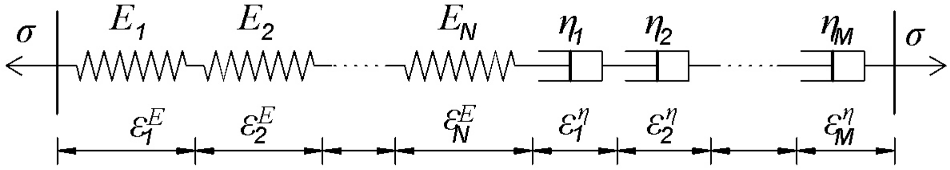

Figure 1: Serial viscoelastic model

The system shown on Figure 1 can be described with Kinematatics, Equilibrium and Constitutive equations. It is usefull to express kinematics equations in terms of strain, thus we can write consistent with the figure that the overall strain is can be expressed as a sum of individual strains of either string or damper:

$$\varepsilon(t) = \varepsilon^{\mathrm{E}}(t) + \varepsilon^{\eta}(t) = \sum_{i=1}^{N} \varepsilon_i^{\mathrm{E}}(t) + \sum_{i=1}^{M} \varepsilon_i^{\eta}(t) \tag{1}$$

where ε can be regarded as overall strain, ε^{E} as a strain caused with elastic strings and similarly ε^{η} describes a stress related to viscous dampers.

Next, Equilibrium equations setting balance in the nodes are following:

$$\sigma(t) = \sigma_i^{\mathrm{E}}(t) = \sigma_j^{\mathrm{E}}(t), \quad i = 1, 2, \ldots, N \text{ and } j = 1, 2, \ldots, M \tag{2}$$

where σ is a stress imposed on the system. Finally, we an write down Constitutive equations connecting stress and strain that can be expressed as a Hook's law in the case of elastic string. The more complicated situation arises in the case of viscous damper. The viscous component can be modeled as dashpots that is a mechanical device, a damper which resists motion via viscous friction. The resulting force is proportional to the velocity, but acts in the opposite direction, slowing the motion and absorbing energy. Hence, the constitutive equations for both elastic string and viscous damper are following:

$$\sigma_i(t) = E_i \varepsilon_i^{\mathrm{E}}(t), \quad i = 1, 2, \ldots, N \tag{3}$$

$$\sigma_i(t) = \eta_i \dot{\varepsilon}_i^{\eta}(t), \quad i = 1, 2, \ldots, M \tag{4}$$

where $E_i \in \mathbb{R}^+$ is elastic stiffness and $\eta_i \in \mathbb{R}^+$ is material constant called viscosity (the set $\mathbb{R}^+$ represents strictly positive real numbers).

Using Equations 3 with combination with Equilibrium Equation (2) and Konstitutive Equation (1) leads to an expression for ε^{E} depending only on imposed stress σ:

$$\varepsilon^{\mathrm{E}}(t) = \sigma(t) \sum_{i=1}^{N} \frac{1}{E_i}$$

Noting that the sumation in the equation can be expressed with a substitute stiffness E, it can be rewritten into:

$$\varepsilon^{\mathrm{E}}(t) = \sigma(t) \frac{1}{E}$$

where $\sum_{i=1}^{N} \frac{1}{E_i} = \frac{1}{E}$.

Analogicaly, the viscous part of the model can be expressed as:

$$\dot{\varepsilon}^{\eta}(t) = \sigma(t) \frac{1}{\eta}$$

where $\sum_{i=1}^{M} \frac{1}{\eta_i} = \frac{1}{\eta}$.

Thus, it can be said that the system described in this section and shown on Figure 1 is possible to express using two constant, one dealing with the elastic string and the second with the viscous damper. This system is called Kelvin-Voigt model [1].

3 Laplace transform

In mathematics, the Laplace transform is one of the best known and most widely used integral transforms. It is commonly used to produce an easily solvable algebraic equation from an ordinary differential equation.

In mathematics, it is used for solving differential and integral equations. In physics, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In this analysis, the Laplace transform is often interpreted as a transformation from the time-domain, in which inputs and outputs are functions of time, to the frequency-domain, where the same inputs and outputs are functions of complex angular frequency, or radians per unit time [3].

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $\hat{f}(s)$, defined by:

$$\mathcal{L}\{f(t)\} = \hat{f}(s) = \lim_{\alpha \to 0^-} \int_{\alpha}^{\infty} f(t) e^{-st} dt \tag{5}$$

Noting that the limit assures the inclusion of the entire Dirac delta function $\delta(t)$ at 0 if there is such an impulse in $f(t)$ at 0. The parameter s is in general complex: $s = a + \mathrm{i}b$.

Inverse Laplace transform is given by the following complex integral:

$$\mathcal{L}^{-1}\{\hat{f}(s)\} = f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} f(s) e^{st} ds \tag{6}$$

where γ is a real number depending on the region of convergence of $\hat{f}(s)$.

This integral transform has a number of properties that make it useful for analyzing linear dynamic systems. The most important is a linearity of operator:

$$\mathcal{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha \mathcal{L}\left\{f(t)\right\} + \beta \mathcal{L}\left\{g(t)\right\} \tag{7}$$

and the most significant advantage is that differentiation and integration become multiplication and division, respectively, by s. Hence, the Laplace transform of functions derivative is written as:

$$\mathcal{L}\left\{\dot{f}(t)\right\} = s\hat{f}(t) - \lim_{\alpha \to 0^-} f(\alpha), \tag{8}$$

where the second term $\lim_{\alpha \to 0^-} f(\alpha)$ denotes an initial condition that is assumed to equal to zero in the models used in this work, thus this member vanishes and it is not included in the next text [3].

3.1 Algorithmization of the models

This section provides algorithmization of the viscoelastic models. First of all, the unambiguous description of the model that can be processed with a computer is characterized in Section

3.1.1 Description of the model

A general linear viscoelastic model is composed from the serial strings, described in Section 2.1. An example of real model is shown in Figure 3. In general case N serial string and P points that are step-by-step number from 1 to N and from 1 to P respectively is considered, noting that points are number in given direction in order to clearly determine direction of forces acting on the points.

Next, the general i^{th} serial string can be described with four values, two of them b_i and e_i ($b_i, e_i \in \{1, 2, \ldots, P\}$ and $b_i < e_i$) determine the beginning and ending point of the string. The two remaining values are elastic and viscous material constant E_i resp. η_i determining the string. The overall system can be recorded in a table as it is shown in Table 1.

The model can be extended by enlarging the definition domain of the material parameters E and η, assuming that $E, \eta \in \mathbb{R}^+ \cup \{\infty\}$. After definition of a following operation with the used symbol ∞:

$$\frac{1}{\infty} \stackrel{\text{def}}{=} 0,$$

it can be forwarded to situation when one of the parameters is equal to that symbol ∞. It falls into models where only elastic string or only viscous damper is used. Noting that it is forbidden to take both of the constants equal to ∞ as it represents the infinitely rigid model which does not allow any deformation.

Table 1: Table describing the model structure

number of string	begin of string	end of string	stiffness of elastic string	viscosity of damper
1	b_1	e_1	E_1	η_1
2	b_2	e_2	E_2	η_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	b_i	e_i	E_i	η_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	b_N	e_N	E_N	η_N

In order to analyse the model, a total stress $\sigma(t)$ is imposed on a lateral points 1 and P while the overall strain $\varepsilon(t)$ is observed or vice versa. Thus, the object of analysis is in finding expression between the stress $\sigma(t)$ and the strain $\varepsilon(t)$.

For that purposes, the auxiliary variables are introduced. Unknown stress in the individual serial string i is marked as σ_i, where $i = 1, 2, \dots, N$ and unknown strain between two adjacent points j and $j+1$ is called $\varepsilon_{j,j+1}$ where $j = 1, 2, \dots, P-1$.

3.1.2 Kinematics, Constitutive and Equlibrium Equations

This section contains equations that describe the model. First group of equations is called Kinematics equation as it puts together the overall strain with strains of adjacent points. An algebraic equation states as:

$$\sum_{j=1}^{P-1} \varepsilon_{j,j+1} = \varepsilon$$

This equation can be rewritten using linearity of Laplace transform into following form:

$$\sum_{j=1}^{P-1} \hat{\varepsilon}_{j,j+1} = \hat{\varepsilon}. \tag{9}$$

The second group of equations is called Constitutive equations as it express the relation between stress of i^{th} string and a related strain corresponding to points where the string is connected. As stated in Section 2.1, a strain of a serial string is composed of its elastic and viscous part:

$$\varepsilon_{b_i,e_i} = \varepsilon^{\mathrm{E}}_{b_i,e_i} + \varepsilon^{\eta}_{b_i,e_i} \tag{10}$$

where subscripts b_i, e_i represents the strain between points b_i and e_i and superscript E, η label its elastic and viscous part. Next, we know from simple equilibrium that stress in both elastic and viscous damper is the same:

$$\sigma_i = \sigma^{\mathrm{E}}_i = \sigma^{\eta}_i \tag{11}$$

Hence, it is possible to write constitutive equations for elastic and viscous member as follows:

$$\varepsilon^{\mathrm{E}}_{b_i,e_i} = \frac{\sigma_i}{E} \tag{12}$$

$$\dot{\varepsilon}^{\eta}_{b_i,e_i} = \frac{\sigma_i}{\eta} \tag{13}$$

In order to eliminate variables $\varepsilon^{\mathrm{E}}_{b_i,e_i}$ and $\varepsilon^{\eta}_{b_i,e_i}$, it is necessary to provide an other algebraic emendations due to the fact that Equation (13) contains derivative. Substitution Equation (13) and derivative of Equation (12) into derivative of Equation (10) leads to following expression of constitutive equation of the serial string:

$$\dot{\varepsilon}_{b_i,e_i} = \frac{\dot{\sigma}_i}{E} + \frac{\sigma_i}{\eta} \tag{14}$$

The left side of the equation can be expressed using kinematics equations as a sum of strains between points b_i and e_i:

$$\varepsilon_{b_i,e_i} = \sum_{j=b_i}^{e_i-1} \varepsilon_{j,j+1} \quad \Leftrightarrow \quad \hat{\varepsilon}_{b_i,e_i} = \sum_{j=b_i}^{e_i-1} \hat{\varepsilon}_{j,j+1} \quad \Leftrightarrow \quad \dot{\varepsilon}_{b_i,e_i} = \sum_{j=b_i}^{e_i-1} \dot{\varepsilon}_{j,j+1} \tag{15}$$

Finally, Equation (14) can be rewritten in Laplace transform. Using Equation (15) and after dividing by Laplace variable s, it follows as:

$$\sum_{j=b_i}^{e_i-1} \hat{\varepsilon}_{j,j+1} = \hat{\sigma}_i \left(\frac{1}{E_i} + \frac{1}{s\eta_i} \right) \tag{16}$$

that is equation in terms of variables introduced at the end of Section 3.1.1.

Last group of equtions is called Equilibrium equations as it set equilibrium in each point. The following equations stepwise introduce the equilibrium in the first point, inner points, and the last point:

$$\sigma = \sum_{j=1}^{P} \delta_{ib_j} \sigma_j \tag{17}$$

$$\sum_{j=1}^{P} \delta_{ib_j} \sigma_j = \sum_{k=1}^{P} \delta_{ie_k} \sigma_k, \quad i = 1, 2, \ldots, P-1 \tag{18}$$

$$\sum_{k=1}^{P} \delta_{ie_k} \sigma_k = \sigma \tag{19}$$

3.2 Setting and solving system of equations

For the sake of algorithmization, it is usefull to write algebraic equations in matrix notation. Because of lucidity, necessery equations, that are described in previous section, are again rewritten in the order that are used in the matrix:

$$-\sum_{j=b_i}^{e_i-1} \hat{\varepsilon}_{j,j+1} + \hat{\sigma}_i \left(\frac{1}{E_i} + \frac{1}{s\eta_i} \right) = 0, \quad i = 1, 2, \ldots, N \tag{20}$$

$$-\sum_{j=1}^{P} \delta_{ib_j} \hat{\sigma}_j + \hat{\sigma} = 0 \tag{21}$$

$$\sum_{j=1}^{P} (\delta_{ib_j} - \delta_{ie_j}) \hat{\sigma}_j = 0, \quad i = 1, 2, \ldots, P-1 \tag{22}$$

$$-\sum_{k=1}^{P} \delta_{ie_k} \hat{\sigma}_k + \hat{\sigma} = 0 \tag{23}$$

$$-\sum_{j=1}^{P-1} \hat{\varepsilon}_{j,j+1} + \hat{\varepsilon} = 0 \tag{24}$$

Next, the matrix notation of those equations looks like:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o} \tag{25}$$

where an array $\boldsymbol{o}$ is a nil vector:

$$\boldsymbol{o} = \begin{Bmatrix} 0 & 0 & \ldots & 0 \end{Bmatrix}^T,$$

next, an array $\boldsymbol{x}$ is a vector of variables:

$$\boldsymbol{x} = \begin{Bmatrix} \hat{\varepsilon}_{1,2} & \hat{\varepsilon}_{2,3} & \ldots & \hat{\varepsilon}_{P-1,P} & \hat{\sigma}_1 & \hat{\sigma}_2 & \ldots & \hat{\sigma}_N & \hat{\sigma} & \hat{\varepsilon} \end{Bmatrix}^T$$

and finally, an array $\boldsymbol{A}$ is a matrix whose elements are rational functions:

$$\boldsymbol{A} \in \mathscr{R}^{(N+P+1)\times(P+N+1)}, \quad \mathscr{R} = \left\{ y(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}, a_i, b_i \in \mathbb{R} \right\}$$

and finally the general form of matrix $\boldsymbol{A}$ looks as follows:

$$\boldsymbol{A} = \left[\begin{array}{cccc|cccc|cc}
-v^1_{b_1e_1} & -v^2_{b_1e_1} & \cdots & -v^{P-1}_{b_1e_1} & \frac{1}{E_1}+\frac{1}{s\eta_1} & 0 & \cdots & 0 & 0 & 0 \\
-v^1_{b_2e_2} & -v^2_{b_2e_2} & \cdots & -v^{P-1}_{b_2e_2} & 0 & \frac{1}{E_2}+\frac{1}{s\eta_2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-v^1_{b_Ne_N} & -v^2_{b_Ne_N} & \cdots & -v^{P-1}_{b_Ne_N} & 0 & 0 & \cdots & \frac{1}{E_N}+\frac{1}{s\eta_N} & 0 & 0 \\
\hline
0 & 0 & \cdots & 0 & -\delta_{1,b_1} & -\delta_{1,b_2} & \cdots & -\delta_{1,b_N} & 1 & 0 \\
0 & 0 & \cdots & 0 & \Delta^2_{b_1,e_1} & \Delta^2_{b_2,e_2} & \cdots & \Delta^2_{b_N,e_N} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \Delta^{P-1}_{b_1,e_1} & \Delta^{P-1}_{b_2,e_2} & \cdots & \Delta^{P-1}_{b_N,e_N} & 0 & 0 \\
0 & 0 & \cdots & 0 & -\delta_{P,e_1} & -\delta_{P,e_2} & \cdots & -\delta_{P,e_N} & 1 & 0 \\
\hline
-1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right]$$

The functions occurring in the matrix $\boldsymbol{A}$ are defined as follows:

$$v^k_{b_i,e_i} = \begin{cases} 1, & \text{if } b_i \le k \le e_i - 1 \\ 0, & \text{otherwise} \end{cases}, \quad \text{where } i = 1,2,\dots,N$$

$$\Delta^k_{b_i,e_i} = \delta_{k,b_i} - \delta_{k,e_i}, \quad \text{where } i = 1,2,\dots,N.$$

Next, the constructed Equation (25) is solved using Gaussian elimination [2] that transfers the matrix $\boldsymbol{A}$ into row echelon form[1]. It can be shown that one of the Equilibrium equations can be expressed as a linear combination of the others, thus after the Gaussian elimination the last row of matrix $\boldsymbol{A}$ is identically equal to zero. Then, the equation getting from the $(P+N)^{\text{th}}$ row of matrix $\boldsymbol{A}$ looks like:

$$\hat{\sigma} A_{N+P,P+N} = \hat{\varepsilon} A_{N+P,P+N+1} \quad \Leftrightarrow \quad \hat{\sigma} = \frac{A_{N+P,P+N+1}}{A_{N+P,P+N}} \hat{\varepsilon} \quad \Leftrightarrow$$

$$\hat{\sigma} = \mathcal{E}\hat{\varepsilon}, \quad \text{where } \mathcal{E} = \frac{A_{N+P,P+N+1}}{A_{N+P,P+N}} \tag{26}$$

that express the relation between the total stress and overall strain in Laplace transform.

3.3 Stress relaxation and creep

Next step is an analysis of Equation (26) that primarily depends on knowledge of either the strain function $\varepsilon(t)$ or stress function $\sigma(t)$. As the expression of either function is known, it can be convert using Laplace transform and next inverse Laplace transform produces the expression of second function.

Some phenomena in viscoelastic materials are:

[1] In linear algebra a matrix is in row echelon form (sometimes called row canonical form) if all nonzero rows are above any rows of all zeroes, and the leading coefficient of a row is always strictly to the right of the leading coefficient of the row above it [4].

- if the stress is held constant, the strain increases with time. It is called creep and the expression for stress function is following:

$$\sigma(t) = \sigma_0 H(t) \tag{27}$$

  where σ_0 is constant and $H(t)$ is called Heaviside or unit step function defined as:

$$H(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$$

- if the strain is held constant, the stress decreases with time It is called relaxation and the expression for strain function is following [5]:

$$\varepsilon(t) = \varepsilon_0 H(t) \tag{28}$$

where ε_0 is konstant.

Next, we can perform Laplace transform on Equation (27) and (28) that heads to:

$$\hat{\sigma}(s) = \frac{\sigma_0}{s} \tag{29}$$

$$\hat{\varepsilon}(s) = \frac{\varepsilon_0}{s} \tag{30}$$

After substitution of Equations (29) and (30) into Equation (26), it heads to:

$$\hat{\sigma}(s) = \frac{\mathcal{E}\varepsilon_0}{s} \tag{31}$$

$$\hat{\varepsilon}(s) = \frac{\sigma_0}{s\mathcal{E}} \tag{32}$$

Inverse Laplace transform used on those two equations follows as:

$$\sigma(t) = \varepsilon_0 \mathcal{L}^{-1}\left\{\frac{\mathcal{E}}{s}\right\} \tag{33}$$

$$\varepsilon(t) = \sigma_0 \mathcal{L}^{-1}\left\{\frac{1}{s\mathcal{E}}\right\} \tag{34}$$

that express the stress function for relaxation and strain function for creep. The product of inverse Laplace transform in the case relaxation (Equation (33)) is marked as:

$$R_0(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{E}}{s}\right\} \tag{35}$$

and it is called relaxation function. In the case of creep, the inverse Laplace transform in Equation (34) is called compliance function:

$$J_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s\mathcal{E}}\right\}. \tag{36}$$

A relationship between relaxation function stated in Equation (35) and compliance function stated in Equation (36) is a quite complicated. The different situation is in the case of Laplace transforms. Thus Equations (35) and (36) leads to:

$$\mathcal{E} = s\mathcal{L}^{-1}\left\{R_0(t)\right\}$$
$$\mathcal{E} = \frac{1}{s\mathcal{L}^{-1}\left\{J_0(t)\right\}}$$

These two equations have to be equal to each other, thus we can write the relationship between relaxation function and compliance function in the terms of Laplace transform:

$$\mathcal{L}\left\{J_0(t)\right\} = \frac{1}{s^2\mathcal{L}\left\{R_0(t)\right\}}$$

Now, we can discuss the inverse Laplace transform of used functions. We can noticed that the term $\mathcal{E}$ has to be from the set of all rational functions $\mathscr{R}$ and thus it can be decomposed using partial fractions. Then the function $\mathcal{E}$ can be expressed as linear combination of following terms:

$$P_n(s), \frac{a}{s+b}, \frac{cs+d}{s^2+es+f}, \frac{g}{s^2+hs+i}$$

where $P_n(s)$ is a polynomial function of order n and $a, b, c, d, e, f, g, h, i$ are constants. The inverse Laplace transform of those expressions is well known, thus all of the models discussed in this work can expressed using basis functions.

4 Application to real viscoelastic models

This section shows illustration of the algorithm on two models. The first model is called Standard linear solid model that is well analysed and described in literature. It is chosen in order to show functionality of the algorithm. The second model is selected randomly and is a more complicated than the previous one.

4.1 Standard linear solid model

The sketch of the Stadard linear solid model with all of the parameters is shown in Figure 2. For the next analysis, it is necessary to describe the model with the Table 2 as it is introduced in Section 3.1.1.

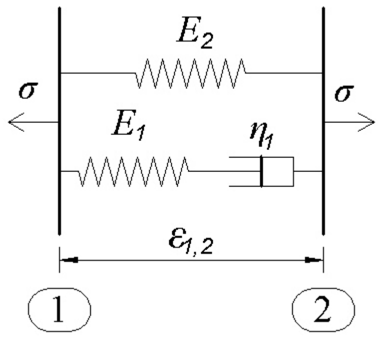

Figure 2: Sketch of a Standard linear solid model

Table 2: Input table of the Standard linear solid model

number of string	begin of string	end of string	stiffness of elastic string	viscosity of damper
1	1	2	E_1	η_1
2	1	2	E_2	∞

Next, it is necessary set the system of equations $\boldsymbol{Ax} = \boldsymbol{o}$ as it is described in Section 3.2:

$$\begin{bmatrix} -1 & \frac{1}{E_1} + \frac{1}{\eta_1 s} & 0 & 0 & 0 \\ -1 & 0 & \frac{1}{E_2} & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{1,2} \\ \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma} \\ \hat{\varepsilon} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Next, it is necessary to provide Gauss elimination that heads to matrix $\boldsymbol{A}$ in the following form:

$$\boldsymbol{A} = \begin{bmatrix} -1 & \frac{1}{E_1}+\frac{1}{\eta_1 s} & 0 & 0 & 0 \\ 0 & -\frac{\eta_1 s+E_1}{E_1\eta_1 s} & \frac{1}{E_2} & 0 & 0 \\ 0 & 0 & -\frac{E_2\eta_1 s+E_2E_1+E_1\eta_1 s}{(\eta_1 s+E_1)E_2} & 1 & 0 \\ 0 & 0 & 0 & -\frac{\eta_1 s+E_1}{E_2\eta_1 s+E_2E_1+E_1\eta_1 s} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equation is identically equal to zero thus the last but one equation put together overall strain and stress in terms of Laplace transform:

$$\hat{\sigma} = \mathcal{E}\hat{\varepsilon} \quad \text{where } \mathcal{E} = \frac{E_2\eta_1 s + E_2E_1 + E_1\eta_1 s}{\eta_1 s + E_1}$$

Then it is possible to analyse relaxation and creep. As it is described in Section 3.3 relaxation and compliance function respectively are calculated using inverse Laplace transform heading to:

$$R_0(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{E}}{s}\right\} = E_1\,\mathrm{e}^{-\frac{E_1 t}{\eta_1}+E_2}$$

$$J_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s\mathcal{E}}\right\} = \frac{E_1 + E_2 - E_1\,\mathrm{e}^{-\frac{E_2E_1 t}{(E_1+E_2)\eta_1}}}{E_2(E_1+E_2)}$$

4.2 Example of solution for arbitrary model

This section provide analysis of randomly chosen model. The sketch of the model is shown in Figure 3.3 and analogically to previous case Table 3 determines the model in a form that can be easily processed.

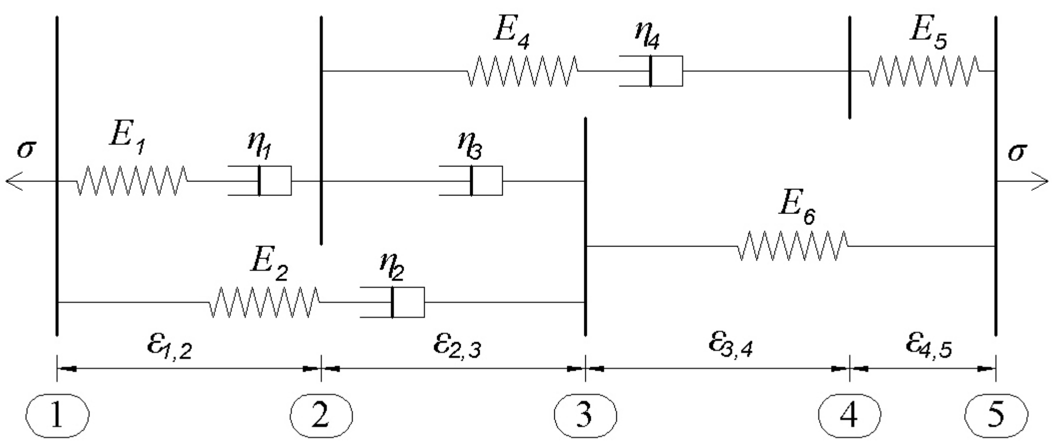

Figure 3: Sketch of a model

Table 3: Input table of the model

number of string	begin of string	end of string	stiffness of elastic string	viscosity of damper
1	1	2	E_1	η_1
2	1	3	E_2	η_2
3	2	3	∞	η_3
4	2	4	E_4	η_4
5	4	5	E_5	∞
6	3	5	E_6	∞

Next, Table 3 is used to set the system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ with coefficents matrix $\boldsymbol{A}$ in the following form:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & \frac{1}{E_1}+\frac{1}{\eta_1 s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & \frac{1}{E_2}+\frac{1}{\eta_2 s} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{\eta_3 s} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{E_4}+\frac{1}{\eta_4 s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{E_5} & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{E_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Next, the Gaussian elimination find general expression between overall stress and strain in term of Laplace transform but the expression is a quite complicated. For the purposes of good arrangement the material constants of the model are taken identically equal to one:

$$E_1 = E_2 = E_4 = E_5 = E_6 = \eta_1 = \eta_2 = \eta_3 = \eta_4 = 1.$$

Then, an expression of the second equation from the end (as the last is identically equal to zero) states:

$$\hat{\sigma} = \mathcal{E}\hat{\varepsilon}, \quad \text{where } \mathcal{E} = \frac{s\,(6\,s^2 + 13\,s + 5)}{7\,s^3 + 19\,s^2 + 14\,s + 3}.$$

Finally, we can find compliance and relaxation function respectively using inverse

Laplace transform.

$$J_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s\mathcal{E}}\right\} = \mathcal{L}^{-1}\left\{\frac{7s^3+19s^2+14s+3}{s^2(6s^2+13s+5)}\right\}$$

$$R_0(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{E}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{6s^2+13s+5}{7s^3+19s^2+14s+3}\right\}$$

The main role plays a decomposition into partial fractions that is bases on finding roots of polynomial function in denominator of $\frac{1}{s\mathcal{E}}$ or $\frac{\mathcal{E}}{s}$ in the case of compliance function or relaxation function respectively. Finally, expressions for both functions are following:

$$J_0(t) = \frac{31}{25} + \frac{3t}{5} - \frac{1}{525}\,\mathrm{e}^{-\frac{5t}{3}} - \frac{1}{14}\,\mathrm{e}^{-\frac{t}{2}}$$

$$R_0(t) = -\frac{1}{257}\sum_{i\in\mathscr{A}}(217i^2+378i+25)e^{-it}$$

where the sum is over the three element set $\mathscr{A} = \{x \in \mathbb{R}; 7x^3+19x^2+14x+3=0\}$.

5 Conclusion

This work provides a description of algorithm for analysing viscoelastic models using Laplace transform. The aim is not provide exhausted source of information dealing with this problem as it include a large area of mathematical problems but it shows the main problems related to this area.

The next work could be done in optimizing algorithm for the use of more complicated models, further the calculation of relaxation and creep could be extended with harmonic stress or strain loading.

A Appendix

A.1 MATLAB algorithm returning stress relaxation and creep function

```
function [J,R]=viscoelasticity(B)
%function that calculate stress relaxation function and creep
%% input
syms s
%% MATRIX SETTINGS_V2
[N,n]=size(B);
P=max(max(double(B(:,[1,2]))));
B1=double(B(:,[1,2]));
%setting A1
A1=zeros(N,P-1);
for i=1:N
    A1(i,[B1(i,1):B1(i,2)-1])=-1;
end
%setting A2
A2=sym(zeros(N,N+2));
for i=1:N
    A2(i,i)=1/B(i,3)+1/(B(i,4)*s);
end
%setting A3
A3=zeros(P,P-1);
%setting A4
A4=zeros(P,N+2);
A4(1,:)=[-kronecker(ones(1,N),B1(:,1)'),1,0];
for i=2:P-1
    A4(i,1:N)=kronecker(i*ones(1,N),B1(:,2)')...
        -kronecker(i*ones(1,N),B1(:,1)');
end
A4(P,:)=[-kronecker(P*ones(1,N),B1(:,2)'),1,0];
%setting A5
A5=[-ones(1,P-1),zeros(1,N),0,1];
%setting A
A=[A1,A2;A3,A4;A5];
clear A1 A2 A3 A4 B1 i m n
%% Gauss elimination
[G]=gauss(A);
%% inverse Laplace transform
C=-G(P+N,P+N)/G(P+N,P+N+1);
if (isequal(class(C),'sym')==1)&&(isequal(findsym(C,1),'s')==1)
    syms t
```

```
    J=ilaplace(C/s,s,t);%compliance function
    R=ilaplace(1/(s*C),s,t);%relaxation function
end
end
```

A.2 MATLAB algorithm of Gaussian elimination

```
function [G]=gauss(A)
%gauss elimination
[m,n]=size(A);
j=1; %column
k=1; %row
if isequal(class(A),'sym')
    while (k<m)&&(j<=n)
        %urceni prvniho clenu
        index=find(A(k+1:m,j)~=0);
        if (A(k,j)==0)&&isequal(index,[])
            j=j+1;
            continue
        elseif A(k,j)==0
            A([k,k+index(1)],:)=A([k+index(1),k],:);%swaping of rows
            index=index(2:length(index)); %row
        end
        for i=k+index %row
                A(i,:)=simplify(A(i,:)-A(i,j)/A(k,j)*A(k,:));
        end
        j=j+1;
        k=k+1;
    end
else
    while (k<m)&&(j<=n)
        %urceni prvniho clenu
        index=find(A(k+1:m,j)~=0);
        if (A(k,j)==0)&&isequal(index,[])
            j=j+1;
            continue
        elseif A(k,j)==0
            M([k,k+index(1)],:)=M([k+index(1),k],:);%swaping of rows
            index=index(2:length(index)); %row
        end
        for i=k+index %row
                A(i,:)=A(i,:)-A(i,j)/A(k,j)*A(k,:);
        end
```

```
        j=j+1;
        k=k+1;
    end
end
G=A;
end
```

References

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