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Optimal time-consistent investment and reinsurance policies for mean-variance insurers☆

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ABSTRACT

This paper investigates the optimal time-consistent policies of an investment-reinsurance problem and an investment-only problem under the mean-variance criterion for an insurer whose surplus process is approximated by a Brownian motion with drift. The financial market considered by the insurer consists of one risk-free asset and multiple risky assets whose price processes follow geometric Brownian motions. A general verification theorem is developed, and explicit closed-form expressions of the optimal polices and the optimal value functions are derived for the two problems. Economic implications and numerical sensitivity analysis are presented for our results. Our main findings are: (i) the optimal time-consistent policies of both problems are independent of their corresponding wealth processes; (ii) the two problems have the same optimal investment policies; (iii) the parameters of the risky assets (the insurance market) have no impact on the optimal reinsurance (investment) policy; (iv) the premium return rate of the insurer does not affect the optimal policies but affects the optimal value functions; (v) reinsurance can increase the mean-variance utility.

1. Introduction

Insurers can control their risks by means of some business activities, such as investing in a financial market, purchasing reinsurance, and acquiring new business (acting as a reinsurer for other insurers). As a result, there have arisen many optimization problems with various objectives in insurance risk management. This topic has been extensively investigated in the literature. For example, Browne (1995) obtains the optimal investment strategies for an insurer who maximizes the expected utility of the terminal wealth or minimizes the ruin probability, where the surplus process of the insurer is modeled by a drifted Brownian motion. Yang and Zhang (2005) study the optimal investment policies for an insurer who maximizes the expected exponential utility of the terminal wealth or maximizes the survival probability, where the surplus process is driven by a jump–diffusion process. Further, Xu et al. (2008), Cao and Wan (2009) and Gu et al. (2010) study the optimal investment-reinsurance policies for an insurer who maximizes the expected utility of the terminal wealth in different situations.

Recently, many scholars consider the optimal investment and/or reinsurance policies for insurers under the mean-variance criterion, which is pioneered by Markowitz (1952) and has long been recognized as the milestone of modern portfolio theory. For example, Bäuerle (2005) considers the optimal proportional reinsurance/new business problem under the mean-variance criterion where the surplus process is modeled by the classical Cramér–Lundberg (CL) model, and derives the optimal policy in closed-form. Delong and Gerrard (2007) consider two optimal investment problems for an insurer: one is the classical mean-variance portfolio selection and the other is the mean-variance terminal objective involving a running cost penalizing deviation of the insurer's wealth from a specified profit-solvency target. They assume that the claim process is a compound Cox process with the intensity described by a drifted Brownian motion and the insurer invests in a financial market consisting of a risk-free asset and a risky asset whose price is driven by a Lévy process. Bai and Zhang (2008) study the optimal investment-reinsurance policies for an insurer under the mean-variance criterion by the linear quadratic

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(LQ) method and the dual method, where they assume that the surplus of the insurer is described by a CL model and a diffusion approximation (DA) model respectively. Zeng et al. (2010) assume that the surplus of an insurer is modeled by a jump–diffusion process, and derive the optimal investment policies explicitly under the benchmark and mean-variance criteria by the stochastic maximum principle.

It is apparent to all that the mean-variance criterion lacks the iterated-expectation property, which results in that continuous-time/multi-period mean-variance problems are time-inconsistent in the sense that the Bellman Optimality Principle does not hold and hence the traditional dynamic programming approach cannot be directly applied. The optimal policies to dynamic mean-variance problems considered in all the literature mentioned above are derived under the implicit assumption that the decision makers pre-commit themselves to follow in the future the policies chosen at the initial time, namely, the decision makers initially choose policies to maximize their objective functions at time 0 and thereafter do not deviate from these policies. Such policies are so-called pre-commitment policies, which are time-inconsistent in that they are optimal only when sitting at the initial time.

However, time consistency of policies is a basic requirement for rational decision making in many situations. A decision maker sitting at time t would consider that, starting from $t+\Delta t$, she will follow the policy that is optimal sitting at time $t+\Delta t$. Namely, the optimal policy derived at time t should agree with the optimal policy derived at time $t+\Delta t$. Strotz (1956) first analytically formalizes time inconsistency and works on time-consistent policies for time-inconsistent problems. He proposes that time-inconsistent problems can be solved either by pre-commitment policies or by time-consistent policies. In very recent times, time-inconsistent stochastic control problems have attracted much attention. Bjök and Murgoci (2009) develop a general theory for Markovian time-inconsistent stochastic control problems with fairly general objectives. They derive an extension of the standard Hamilton–Jacobi–Bellman (HJB) equation in the form of a system of non-linear PDFs. Wang and Forsyth (submitted for publication) study the time-consistent policy and the pre-commitment policy of a continuous-time mean-variance asset allocation problem and develop a numerical scheme which can determine the optimal policy whatever type of constraint is applied to the investment behavior. Bjök et al. (2010) consider a mean-variance portfolio optimization problem with state-dependent risk aversion in a continuous-time setting. Basak and Chabakauri (2010) study a dynamic mean-variance asset-allocation problem within a Wiener driven framework and derive the explicit time-consistent policy by solving the extended HJB equation.

As far as we know, there is no literature on the optimal investment and reinsurance problems for mean-variance insurers who are concerned about the time-consistent policies. In this paper we try to pioneer this study. Specifically, we consider the optimal time-consistent policies of an investment-reinsurance problem and an investment-only problem for a mean-variance insurer. In the first problem, the insurer is allowed to invest in a financial market and purchase proportional reinsurance/acquire new business. In the second problem, the insurer is only allowed to invest in a financial market but not allowed to purchase proportional reinsurance/acuqire new business. In both problems, the insurer is of mean-variance preference, the surplus process of the insurer is modeled by a DA model, and the financial market consists of one risk-free asset and multiple risky assets whose price processes are driven by geometric Brownian motions. We develop a general verification theorem and derive closed-form expressions for the optimal time-consistent policies and the optimal value functions of the two problems. We also present economic implications of our results and provide sensitivity analysis by a numerical example.

The rest of this paper is organized as follows. Section 2 describes the model and some assumptions. Section 3 formulates the optimization problems and gives a general verification theorem. The investment-reinsurance problem and the investment-only problem under the mean-variance criterion without pre-commitment are solved in Section 4. Section 5 provides a numerical sensitivity analysis and Section 6 concludes the paper.

2. Model and assumptions

We start with a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{P})$, where T is a finite and positive constant, representing the time horizon, $\mathcal{F}_t$ stands for the information available at time t, and any decision made at time t is based upon such information. All stochastic processes introduced below are supposed to be well-defined and adapted processes in this space.

2.1. Surplus process

We consider an insurer whose surplus process is modeled by a DA model. To understand the DA model better, it is advantageous to start from the classical CL model. In the CL model the claims arrive according to a homogeneous Poisson process $\{N_t\}$ with intensity λ; the individual claim sizes are $Z_i,\ i=1,2,\ldots$, which are assumed to be independent of $\{N_t\}$ and be independent and identically distributed (i.i.d.) positive random variables with finite first and second-order moments given by μ_∞ and σ_∞^2, respectively. Then the surplus process of the insurer without reinsurance and investment follows

$$\mathrm{d}R(t) = c\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N_t} Z_i, \tag{1}$$

where c is the premium rate which is assumed to be calculated according to the expected value principle, i.e., $c=(1+\eta)\lambda\mu_\infty$, and here $\eta>0$ is the relative safety loading of the insurer. By Grandll (1991), the CL model can be approximated by the following diffusion model

$$\mathrm{d}R(t) = \mu\mathrm{d}t + \sigma_0\mathrm{d}W_0(t), \tag{2}$$

where $\mu=\eta\lambda\mu_\infty$ can be regarded as the premium return rate of the insurer, $\sigma_0^2=\lambda\sigma_\infty^2$ measures the volatility of the insurer's surplus, $\{W_0(t)\}$ is a standard Brownian motion. It is worth pointing out that the DA model (2) works well for large insurance portfolios, where an individual claim is relatively small compared to the size of surplus. The DA model has been used in much existing literature, for example, Browne (1995), Promislow and Young (2005), Gerber and Shiu (2006), Bai and Guo (2008), Cao and Wan (2009), Chen et al. (2010), Gu et al. (2010), and so on.

In addition, the insurer is allowed to purchase proportional reinsurance or acquire new business (for example, acting as a reinsurer of other insurers, see Bäuerle (2005)) at each moment in order to control insurance business risk. The proportional reinsurance/new business level is associated with the value of risk exposure $a(t)\in[0,+\infty)$ at any time $t\in[0,T]$. $a(t)\in[0,1]$ corresponds to a proportional reinsurance cover and shows that the cedent should divert part of the premium to the reinsurer at the rate of $(1-a(t))\theta$, where θ can be regarded as the premium return rate of the reinsurer. Meanwhile, the insurer should pay $100a(t)\%$ while the rest $100(1-a(t))\%$ is paid by the reinsurer for each claim occurring at time t. The proportional reinsurance is called cheap if $\theta=\mu$ while being not cheap if $\theta>\mu$. $a(t)\in(1,+\infty)$ corresponds to acquiring new business. For convenience, we call the process of risk exposure $\{a(t):t\in[0,T]\}$ as a reinsurance policy. When a reinsurance policy $\{a(t):t\in[0,T]\}$ is adopted, the corresponding DA dynamics for the surplus process becomes

$$\mathrm{d}R(t) = [\mu-(1-a(t))\theta]\mathrm{d}t + \sigma_0 a(t)\mathrm{d}W_0(t). \tag{3}$$

2.2. Financial market

The financial market in the present paper is assumed to consist of one risk-free asset (bond or bank account) and n risky assets (stocks or mutual funds). Assume that the price process $S_0(t)$ of the risk-free asset evolves according to the ordinary differential equation (ODE)

$$\mathrm{d}S_0(t) = r_0(t)S_0(t)\mathrm{d}t, \quad S_0(0) = s_0, \tag{4}$$

where s_0 is the initial price; $r_0(t)$ is a positive continuous bounded deterministic function and represents the risk-free rate. And assume that the price processes $S_i(t)$ of the ith risky asset ($i = 1, 2, \ldots, n$) follows the geometric Brownian motion

$$\mathrm{d}S_i(t) = S_i(t)\left[r_i(t)\mathrm{d}t + \sum_{j=1}^{k}\sigma_{ij}(t)\mathrm{d}W_j(t)\right], \quad S_i(0) = s_i, \tag{5}$$

where s_i is the initial price of the ith risky asset; $r_i(t)$ and $\sigma_{ij}(t)$ are positive continuous bounded deterministic functions; $W(t) := (W_1(t), \ldots, W_k(t))'$ is a k-dimensional standard Brownian motion. Here the superscript "′" denotes the transpose of a matrix or vector and $k \geq n$. Assume that the process $\{W_0(t)\}$ is independent of the process $\{W(t)\}$ and $r_i(t) > r_0(t)$ for each $i = 1, 2, \ldots, n$ and $t \in [0, T]$.

2.3. Wealth process

Assume that the insurer can dynamically purchase proportional reinsurance/acquire new business and invest in the financial market over the time interval $[0, T]$ and that there is no transaction cost in the financial market and the insurance market. A policy is a stochastic process $\pi = \{(a^\pi(t), b^\pi(t)) : t \in [0, T]\}$, where $a^\pi(t)$ corresponds to the value of risk exposure at time t, $b^\pi(t) := (b_1^\pi(t), b_2^\pi(t), \ldots, b_n^\pi(t))'$, $b_i^\pi(t)$ is the dollar amount invested in the ith risky asset at time t. The dollar amount invested in the risk-free asset at time t is then $X^\pi(t) - \sum_{i=1}^{n} b_i^\pi(t)$, where $X^\pi(t)$ is the corresponding wealth process when a policy π is adopted. We assume that the initial wealth of the insurer is x_0. Then the dynamics for $X^\pi(t)$ is given by

$$\begin{aligned}\mathrm{d}X^\pi(t) &= [r_0(t)X^\pi(t) + \theta a^\pi(t) + r(t)'b^\pi(t) + m]\mathrm{d}t \\ &\quad + \sigma_0 a^\pi(t)\mathrm{d}W_0(t) + b^\pi(t)'\sigma(t)\mathrm{d}W(t),\end{aligned} \tag{6}$$

with $X^\pi(0) = x_0$, where $m = \mu - \theta$, $r(t) = (r_1(t) - r_0(t), r_2(t) - r_0(t), \ldots, r_n(t) - r_0(t))'$, $\sigma(t) = (\sigma_{ij}(t))_{n\times k}$. Moreover, denote $\Sigma(t) = \sigma(t)\sigma(t)'$ and assume that $\Sigma(t)$ is nonsingular for all $t \in [0, T]$. In this paper, a policy $\pi = \{(a^\pi(t), b^\pi(t))' : t \in [0, T]\}$ is said to be admissible if it satisfies the following conditions: (i) $\forall t \in [0, T]$, $(a^\pi(t), b^\pi(t))$ is $\mathscr{F}_t$ progressively measurable and $a^\pi(t) \geq 0$; (ii) $\int_0^T [a^\pi(t)^2\sigma_0^2 + b^\pi(t)'\Sigma(t)b^\pi(t)]\mathrm{d}t < +\infty$; (iii) the stochastic different Eq. (6) has a unique solution X^π on $[0, T]$. Denote by Π_1 the set of all admissible policies and $\Pi_2 = \{\pi \in \Pi_1 : a^\pi(t) \equiv 1, \forall t \in [0, T]\}$.

3. Problems formulation and verification theorem

In this paper, we consider two optimization problems for the insurer denoted by (P_{IR}) and (P_I), where (P_{IR}) is the investment-reinsurance problem and (P_I) is the investment-only problem. For (P_{IR}), we assume that the insurer is allowed to invest in the financial market described in Section 2.2 and purchase proportional reinsurance/acquire new business. For (P_I), we assume that the insurer only invests in the financial market rather than purchases reinsurance/acquires new business. In both problems, the insurer adopts the mean-variance criterion to choose the optimal policies. Quite different from the existing literature in this field, this paper aims to obtain the optimal time-consistent policies instead of the pre-commitment policies for the insurer.

In order to understand the mean-variance optimization problem without pre-commitment well, we start firstly from the mean-variance optimization problem with pre-commitment. A mean-variance optimization problem with pre-commitment can be described as maximizing

$$J_0(0, X_0, \pi) = \mathrm{E}_{0,x_0}[X^\pi(T)] - \frac{\gamma}{2}\mathrm{Var}_{0,x_0}[X^\pi(T)] \tag{7}$$

over all admissible policies, where $E_{t,x}[\cdot] = E[\cdot|X^\pi(t) = x]$, γ is a positive constant representing the degree of risk aversion of the insurer. The term "pre-commitment" involves the target given implicitly by considering the variance as the quadratic derivation from the target $E[X^\pi(T)]$, i.e., the insurer pre-commits to herself to the target $E_{0,x_0}[X^\pi(T)]$ determined at time 0 but does not update her target at subsequent dates.

However, for the mean-variance optimization problem without pre-commitment, the insurer updates her target at each state (t, x) and the value function is given by

$$\begin{aligned}J(t, x, \pi) &= \mathrm{E}_{t,x}[X^\pi(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x}[X^\pi(T)] \\ &= \mathrm{E}_{t,x}[X^\pi(T)] - \frac{\gamma}{2}\left(\mathrm{E}_{t,x}[X^\pi(T)^2] - (\mathrm{E}_{t,x}[X^\pi(T)])^2\right).\end{aligned} \tag{8}$$

The target of the insurer is to find the optimal value function

$$V(t, x) = \sup_{\pi\in\Pi} J(t, x, \pi), \tag{9}$$

and the optimal policy $\pi^* = \{(a^{\pi^*}(t), b^{\pi^*}(t)), t \in [0, T]\}$ such that $V(t, x) = J(t, x, \pi^*)$, where $\Pi = \Pi_1$ for (P_{IR}) and $\Pi = \Pi_2$ for (P_I).

We are now going to provide a verification theorem for the mean-variance problem (9) without pre-commitment. To generalize our verification theorem, we consider a general optimization problem of the form

$$V(t, x) = \sup_{\pi\in\Pi} f(t, x, y^\pi(t, x), z^\pi(t, x)), \tag{10}$$

where $f : [0, T] \times R^3 \to R$ is a function in $C^{1,2,2,2}$, and

$$y^\pi(t, x) = \mathrm{E}_{t,x}[X^\pi(T)], \tag{11}$$

$$z^\pi(t, x) = \mathrm{E}_{t,x}[X^\pi(T)^2]. \tag{12}$$

In particular, if we let

$$f(t, x, y, z) = y - \frac{\gamma}{2}(z - y^2), \tag{13}$$

then the problem (10) reduces to our concerned mean-variance optimization problem without pre-commitment, (9). In addition, if we set

$$f(t, x, y, z) = y - \frac{\gamma(t, x)}{2}(z - y^2),$$

then the problem (10) reduces to the mean-variance optimization problem without pre-commitment but with state-dependent risk aversion coefficient $\gamma(t, x)$. If we let

$$f(t, x, y, z) = y - \frac{\gamma(t, x)}{2}(z - y^2)^{\frac{1}{2}},$$

then the problem (10) reduces to the mean-standard derivation optimization problem without pre-commitment but with state-dependent risk aversion coefficient.

Theorem 1 (*Verification Theorem*). *For the optimization problem (10), if there exist three real value functions $F, G, H : [0, T] \times R \to R$ satisfying the following extended HJB system: $\forall (t, x) \in [0, T] \times R$,*

$$\sup_{\pi\in\Pi}\Big\{F_t - f_t + (F_x - f_x)[r_0(t)x + \theta a^\pi(t) + r(t)'b^\pi(t) + m] + \frac{1}{2}(F_{xx} - U)[a^\pi(t)^2\sigma_0^2 + b^\pi(t)'\Sigma(t)b^\pi(t)]\Big\} = 0, \quad (14)$$

$$F(T, x) = f(T, x, x, x^2), \quad (15)$$

$$G_t + G_x[r_0(t)x + \theta a^{\pi^*}(t) + r(t)'b^{\pi^*}(t) + m] + \frac{1}{2}G_{xx}[a^{\pi^*}(t)^2\sigma_0^2 + b^{\pi^*}(t)'\Sigma(t)b^{\pi^*}(t)] = 0, \quad (16)$$

$$G(T, x) = x, \quad (17)$$

$$H_t + H_x[r_0(t)x + \theta a^{\pi^*}(t) + r(t)'b^{\pi^*}(t) + m] + \frac{1}{2}H_{xx}[a^{\pi^*}(t)^2\sigma_0^2 + b^{\pi^*}(t)'\Sigma(t)b^{\pi^*}(t)] = 0, \quad (18)$$

$$H(T, x) = x^2, \quad (19)$$

where

$$U(f, y, z) = f_{xx} + 2f_{xy}y_x + 2f_{xz}z_x + f_{yy}y_x^2 + 2f_{yz}y_xz_x + f_{zz}z_x^2 \quad (20)$$

with $y = y^\pi(t, x)$ and $z = z^\pi(t, x)$ for short, and

$$\pi^* = \arg\sup_{\pi\in\Pi}\{F_t - f_t + (F_x - f_x)[r_0(t)x + \theta a^\pi(t) + r(t)'b^\pi(t) + m] + \frac{1}{2}(F_{xx} - U)[a^\pi(t)^2\sigma_0^2 + b^\pi(t)'\Sigma(t)b^\pi(t)]\}, \quad (21)$$

then $V(t, x) = F(t, x)$, $y^{\pi^*}(t, x) = G(t, x)$, $z^{\pi^*}(t, x) = H(t, x)$, and the optimal policy is given by π^*.

Proof. See Appendix. □

The verification theorem is suitable for (P_{IR}) if replace Π with Π_1, and for (P_I) if replace Π and $a^\pi(t)$ with Π_2 and 1, respectively.

4. Solving (P_{IR}) and (P_I)

This section works on solving the investment-reinsurance optimization problem (P_{IR}) and the investment-only optimization problem (P_I) under the mean-variance criterion without pre-commitment. In this case, the function f is given by (13), and hence we have

$$f_y = 1 + \gamma y, \qquad f_{yy} = \gamma, \qquad f_z = -\frac{\gamma}{2}, \qquad f_t = f_x = f_{xx} = f_{xy} = f_{xz} = f_{yz} = f_{zz} = 0. \quad (22)$$

4.1. Constructing the solution of investment-reinsurance problem (P_{IR})

In this subsection, the solution of the investment-reinsurance optimization problem (P_{IR}) is constructed. Assume that there exist three real value functions $F(t, x)$, $G(t, x)$ and $H(t, x)$ satisfying the extended HJB system (14)–(21) and $U > F_{xx}$ for all $(t, x) \in [0, T] \times R$. According to Theorem 1, we know by inserting (22) into (20) that

$$U(f, y^{\pi^*}, z^{\pi^*}) = U(f, G, H) = \gamma G_x^2, \quad (23)$$

and by (10) and (13) that

$$F(t, x) = V(t, x) = f\left(t, x, y^{\pi^*}, z^{\pi^*}\right) = E_{t,x}[X^{\pi^*}(T)] - \frac{\gamma}{2}\left(E_{t,x}[X^{\pi^*}(T)^2] - (E_{t,x}[X^{\pi^*}(T)])^2\right) = G(t, x) - \frac{\gamma}{2}(H(t, x) - G(t, x)^2),$$

which tells us that

$$H(t, x) = G(t, x)^2 + \frac{2}{\gamma}(G(t, x) - F(t, x)). \quad (24)$$

By differentiating the inside of the bracket of (14) with respect to $a^\pi(t)$ and $b^\pi(t)$ respectively, and by using (22) and (23), we can find the optimal policy

$$\pi^* = (a^{\pi^*}(t), b^{\pi^*}(t)), \qquad a^{\pi^*}(t) = -\frac{\theta F_x}{\sigma_0^2(F_{xx} - \gamma G_x^2)}, \qquad b^{\pi^*}(t) = -\frac{\Sigma(t)^{-1}r(t)F_x}{F_{xx} - \gamma G_x^2}, \quad (25)$$

with

$$F_t + (r_0(t)x + m)F_x - \frac{l(t)^2F_x^2}{2(F_{xx} - \gamma G_x^2)} = 0, \quad (26)$$

$$G_t + \left(r_0(t)x + m - \frac{l(t)^2F_x}{F_{xx} - \gamma G_x^2}\right)G_x + \frac{l(t)^2F_x^2G_{xx}}{2(F_{xx} - \gamma G_x^2)^2} = 0, \quad (27)$$

where $l(t)^2 = \theta^2/\sigma_0^2 + r(t)'\Sigma(t)^{-1}r(t)$.

In addition, given the linear structure of the dynamics, as well as the boundary conditions, it is natural to guess that

$$F(t, x) = A(t)x + B(t), A(T) = 1, B(T) = 0, \quad (28)$$

$$G(t, x) = \alpha(t)x + \beta(t), \alpha(T) = 1, \beta(T) = 0. \quad (29)$$

The partial derivatives are

$$F_t = A_tx + B_t, \qquad F_x = A(t), \qquad F_{xx} = 0,$$
$$G_t = \alpha_tx, \qquad G_x = \alpha(t), \qquad G_{xx} = 0.$$

Inserting them into (25)–(27), we have

$$\pi^* = (a^{\pi^*}(t), b^{\pi^*}(t)), \qquad a^{\pi^*}(t) = \frac{\theta A(t)}{\gamma\sigma_0^2\alpha(t)^2}, \qquad b^{\pi^*}(t) = \frac{\Sigma(t)^{-1}r(t)A(t)}{\gamma\alpha(t)^2}, \quad (30)$$

$$A_tx + B_t + (r_0(t)x + m)A(t) + \frac{l(t)^2A(t)^2}{2\gamma\alpha(t)^2} = 0, \quad (31)$$

$$\alpha_tx + \beta_t + (r_0(t)x + m)\alpha(t) + \frac{l(t)^2A(t)}{\gamma\alpha(t)} = 0. \quad (32)$$

By separating the variables with and without x, we can derive the following system of ODEs

$$A_t + r_0(t)A(t) = 0, \quad A(T) = 1;$$
$$B_t + mA(t) + \frac{l(t)^2A(t)^2}{2\gamma\alpha(t)^2} = 0, \quad B(T) = 0;$$
$$\alpha_t + r_0(t)\alpha(t) = 0, \quad \alpha(T) = 1;$$
$$\beta_t + m\alpha(t) + \frac{l(t)^2A(t)}{\gamma\alpha(t)} = 0, \quad B(T) = 0.$$

Then we can easily obtain

$$A(t)=\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}, \tag{33}$$

$$B(t)=m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u+\frac{1}{2\gamma}\int_t^T l(s)^2\mathrm{d}s, \tag{34}$$

$$\alpha(t)=\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}, \tag{35}$$

$$\beta(t)=m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u+\frac{1}{\gamma}\int_t^T l(s)^2\mathrm{d}s. \tag{36}$$

Inserting (33) and (35) into (30) leads to the optimal policy

$$\begin{aligned}&\pi^*=(a^{\pi^*}(t),b^{\pi^*}(t)),\qquad a^{\pi^*}(t)=\frac{\theta}{\gamma\sigma_0^2}\mathrm{e}^{-\int_t^T r_0(s)\mathrm{d}s},\\ &b^{\pi^*}(t)=\frac{\Sigma^{-1}(t)r(t)}{\gamma}\mathrm{e}^{-\int_t^T r_0(s)\mathrm{d}s}.\end{aligned} \tag{37}$$

Substituting (33) and (34) into (28) yields

$$\begin{aligned}F(t,x)&=x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\frac{1}{2\gamma}\int_t^T l(s)^2\mathrm{d}s.\end{aligned} \tag{38}$$

Inserting (35) and (36) into (29), we can get

$$\begin{aligned}G(t,x)&=x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\frac{1}{\gamma}\int_t^T l(s)^2\mathrm{d}s.\end{aligned} \tag{39}$$

According to (24), (38) and (39), we have

$$\begin{aligned}H(t,x)&=\left[x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u+\frac{1}{\gamma}\int_t^T l(s)^2\mathrm{d}s\right]^2\\ &\quad+\frac{1}{\gamma^2}\int_t^T l(s)^2\mathrm{d}s.\end{aligned} \tag{40}$$

According to the above results and Theorem 1, we have the following theorem.

Theorem 2. *For the investment-reinsurance problem (P_{IR}), the optimal policy is given by (37); the optimal value function is given by (38); the expected terminal wealth under the optimal policy is given by (39); and the variance of the terminal wealth under the optimal policy is given by*

$$\mathrm{Var}_{t,x}[X^{\pi^*}(T)]=\frac{2}{\gamma}(G(t,x)-F(t,x))=\frac{1}{\gamma^2}\int_t^T l(s)^2\mathrm{d}s. \tag{41}$$

Remark 1. We find that (1) the optimal policy does not depend on the wealth process $X^{\pi^*}(t)$; (2) the premium return rate of the insurer μ has no impact on the optimal policy but has impact on the value function; (3) the parameters of risky assets have no influence on the optimal reinsurance policy $a^{\pi^*}(t)$, and the parameters of the insurance market have no impact on the optimal investment policy $b^{\pi^*}(t)$; (4) when $\mu=\theta=\sigma_0=0$ and the parameters of the risk-free asset and risky assets are constants, our results are the same as Proposition 5.1 in Bjök and Murgoci (2009).

Remark 2. From (39) and (41), we can get the relationship between the expectation and the variance of the terminal wealth under the optimal policy of the problem (P_{IR}) as below:

$$\begin{aligned}\mathrm{E}_{t,x}[X^{\pi^*}(T)]&=x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+m\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]\int_t^T l(s)^2\mathrm{d}s}.\end{aligned}$$

This relationship is known as the efficient frontier of the problem (P_{IR}) at time t, as in the modern portfolio theory. The efficient frontier is a straight line in the mean-standard deviation plane, no matter at what time. Evidently, at any time, the cheaper the reinsurance (i.e., the large the m) is, the better (i.e., the higher) the efficient frontier will be. This is consistent with our intuition.

4.2. Constructing the solution of investment-only problem (P_I)

This subsection is devoted to constructing the solution of the investment-only optimization problem (P_I) under the mean-variance criterion without pre-commitment. In this problem, the insurer neither purchases reinsurance nor acquires new business, that is to say, the value of risk exposure $a^\pi(t)\equiv 1$ for all $t\in[0,T]$ and the set of all admissible policies is Π_2. When an admissible policy $\pi\in\Pi_2$ is adopted, the dynamics for the wealth process $X^\pi(t)$ is given by

$$\begin{aligned}\mathrm{d}X^\pi(t)&=[r_0(t)X^\pi(t)+\mu+r(t)'b^\pi(t)]\mathrm{d}t\\ &\quad+\sigma_0\mathrm{d}W_0(t)+b^\pi(t)'\sigma(t)\mathrm{d}W(t),\ X^\pi(0)=x_0.\end{aligned} \tag{42}$$

Similar to the previous subsection, for the investment-only problem we can derive the optimal investment policy and three real value functions $\tilde{F},\tilde{G},\tilde{H}$ satisfying (14)–(21) in Theorem 1, which are described as below:

$$\pi^*=(1,b^{\pi^*}(t)),\qquad b^{\pi^*}(t)=\frac{\Sigma(t)^{-1}r(t)}{\gamma}\mathrm{e}^{-\int_t^T r_0(s)\mathrm{d}s}, \tag{43}$$

$$\begin{aligned}\tilde{F}(t,x)&=x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+\mu\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad-\frac{\gamma\sigma_0^2}{2}\int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\frac{1}{2\gamma}\int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s,\end{aligned} \tag{44}$$

$$\begin{aligned}\tilde{G}(t,x)&=x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+\mu\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\frac{1}{\gamma}\int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s,\end{aligned} \tag{45}$$

$$\begin{aligned}\tilde{H}(t,x)&=\Bigg[x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s}+\mu\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u\\ &\quad+\frac{1}{\gamma}\int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s\Bigg]^2\\ &\quad+\frac{1}{\gamma^2}\int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s\\ &\quad+\sigma_0^2\int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u.\end{aligned} \tag{46}$$

According to Theorem 1, we have the following result.

Theorem 3. *For the investment-only problem (P_I), the optimal policy is given by (43); the optimal value function is given by (44); the expected terminal wealth under the optimal policy is given by (45); and the variance of the terminal wealth under the optimal policy is given by*

$$\mathrm{Var}_{t,x}[X^{\pi^*}(T)] = \frac{1}{\gamma^2}\int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s + \sigma_0^2\int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u. \quad (47)$$

Remark 3. (1) The optimal investment policy $b^{\pi^*}(t)$ of the investment-only problem (P_I) is the same as that of the investment-reinsurance problem. This property implies that the optimal investment and reinsurance strategy can be separated and the economic implications on the optimal investment strategy stated after Theorem 2 apply to this model as well. (2) When $\mu = \sigma_0 = 0$ and the parameters of the risk-free asset and risky assets are constants, our results are again the same as Proposition 5.1 of Bjök and Murgoci (2009).

Remark 4. From (45) and (47), we can get the relationship between the expectation and the variance of the terminal wealth under the optimal policy of the problem (P_I) as below: $\mathrm{E}_{t,x}[X^{\pi^*}(T)]$ is given in Box I where $\mathrm{Var}_{t,x}[X^{\pi^*}(T)] \geq \sigma_0^2 \int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u$, which is guaranteed by (47). This efficient frontier of the investment-only problem (P_I) at time t is no longer a straight line but a hyperbola in the mean-standard derivation plane. It is a straight line only when $\sigma_0 = 0$. This reveals that due to the existence of the volatility σ_0 of the insurer's surplus process, the risk of insurance business cannot be completely hedged by only investment.

Corollary 1. *The optimal value function of the investment-reinsurance problem (P_{IR}) is larger than that of the investment-only problem (P_I). In other words, reinsurance can increase mean-variance utility.*

Proof. According to (38) and (44), it follows that

$$\begin{aligned} F(t,x) - \tilde{F}(t,x) &= \frac{1}{2\gamma}\int_t^T \frac{\theta^2}{\sigma_0^2}\mathrm{d}u - \theta\int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u \\ &\quad + \frac{\gamma\sigma_0^2}{2}\int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u \\ &= \frac{1}{2\gamma}\int_t^T \left(\frac{\theta}{\sigma_0} - \gamma\sigma_0\mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\right)^2 \mathrm{d}u \geq 0. \quad \square \end{aligned}$$

5. Numerical sensitivity analysis

In this section, we provide a numerical example to analyze how the parameters of the insurance market and the coefficient of the insurer's risk aversion impact on the optimal time-consistent policies, the optimal value functions and the efficient frontiers of the investment-reinsurance problem and the investment-only problem. For convenience but without loss of generality, this example only considers one risky asset and the parameters of the financial market are constants. Throughout the numerical analysis, unless otherwise stated, the basic parameters are given by: $\mu = 0.5, \sigma_0 = 1, \theta = 0.8, \gamma = 0.6, r_0 = 0.05, r_1 = 0.1, \sigma_1 = 0.2, T = 10, t = 0, x_0 = 1$.

5.1. Impact of parameters on the optimal policies

This subsection works on analyzing how the parameters of the insurance market and the coefficient of the insurer's risk aversion impact on the optimal time-consistent policies of the two optimization problems.

Fig. 1 shows that the optimal dollar amount invested in the risky asset and the optimal reinsurance proportion both increase with respect to (w.r.t.) time t, namely, as time elapses, the insurer should invest more money in the risky asset and keep more insurance business. In addition, the subgraph (a) illustrates that the optimal investment policy $b^{\pi^*}(t)$ is decreasing w.r.t. the coefficient of risk aversion γ, i.e., the more the insurer dislikes risk, the less amount the insurer invests in the risky asset; the subgraph (b) tells us that the optimal reinsurance policy $a^{\pi^*}(t)$ is also decreasing w.r.t. the coefficient of risk aversion γ, that is to say, the more risk averse the insurer is, the less insurance business the insurer keeps; the subgraph (c) displays that the optimal reinsurance policy $a^{\pi^*}(t)$ decreases with the volatility σ_0 of the insurer's surplus, namely, when the risk of the insurer's surplus becomes bigger, the insurer will keep less insurance business; the subgraph (d) reveals that the optimal reinsurance policy $a^{\pi^*}(t)$ is increasing w.r.t. the premium return rate θ of the reinsurer, i.e., the more expensive the reinsurance is, the more insurance business the insurer keeps.

5.2. Impact of parameters on the optimal value functions

This subsection is devoted to analyzing how the parameters of the insurance market and the coefficient of the insurer's risk aversion impact on the optimal value functions of the two problems. For simplicity but without loss of generality, we only consider the optimal value functions at time 0.

The subgraphs (a), (b) and (c) in Fig. 2 all demonstrate that the optimal value function of the investment-reinsurance problem (P_{IR}) is larger than that of the investment-only problem (P_I) no matter how much of the values of the parameters, as shown in Corollary 1. Further, the subgraph (a) indicates that the two optimal value functions are both decreasing w.r.t. the risk aversion coefficient γ, namely, the more risk averse the insurer, the smaller the optimal mean-variance utilities in both two problems. The subgraph (b) shows that the two optimal value functions are increasing w.r.t. the premium return rate μ of the insurer, which implies that the higher the premium return rate of the insurer, the larger the optimal utilities of the two problems. The subgraph (c) illustrates that as the volatility σ_0 of the insurer's surplus increases, the two optimal value functions both decrease, i.e., the more the surplus risk of the insurer, the less the optimal utilities of the two problems. The subgraph (d) depicts the impact of the premium return rate θ of the reinsurer on the optimal value function of the investment-reinsurance problem (P_{IR}), which shows that as the premium return rate θ of the reinsurer increases, the optimal value function first decreases and then increases, implying that both higher and lower premium return rates of the reinsurer can yields higher optimal utilities and there is a reinsurance premium return rate that minimizes the optimal utilities.

5.3. Impact of parameters on the efficient frontiers

In this subsection, we analyze how the parameters of the insurance market impact on the efficient frontiers of the two problems. Without loss of generality we only consider the efficient frontiers at time 0.

In Fig. 3, the subgraph (a) shows that the efficient frontiers of the two problems both move up as the premium return rate μ of the insurer increases, that is, the higher the premium return rate of the insurer, the bigger the expected terminal wealth with the same variance of the terminal wealth; the subgraph (b) demonstrates that as the volatility σ_0 of the insurer's surplus increases, the efficient frontiers of the two problems both move down, in other words, the larger the insurer's surplus volatility, the higher the variance of the terminal wealth with the same expected terminal wealth; both the subgraphs (a) and (b) show that the efficient frontier of the investment-reinsurance problem is better (higher) than that of the investment-only problem; the subgraph (c) displays that the slope of the efficient frontier for the

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = x\mathrm{e}^{\int_t^T r_0(s)\mathrm{d}s} + \mu \int_t^T \mathrm{e}^{\int_u^T r_0(s)\mathrm{d}s}\mathrm{d}u + \sqrt{\left(\mathrm{Var}_{t,x}[X^{\pi^*}(T)] - \sigma_0^2 \int_t^T \mathrm{e}^{\int_u^T 2r_0(s)\mathrm{d}s}\mathrm{d}u\right) \int_t^T r(s)'\Sigma(s)^{-1}r(s)\mathrm{d}s}$$

Box I.

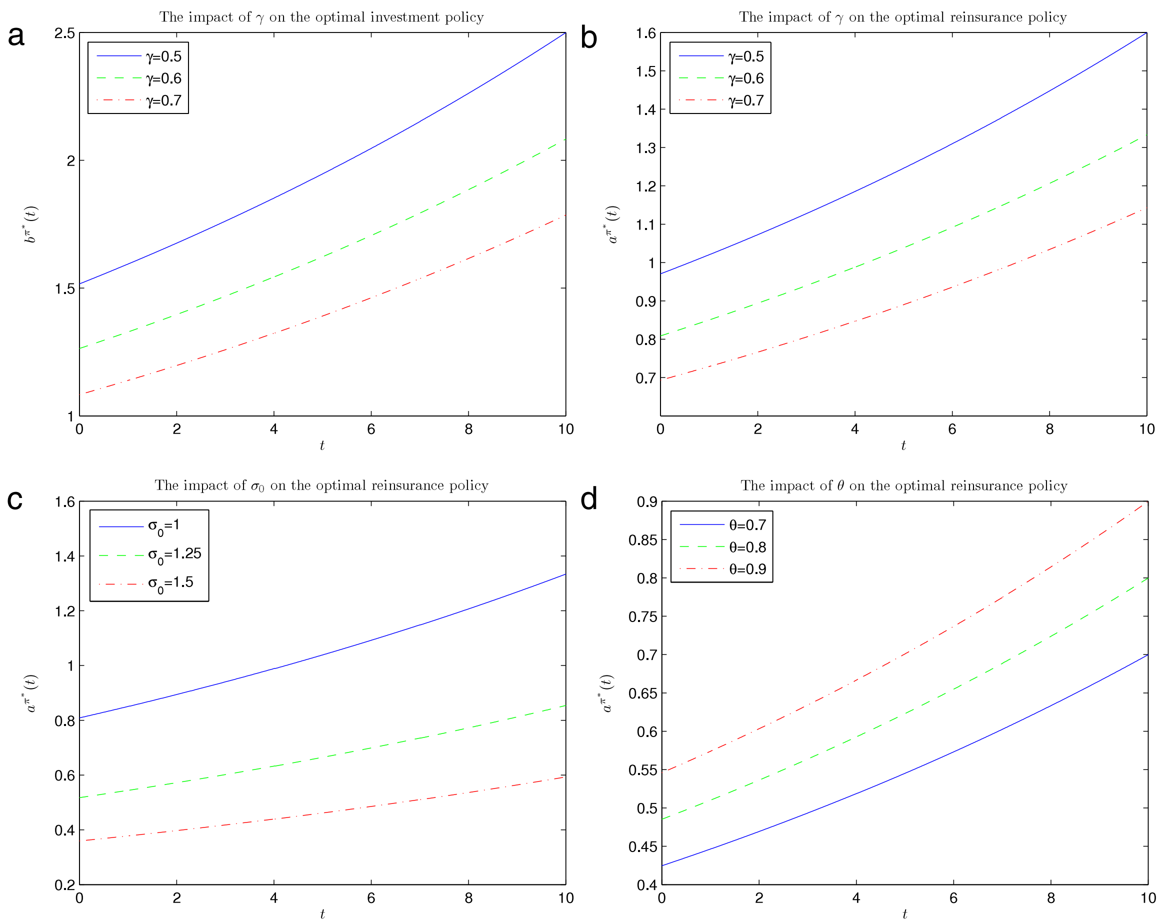

Fig. 1. The impact of parameters on the optimal investment policy and the optimal reinsurance policy.

investment-reinsurance problem is increasing with the premium return rate θ of the reinsurer, namely, the higher the premium return rate of the reinsurer, the larger the expected terminal wealth with the same variance of the terminal wealth.

6. Conclusions

In this paper we study two mean-variance optimization problems with time-consistent policies for an insurer. One is an investment-reinsurance problem and the other is an investment-only problem. In the first problem the insurer is allowed to invest in a financial market and purchase proportional reinsurance/acquire new business. In the second problem the insurer is only allowed to invest in a financial market but not allowed to purchase reinsurance/acquire new business. The surplus process of the insurer is assumed to follow a DA model and the financial market consists of one risk-free asset and multiple risky assets whose price processes are governed by geometric Brownian motions. A verification theorem is developed for a more general optimization problem including the two problems as special cases. And explicit closed-form expressions for the optimal time-consistent policies and the optimal value functions of the two problems are derived. Impact of the parameters of the insurance market and the coefficient of the insurer's risk aversion on the optimal policies, the optimal value functions and the efficient frontiers are analyzed by a numerical example. Some interesting results are found through our theoretical derivation and numerical sensitivity analysis.

We are the first to study time-consistent investment and reinsurance policies for mean-variance insurers. Our work is just a basic framework. There are still many works needed to be investigated in this direction. For example, (1) as the insurer updates its policy continuously in our time-inconsistent problems, it may be more interesting to consider a time and state-dependent coefficient of risk aversion, instead of a constant one, in order to analyze how time-inconsistent risk aversions modify the optimal policy when the risk aversion is constant. (2) In our problems the time horizon is pre-given and fixed. It may be interesting to take into account an uncertain exit time. (3) This paper assumes that the risky assets' price processes are driven by diffusion processes in order to derive closed-form solutions, it is noteworthy to extend this work to a jump–diffusion case because the real financial markets are often of such cases. (4) It may be also of interest to relax the assumption that the risk source of reinsurance market is independent of the risk source of the financial market.

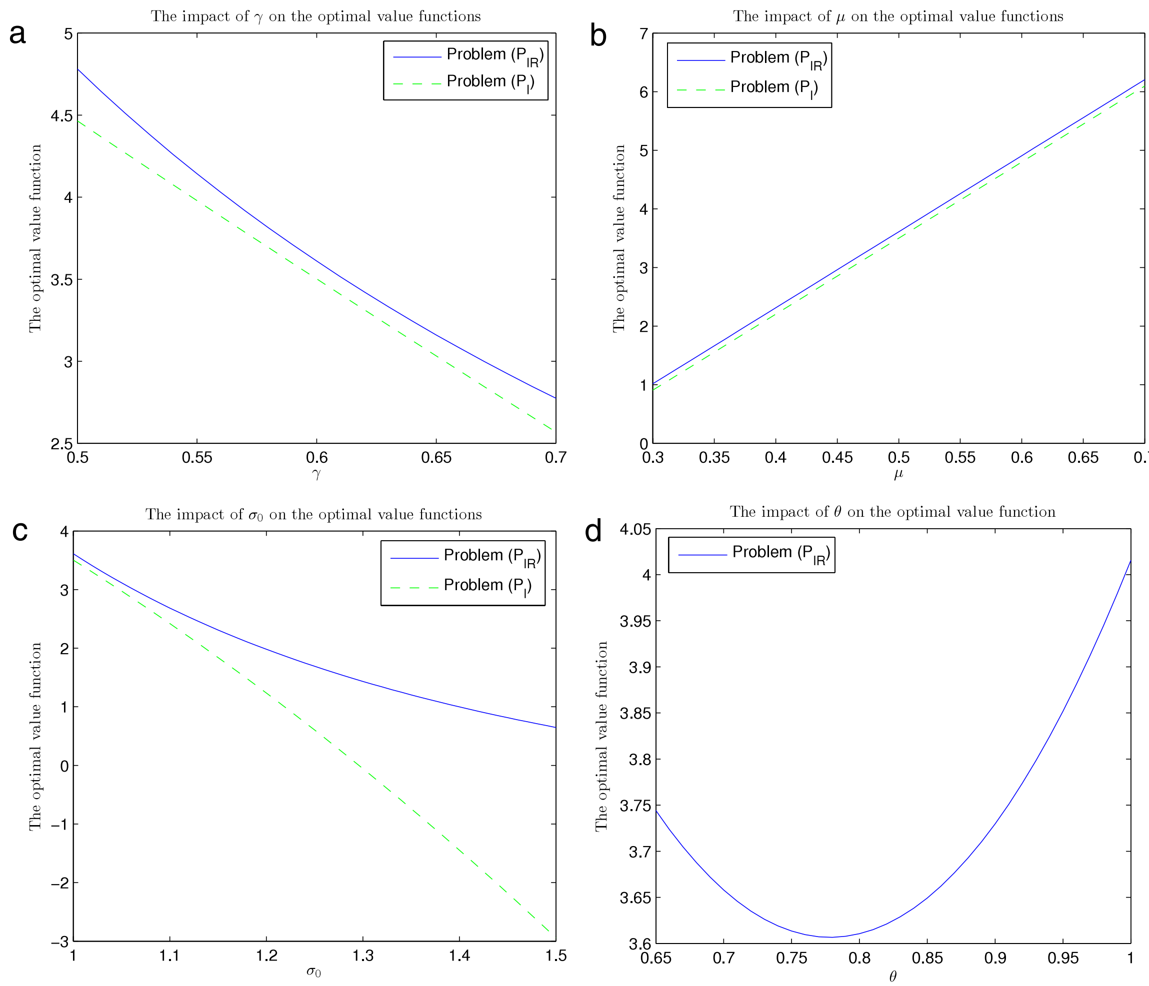

Fig. 2. The impact of parameters on the optimal value functions of the investment-reinsurance problem and the investment-only problem.

(5) This paper measures the risk of the terminal wealth by its variance. One can consider other risk measures, such as Value-at-Risk. (6) In addition to investment and reinsurance, insurers may be concerned with consumption. It is also worth investigating the optimal investment-consumption-reinsurance policies under a time-consistent framework.

Appendix

Proof of Theorem 1. We prove it in a similar way as the one of Kryger and Steffensen (2010).

(i) Consider an arbitrary admissible policy π.

First, we claim that if there exists a real value function $Y^\pi(t,x)$ such that $\forall(t,x)\in[0,T]\times R$,

$$Y_t^\pi + Y_x^\pi[r_0(t)x+\theta a^\pi(t)+r(t)'b^\pi(t)+m] + \frac{1}{2}Y_{xx}^\pi[a^\pi(t)^2\sigma_0^2+b^\pi(t)'\Sigma(t)b^\pi(t)]=0, \tag{48}$$

$$Y^\pi(T,x)=x, \tag{49}$$

then

$$Y^\pi(t,x)=y^\pi(t,x). \tag{50}$$

In fact, according to the Itô formula and the Eq. (6), we have

$$\begin{aligned}
Y^\pi(t,X^\pi(t)) &= Y^\pi(T,X^\pi(T)) - \int_t^T \mathrm{d}Y^\pi(s,X^\pi(s)) \\
&= Y^\pi(T,X^\pi(T)) - \int_t^T \Big\{Y_s^\pi + Y_x^\pi[r_0(s)X^\pi(s)+\theta a^\pi(s) + r(s)'b^\pi(s)+m] + \frac{1}{2}Y_{xx}^\pi[a^\pi(s)^2\sigma_0^2 + b^\pi(s)'\Sigma(s)b^\pi(s)]\Big\}\mathrm{d}s \\
&\quad - \int_t^T Y_x^\pi[a^\pi(s)\sigma_0\mathrm{d}W_0(s)+b^\pi(s)'\sigma(s)\mathrm{d}W(s)].
\end{aligned}$$

Substituting (48) and (49) into the above equation gives

$$Y^\pi(t,X^\pi(t)) = X^\pi(T) - \int_t^T Y_x^\pi[a^\pi(s)\sigma_0\mathrm{d}W_0(s) + b^\pi(s)'\sigma(s)\mathrm{d}W(s)].$$

Taking conditional expectation on both sides yields

$$Y^\pi(t,x)=E_{t,x}[Y^\pi(t,X^\pi(t))]=E_{t,x}[X^\pi(T)]=y^\pi(t,x).$$

Similarly (replacing Y and y by Z and z, respectively), we can show the claim that if there exists a real value function $Z^\pi(t,x)$ such that $\forall(t,x)\in[0,T]\times R$,

$$Z_t^\pi + Z_x^\pi[r_0(t)x+\theta a^\pi(t)+r(t)'b^\pi(t)+m] + \frac{1}{2}Z_{xx}^\pi[a^\pi(t)^2\sigma_0^2+b^\pi(t)'\Sigma(t)b^\pi(t)]=0, \tag{51}$$

$$Z^\pi(T,x)=x^2, \tag{52}$$

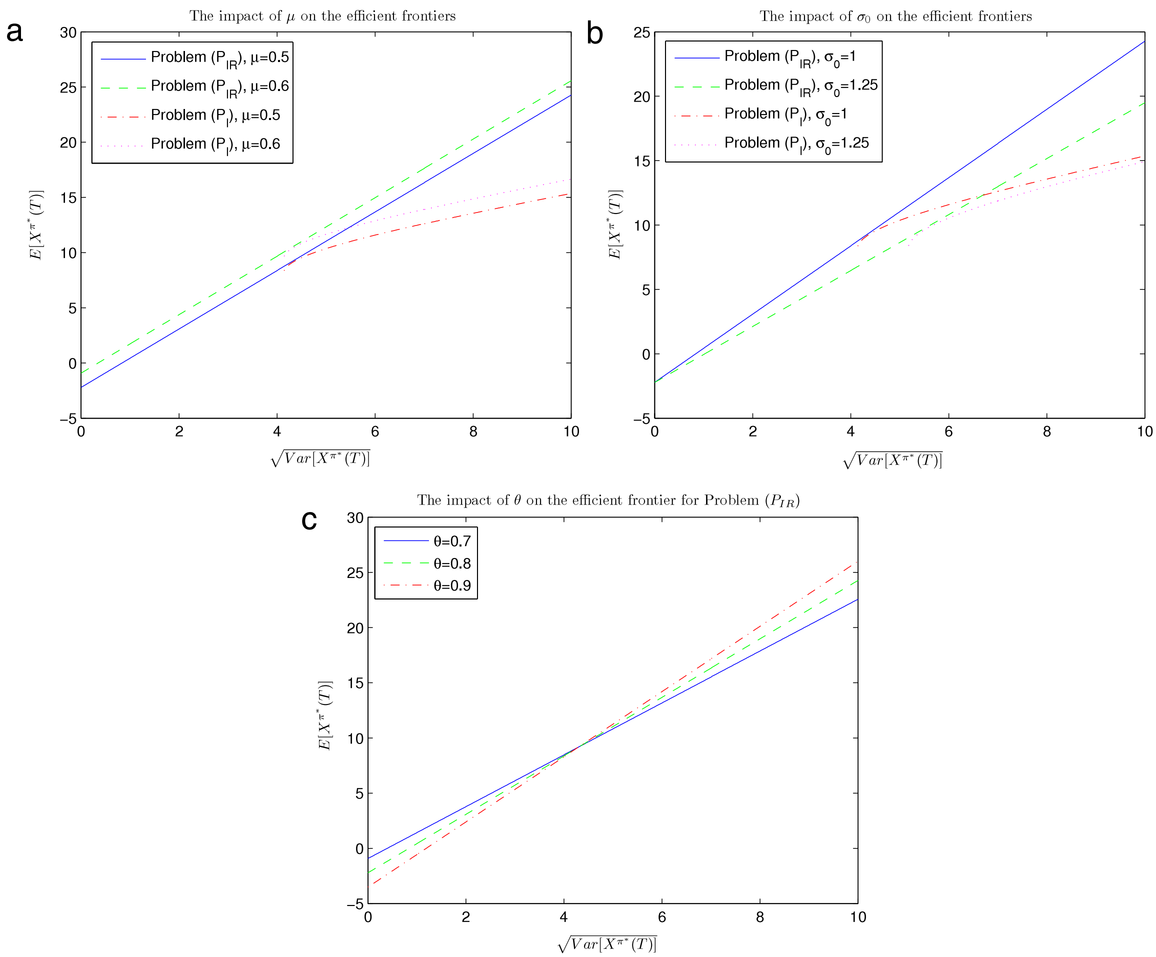

Fig. 3. The impact of parameters on the efficient frontiers of the investment-reinsurance problem and the investment-only problem.

then

$$Z^\pi(t,x) = z^\pi(t,x). \tag{53}$$

Second, given $Y^\pi(t,x)$ and $Z^\pi(t,x)$ that satisfy (48)–(49) and (51)–(52) respectively, we are going to give an expression for

$$f(T, X^\pi(T), Y^\pi(T, X^\pi(T)), Z^\pi(T, X^\pi(T))).$$

From (50) and (53), it follows that

$$\begin{aligned} &f(t, X^\pi(t)), y^\pi(t, X^\pi(t)), z^\pi(t, X^\pi(t)) \\ &\quad = f(t, X^\pi(t), Y^\pi(t, X^\pi(t)), Z^\pi(t, X^\pi(t))). \end{aligned}$$

Since $f \in C^{1,2,2,2}$, by the Itô formula and the Eq. (6), we have

$$\begin{aligned} &f\big(T, X^\pi(T), Y^\pi(T, X^\pi(T)), Z^\pi(T, X^\pi(T))\big) \\ &\quad = f\big(t, X^\pi(t), y^\pi(t, X^\pi(t)), z^\pi(t, X^\pi(t))\big) \\ &\qquad + \int_t^T \mathrm{d}f\big(s, X^\pi(s), Y^\pi(s, X^\pi(s)), Z^\pi(s, X^\pi(s))\big) \\ &\quad = f\big(t, X^\pi(t), y^\pi(t, X^\pi(t)), z^\pi(t, X^\pi(t))\big) \\ &\qquad + \int_t^T \Big\{f_s + f_y Y_s^\pi + f_z Z_s^\pi + (f_x + f_y Y_x^\pi + f_z Z_x^\pi) \\ &\qquad \times \big[r_0(s)X^\pi(s) + \theta a^\pi(s) + r(s)'b^\pi(s) + m\big] \\ &\qquad + \frac{1}{2}\big[f_{xx} + 2f_{xy}Y_x^\pi + 2f_{xz}Z_x^\pi + f_{yy}(Y_x^\pi)^2 \\ &\qquad + 2f_{yz}Y_x^\pi Z_x^\pi + f_{zz}(Z_x^\pi)^2 \\ &\qquad + f_y Y_{xx}^\pi + f_z Z_{xx}^\pi\big]\big[a^\pi(s)^2\sigma_0^2 + b^\pi(s)'\Sigma(s)b^\pi(s)\big]\Big\}\mathrm{d}s \\ &\qquad + \int_t^T (f_x + f_y Y_x^\pi + f_z Z_x^\pi)\big[a^\pi(s)\sigma_0 \mathrm{d}W_0(s) \\ &\qquad + b^\pi(s)'\sigma(s)\mathrm{d}W(s)\big]. \end{aligned}$$

Substituting (48) and (51) into the above formula, we obtain

$$\begin{aligned} &f\big(T, X^\pi(T), Y^\pi(T, X^\pi(T)), Z^\pi(T, X^\pi(T))\big) \\ &\quad = f\big(t, X^\pi(t), y^\pi(t, X^\pi(t)), z^\pi(t, X^\pi(t))\big) \\ &\qquad + \int_t^T \Big\{f_s + f_x\big[r_0(s)X^\pi(s) + \theta a^\pi(s) + r(s)'b^\pi(s) + m\big] \\ &\qquad + \frac{1}{2}U(f, Y^\pi, Z^\pi)\big[a^\pi(s)^2\sigma_0^2 + b^\pi(s)'\Sigma(s)b^\pi(s)\big]\Big\}\mathrm{d}s \\ &\qquad + \int_t^T (f_x + f_y Y_x^\pi + f_z Z_x^\pi)\big[a^\pi(s)\sigma_0 \mathrm{d}W_0(s) \\ &\qquad + b^\pi(s)'\sigma(s)\mathrm{d}W(s)\big]. \end{aligned} \tag{54}$$

Third, based on (54) we show that

$$F(t,x) \geq \sup_{\pi\in\Pi} f(t, x, y^\pi(t,x), z^\pi(t,x)). \tag{55}$$

Applying the Itô formula to F and using (6), we have

$$\begin{aligned} F(t, X^\pi(t)) &= F(T, X^\pi(T)) - \int_t^T \mathrm{d}F(s, X^\pi(s)) \\ &= F(T, X^\pi(T)) - \int_t^T \Big\{F_s + F_x\big[r_0(s)X^\pi(s) \\ &\quad + \theta a^\pi(s) + r(s)'b^\pi(s) + m\big] \end{aligned}$$

$$+\frac{1}{2}F_{xx}\big[a^{\pi}(s)^2\sigma_0^2+b^{\pi}(s)'\Sigma(s)b^{\pi}(s)\big]\Big\}\mathrm{d}s - \int_t^T F_x\big[a^{\pi}(s)\sigma_0\mathrm{d}W_0(s) + b^{\pi}(s)'\sigma(s)\mathrm{d}W(s)\big]. \quad (56)$$

Moreover, (14) implies that $\forall(t,x)\in[0,T]\times R$,

$$F_t \le f_t - [r_0(t)x+\theta a^{\pi}(t)+r(t)'b^{\pi}(t)+m](F_x-f_x) - \frac{1}{2}[a^{\pi}(t)^2\sigma_0^2+b^{\pi}(t)'\Sigma(t)b^{\pi}(t)](F_{xx}-U). \quad (57)$$

Inserting (15) with $x=X^{\pi}(T)$ and (57) with $x=X^{\pi}(t)$ into (56) yields

$$F(t,X^{\pi}(t)) \ge f\big(T,X^{\pi}(T),X^{\pi}(T),X^{\pi}(T)^2\big) - \int_t^T \Big\{f_s+f_x[r_0(s)X^{\pi}(s)+\theta a^{\pi}(s)+r(s)'b^{\pi}(s)+m] + \frac{1}{2}U(f,Y^{\pi},Z^{\pi})[a^{\pi}(s)^2\sigma_0^2+b^{\pi}(s)'\Sigma(s)b^{\pi}(s)]\Big\}\mathrm{d}s - \int_t^T F_x\big[a^{\pi}(s)\sigma_0\mathrm{d}W_0(s)+b^{\pi}(s)'\sigma(s)\mathrm{d}W(s)\big]. \quad (58)$$

According to the Eqs. (49) and (52),

$$Y^{\pi}(T,X^{\pi}(T)) = X^{\pi}(T), \qquad Z^{\pi}(T,X^{\pi}(T)) = X^{\pi}(T)^2.$$

With them, substituting (54) into (58) leads to

$$F(t,X^{\pi}(t)) \ge f\big(t,X^{\pi}(t),y^{\pi}(t,X^{\pi}(t)),z^{\pi}(t,X^{\pi}(t))\big) + \int_t^T (f_x+f_yY_x^{\pi}+f_zZ_x^{\pi}-F_x) \times\big[a^{\pi}(s)\sigma_0\mathrm{d}W_0(s)+b^{\pi}(s)'\sigma(s)\mathrm{d}W(s)\big]. \quad (59)$$

On both sides of the above formula, taking conditional expectation and thereafter supremum over Π, we obtain formula (55).

(ii) Consider the specific policy π^*.

First, according to the assumption of the theorem, functions G and H satisfy the conditions of the two claims in (i) with policy π^*. Hence, $G(t,x)=y^{\pi^*}(t,x)$ and $H(t,x)=z^{\pi^*}(t,x)$.

Second, for the specific policy π^*, the inequalities (57)–(59) become equations. Hence we have

$$F(t,X^{\pi}(t)) = f\big(t,X^{\pi^*}(t),y^{\pi^*}(t,X^{\pi^*}(t)),z^{\pi^*}(t,X^{\pi^*}(t))\big) + \int_t^T (f_x+f_yY_x^{\pi^*}+f_zZ_x^{\pi^*}-F_x) \times\big[a^{\pi^*}(s)\sigma_0\mathrm{d}W_0(s)+b^{\pi^*}(s)'\sigma(s)\mathrm{d}W(s)\big]. \quad (60)$$

Taking conditional expectation on both sides yields

$$F(t,x) = f\big(t,X^{\pi^*}(t),y^{\pi^*}(t,X^{\pi^*}(t)),z^{\pi^*}(t,X^{\pi^*}(t))\big) \le \sup_{\pi\in\Pi} f(t,x,y^{\pi}(t,x),z^{\pi}(t,x)). \quad (61)$$

This together with (55) finally gives

$$F(t,x) = f\big(t,X^{\pi^*}(t),y^{\pi^*}(t,X^{\pi^*}(t)),z^{\pi^*}(t,X^{\pi^*}(t))\big) = \sup_{\pi\in\Pi} f(t,x,y^{\pi}(t,x),z^{\pi}(t,x)).$$

This means that π^* is the optimal policy and the supremum is given by $F(t,x)$. □

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