



## Optimal investment decisions when time-horizon is uncertain

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Received 6 May 2005; received in revised form 5 May 2007; accepted 25 September 2007

Available online 6 January 2008

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### Abstract

Many investors do not know with certainty when their portfolio will be liquidated. Should their portfolio selection be influenced by the uncertainty of exit time? In order to answer this question, we consider a suitable extension of the familiar optimal investment problem of Merton [Merton, R.C., 1971. Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3, 373–413], where we allow the conditional distribution function of an agent's time-horizon to be stochastic and correlated to returns on risky securities. In contrast to existing literature, which has focused on an independent time-horizon, we show that the portfolio decision is affected.

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*Keywords:* Uncertain time-horizon; Dynamic portfolio selection

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There is an interesting discrepancy between finance theory and practice. On the one hand, most of standard financial economics is based on the assumption that, at the moment of making an investment decision, an investor knows with certainty the time of eventual exit. Such an assumption can be traced back to the origins of modern financial economics, and, in particular, to the development of portfolio selection theory by Markowitz (1952). On the other hand, most investors would acknowledge the fact that, upon entering the market, they never know with certainty the time of exiting the market. Factors which can potentially affect the time of exit are, for example, securities markets behavior, changes in the opportunity set, uncertainty of order execution time, changes in an investor's endowment, or time of an exogenous shock to an investor's consumption process (e.g., purchasing or selling of a house), for example.

Since it is obvious that an investment horizon is hardly ever known with certainty at the date when the initial investment decisions are made, it is both of practical and theoretical interest to develop a comprehensive theory of optimal investment and consumption under uncertain time-horizon. One would expect uncertainty over time-of-exit to be a serious complication, because, as underlined in the afore-mentioned examples, it can be in general dependent on risky securities returns. Existing literature, however, has only focused on an independent time-horizon, and, as a

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result, has led to the rather misleading conclusion that the optimal portfolio selection is essentially not affected by uncertainty over exit time. Research on the subject started as early as Yaari (1965), who addresses the problem of optimal consumption for an individual with uncertain date of death, in a simple setup with a pure deterministic investment environment. Hakansson (1969, 1971) extends this work to a discrete-time setting under uncertainty including risky assets. Merton (1971), as a special case, also addresses a dynamic optimal portfolio selection problem for an investor retiring at an uncertain date, defined as the date of the first jump of an independent Poisson process with constant intensity. This was done in a continuous-time setting with no bequest motive. In a subsequent related work, Richard (1975) solves in closed-form an optimal portfolio choice problem with uncertain time of death and the presence of life insurance. In all these papers, the random time-horizon is assumed to be independent of all other sources of uncertainty, and as a result the presence of a random time-horizon is shown to have no impact on the optimal portfolio decision.<sup>1</sup> To the best of our knowledge, the only exception is Karatzas and Wang (2001), who solve the optimal dynamic investment problem in the case of complete markets and when the uncertain time-horizon is a stopping time of asset price filtration. This, however, is also an extremely stylized assumption, which in essence states that randomness of time-horizon is fully dependent upon asset prices and induces no new uncertainty in the economy.

The present paper can be seen as an attempt to cover some of the ground between these two extreme assumptions, an independent time-horizon on the one hand, and a time-horizon that is a stopping time of the asset price filtration on the other hand. Our results can be summarized as follows. We first provide an elegant sufficient condition for optimality in the presence of an uncertain time-horizon. We then apply the result to solve the optimal investment problem in a setup with CRRA preferences, constant expected return and drift parameters, and a deterministic distribution function of time-horizon. In this framework, we confirm and extend a result obtained by Merton (1971) and Richard (1975), as we show that the optimal portfolio selection is not affected by the presence of an uncertain time-horizon, even though the value function is not identical to the one corresponding to the standard fixed-horizon case. One of our contributions is to obtain explicit solutions not only for the optimal strategy but also for the wealth process as a function of the distribution of the uncertain-time-horizon in the case of CRRA utility functions. We also consider the case of an economy with an infinite time span. We then strongly depart from existing literature by relaxing the assumption of an independent time-horizon. In a stylized model where the conditional distribution function of the random time-horizon is assumed to follow an Itô process correlated to stock returns, we confirm that a serious complication occurs in that the portfolio decision is affected. For CRRA preferences, we show however that a solution formally similar to the one obtained in the case of a constant time-horizon can be recovered at the cost of a suitable adjustment to the drift process of the risky assets. We find that, if the probability of exiting the market increases (respectively, decreases) with the return on the risky asset, then the fraction invested in the risky asset is lower than (respectively, greater than) in the case of a certain time-horizon. Our results have natural interpretations and important potential implications for optimal investment decisions when exit time can be impacted by returns on risky securities. As such they extend to the dynamic continuous-time setting recent findings by Martellini and Urošević (2006), who show in a static mean-variance setting that when the exit time is dependent on asset returns, the set of optimal portfolios becomes a function of the exit time distribution so that the standard Markowitz portfolio selection method leads in general to sub-optimal portfolio allocations.<sup>2</sup>

The rest of the paper is organized as follows. In Section 1, we introduce the model of an economy with an uncertain time-horizon. Section 2 is devoted to the problem of optimal dynamic investment decision in the presence of an uncertain time-horizon when the conditional probability of exiting is deterministic. In Section 3, we present explicit solutions in the case of CRRA utility functions. In Section 4, we discuss the case of a non-bounded time-horizon. In Section 5, we extend the setup to a stochastically time-varying conditional distribution of the time-horizon. Our conclusions are presented in Section 6.

## 1. The economy

In this section, we introduce a general model for the economy in the presence of an uncertain time-horizon. Let  $[0, T]$ , with  $T \in \mathbb{R}_*^+$ , denote the (finite) time span of the economy. Uncertainty in the economy is described through a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  on which is defined a  $n$ -dimensional standard Brownian motion  $W$ .

<sup>1</sup> Somewhat related also are recent papers by Collin Dufresne and Hugonnier (2004) and Blanchet-Scaillet et al. (2005) who study the problem of valuation and hedging of cash-flows affected by some event occurring at a random time.

<sup>2</sup> See also Huang et al. (forthcoming) for an analysis of static portfolio decisions when worst-case CVaR is used as a risk measure.

1.1. Asset prices

We consider  $n$  risky assets, the prices of which,  $S_t^i$ ,  $i = 1, \dots, n$ , are given by

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \right), \quad i = 1, \dots, n \tag{1}$$

We shall sometimes use the shorthand notation  $\mu = (\mu^i)_{i=1, \dots, n}$  and  $\sigma = (\sigma^{i,j})_{i,j=1, \dots, n}$ . A risk-free asset is also traded in the economy. The return on that asset, typically a default free bond, is given by  $(dB_t/B_t) = r_t dt$ , where  $r$  is the risk-free rate in the economy. Agents' basic information set is captured by the filtration  $\mathbb{F} = \{\sigma(S_s, s \leq t); t \geq 0\}$ , with  $\mathcal{F}_\infty \subset \mathcal{A}$  and  $\mathcal{F}_0$  is trivial. We also assume that:

- (i) the coefficients  $\mu, r$  are bounded and deterministic and  $r_t \geq 0$ ,
- (ii) the coefficient  $\sigma$  is bounded, invertible, deterministic and the inverse  $\sigma^{-1}$  is also a bounded function,
- (iii)  $W = (W^i)_{i=1, \dots, n}$  is an  $\mathbb{F}$ -Brownian motion.

Under these assumptions, the market is arbitrage-free (see for example, Karatzas, 1996). We denote by  $\mathbb{P}^0$  the equivalent martingale measure which the Radon–Nikodym density with respect to  $\mathbb{P}$ ,  $Z$ , is the solution to  $dZ_t = -Z_t \theta_t dW_t$ , where  $\theta_t = \sigma_t^{-1}(\mu_t - \mathbf{1}r_t)$ , where  $\mathbf{1}$  is a  $n$ -dimensional vector of ones.

A  $n$ -dimensional process  $\pi$  is said to be a *weak admissible* strategy if  $\pi$  is an  $\mathbb{F}$ -predictable almost surely square integrable process, i.e., if  $\sum_{i=1}^n \int_0^T (\pi_s^i)^2 ds < +\infty$ , almost surely. Consider now an investor who uses a weak admissible strategy  $\pi$ , i.e., invests the amount  $\pi_s^i$  in each of the risky securities at date  $s$ . If the strategy  $\pi$  is used in a self-financing way, i.e., if the wealth invested in the riskless asset is  $X_s - \sum_{i=1}^n \pi_s^i$ , then the wealth process  $(X_s^{t,\pi,x}, t \leq s \leq T)$ , with  $X_t^{t,\pi,x} = x$ , evolves according to the following stochastic differential equation

$$dX_s^{t,\pi,x} = X_s^{t,\pi,x} r_s ds + \pi_s [(\mu_s - \mathbf{1}r_s) ds + \sigma_s dW_s]. \tag{2}$$

**Definition 1.** A portfolio  $(\pi_t, 0 \leq t \leq T)$  is said to be admissible if it is a weak admissible strategy and if the associated wealth process is non-negative.

Let  $P(t, x)$  denote the set of admissible portfolios starting from a wealth level  $x$  at date  $t$

$$P(t, x) = \{\pi_u, t \leq u \leq T, X_t^\pi = x, \pi \text{ is weak admissible and } X_u^\pi \geq 0\}$$

where  $X_u^\pi := X_u^{t,\pi,x}$ .

1.2. Timing uncertainty

In this paper, we assume that an agent's time-horizon  $\tau$ , i.e., “the maximum length of time for which the investor gives any weight in his utility function” (Merton, 1975), is a positive random variable measurable with respect to the sigma-algebra  $\mathcal{A}$ .

Importantly, we do *not* assume that  $\tau$  is a stopping time of the filtration  $\mathbb{F}$  generated by asset prices. In other words, we do not assume that observing asset prices up to date  $t$  implies full knowledge about whether  $\tau$  has occurred or not by time  $t$ . Formally, it means that there are some dates  $t \geq 0$  such that the event  $\{t < \tau\}$  does not belong to  $\mathcal{F}_t$ . When  $\tau$  is an  $\mathbb{F}$ -stopping time, e.g., the first hitting time of a deterministic barrier by asset prices, it is possible, although sometimes difficult, to apply the standard tools of dynamic valuation and optimization problems (see Karatzas and Wang, 2001). In this paper, we are instead interested in situations such that the presence of an uncertain time-horizon induces some *new* uncertainty in the economy.<sup>3</sup>

<sup>3</sup> This does not imply that a random time under consideration in this paper may not be dependent upon asset prices behaviors. On the contrary, it is, in general, dependent upon asset prices, yet not depend *only* on asset prices as in the case of a stopping time of the asset price filtration.

There are two sources of uncertainty related to optimal investment in the presence of an uncertain time-horizon, one stemming from the randomness of prices (market risk), the other stemming from the randomness of the timing of exit  $\tau$  (timing risk). A serious complication, which we explicitly address in Section 4, is that, in general, these two sources of uncertainty are not independent. Separating out these two sources of uncertainty is a useful operation that may be achieved as follows. Conditioning upon  $\mathcal{F}_t$  allows one to isolate a pure timing uncertainty component. Since  $\mathcal{F}_t$  contains information about risky asset prices up to time  $t$ ,  $\mathbb{P}[\tau > t|\mathcal{F}_t]$ , for example, is the probability that the agent has not reached his time-horizon at date  $t$ , given all possible information about asset prices. We denote by  $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ , the conditional distribution function of timing uncertainty.

We further make the following assumption.

**[G]** :  $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$  is an increasing absolutely continuous process with respect to Lebesgue measure, with a density denoted by  $f$ , e.g.,  $F_t = \int_0^t f_s ds$ .

**Remark 1.** A sufficient condition for **(G)** to hold is  $\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty)$ , i.e., when  $\tau$  is modelled as a Cox process. On the other hand, one can find examples such that **(G)** holds and  $\mathbb{P}(\tau \leq t|\mathcal{F}_t) \neq \mathbb{P}(\tau \leq t|\mathcal{F}_\infty)$  (cf. Nikeghbali and Yor, 2005).

### 1.3. Preferences

The agent's preferences are captured by an utility function  $U$ , which is a continuous, strictly increasing, strictly concave and continuously differentiable function defined on  $(0, \infty) \rightarrow \mathbb{R}$ , satisfying the following two conditions:  $\lim_{x \rightarrow +\infty} U'(x) = 0$  and  $\lim_{x \rightarrow 0} U'(x) = +\infty$ . Under these assumptions, the function  $U'$  is invertible; the inverse function is denoted as  $I$  defined on  $\mathbb{R}_*^+$ . The agent's portfolio problem choice problem is to find an admissible strategy  $\pi$  which maximizes the expected utility of terminal wealth

$$V(x) = \sup_{\pi \in P(0,x)} \mathbb{E}[U(X_{\tau \wedge T}^{\pi,x})] \tag{3}$$

where  $\tau$  is an agent uncertain time-horizon, e.g., the date of death of the agent.

Using  $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ , and a result from Dellacherie (1972), this problem can be re-written as

$$V(x) = \sup_{\pi \in P(0,x)} \mathbb{E} \left[ \int_0^\infty U(X_{t \wedge T}^{\pi,x}) dF_t \right] = \sup_{\pi \in P(0,x)} \mathbb{E} \left[ \int_0^T U(X_u^{\pi,x}) dF_u + U(X_T^{\pi,x})(1 - F_T) \right]$$

## 2. Deterministic probability of exiting

In this section, we consider the case where the process  $(F_t, t \geq 0)$  is a deterministic function, hence equals the cumulative function of  $\tau$ , with a derivative  $f$ . Note that a necessary and sufficient condition for the conditional distribution  $F_t$  to be a deterministic function of time is to have  $\tau$  independent of  $\mathbb{F}$ , the case on which previous literature has focused on. As in the case of fixed time-horizon, the problem can be solved either using dynamic programming or the martingale/duality approach to utility maximization (Cox and Huang, 1989; Karatzas et al., 1987).

### 2.1. Dynamic programming approach

Using the density  $f$  of random time  $\tau$ , we may re-write the value function at the initial date as

$$V(0, x) = \sup_{\pi \in P(0,x)} \mathbb{E} \left[ \int_0^\infty f(t)U(X_{t \wedge T}^{\pi,x}) dt \right]$$

Let us introduce the value function  $V$  at any time  $t$

$$V(t, x) = \max \mathbb{E} \left\{ \int_t^T ds f(s)U(X_s^{t,\pi,x}) + (1 - F(T))U(X_T^{t,\pi,x}) \right\} \tag{4}$$

**Lemma 1.** For all  $t$  in  $[0, T]$ , the function  $V(t, \cdot)$  is increasing and strictly concave.

**Proof.** If  $x \leq y$ , then  $P(t, x) \subset P(t, y)$  and the result is based on the properties of increasing and concavity of the function  $U$  and the linearity property of  $X$  in  $x$ .  $\square$

We recall some facts on dynamic programming (cf. Fleming and Soner (1993), p. 163, Theorem 3.1).

**Theorem 1.**

(i) If  $Y$  is the  $C^{1,2}$  solution to the following Hamilton–Jacobi–Bellman equation

$$0 = f(t)U(x) + \left( \frac{\partial Y(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^n} A(t, x, \pi) \right) \tag{5}$$

where  $A(t, x, \pi) = [xr_t + \pi(\mu_t - r_t)]Y_x(t, x) + (1/2)\pi^2\sigma_t^2 Y_{xx}(t, x)$  and  $Y_x$  (resp.  $Y_{xx}$ ) denotes the first (resp. second) derivative with respect to the space variable, the boundary condition being

$$Y(T, x) = U(x)(1 - F(T)),$$

then  $V = Y$ .

(ii) If the conditions of (i) hold, the optimal portfolio strategy is given by

$$\pi_t = -\sigma_t^{-1}\theta_t \frac{V_x(t, x)}{V_{xx}(t, x)}$$

These conditions are the usual optimality conditions. One still needs to solve Eq. (5) subject to the appropriate boundary condition to obtain an explicit solution to the problem of optimal investment when time-horizon is uncertain, which is not easy in general. The martingale approach to optimal investment problem allows us to provide an explicit characterization of the optimal wealth process.

2.2. Martingale approach—sufficient condition

The following theorem, which is the main result of this section, provides a sufficient condition for optimality.<sup>4</sup>

**Theorem 2.** Define  $I$  as the inverse of the first derivative of the utility function, i.e.,  $I(x) = (U')^{-1}(x)$ . If  $I$  is  $C^2$  and if there exists a deterministic function  $v$  satisfying  $I(v(0)) = x$  such that the process  $(H_t I(v(t)H_t), t \geq 0)$  is a martingale, then the wealth process  $X^*$  defined by  $X_t^* = I(v(t)H_t)$  is optimal and the portfolio strategy  $\pi^*$  is defined by

$$\pi_t^* = -I'(v(t)H_t)v(t)H_t\sigma_t^{-1}\theta_t$$

where  $H_t = \exp(-\int_0^t r_s ds)Z_t$ .

**Proof.** Let  $X$  be a non-negative wealth process with initial value  $x$ . The concavity of  $U$  implies that

$$\mathbb{E}[U(X_{\tau \wedge T}) - U(X_{\tau \wedge T}^*)] \leq \mathbb{E}[(X_{\tau \wedge T} - X_{\tau \wedge T}^*)U'(X_{\tau \wedge T}^*)]$$

Now,

$$\begin{aligned} \mathbb{E}[(X_{\tau \wedge T} - X_{\tau \wedge T}^*)U'(X_{\tau \wedge T}^*)] &= \mathbb{E} \left[ \int_0^T f(t)(X_t - X_t^*)v(t)H_t dt + (1 - F(T))(X_T - X_T^*)v(T)H_T \right] \\ &= \int_0^\infty dt f(t)v(t)E[(X_t - X_t^*)H_t] + (1 - F(T))v(T)E[(X_T - X_T^*)H_T] \leq 0 \end{aligned}$$

<sup>4</sup> See Bouchard and Pham (2004) for a related result.

since  $HX^*$  is a martingale and  $HX$  a supermartingale with same initial value  $x$  (and using the fact that  $f$  and  $v$  are two deterministic functions of time). The optimal portfolio  $\pi^*$  is obtained by applying Itô's lemma to the process  $I(v(t)H_t)$  and identifying the  $dW$  terms.  $\square$

This result provides an explicit characterization of the optimal strategy. One needs, however, to first verify the conditions for existence of such a function  $v$ . It is in general difficult to give a closed-form solution for  $v$ . One can actually show that for constant coefficients only CRRA utility functions are consistent with the existence of  $v$ .

**Proposition 1.** *If  $r \neq 0$  and if the coefficients  $\mu, r, \sigma$  are constant, the existence of a differentiable function  $v$  satisfying  $I(v(0)) = x$  such that the process  $H_t I(v(t)H_t)$  is a martingale can only be obtained for logarithmic and power utility functions  $U$ .*

**Proof.** To see this, first apply Itô's formula to the process  $H_t I(v(t)H_t)$  for a deterministic function  $v$  to obtain

$$dH_t I(\xi_t) = H_t \theta [-\xi_t I'(\xi_t) - I(\xi_t)] dW_t + \{-r H_t I(\xi_t) + I'(\xi_t) \xi_t H_t (\theta^t \theta - r) + H_t^2 I'(\xi_t) v'(t) + \frac{1}{2} \xi_t^2 H_t \theta^t \theta I''(\xi_t)\} dt$$

where we denote by  $\xi_t = v(t)H_t$ .

The martingale condition implies that (after dividing by  $H$ )

$$-r I(\xi_t) + I'(\xi_t) \xi_t (\theta^t \theta - r) + \frac{1}{2} \xi_t^2 \theta^t \theta I''(\xi_t) + I'(\xi_t) \xi_t \frac{v'(t)}{v(t)} = 0 \tag{6}$$

For a fixed  $t$ ,  $\{\xi_t = v(t)H_t(\omega), \omega \in \Omega\}$  describes the set  $\mathbb{R}^+$ . So, if the coefficients  $r, \mu, \sigma$  are constant, then Eq. (6) has a deterministic solution  $v$  if and only if the function  $I$  satisfies the following equation:

$$-r I(z) + \frac{1}{2} z^2 \theta^t \theta I''(z) = C z I'(z), \quad \forall z \geq 0 \text{ and } \frac{v'}{v} = \text{constant}$$

where  $C$  is a constant. If  $r \neq 0$ , this equation is only satisfied for those functions  $I$  which are power functions, which corresponds to the logarithmic and power utility functions  $U$ . If  $r = 0$ , this equation is only satisfied for those functions  $I'$  which are power functions, which corresponds to the power, logarithmic and the exponential utility functions  $U$ .  $\square$

A generalization of this result is as follows.

**Proposition 2.**

- (a) *If the problem (3) admits a solution  $X^*$ , then  $X_t^* = I(v_t H_t)$  where  $v$  is an adapted process which satisfies*
  - (i) *the process  $H_t I(v_t H_t)$  is a local martingale,*
  - (ii) *the random variable  $\int_0^T v_s f(s) ds + (1 - F(T)) H_T^{-1} U'(X_T^*)$  is a constant.*
- (b) *If there exists an adapted process  $v$  such that (i) and (ii) hold, then  $X_t = I(v_t H_t)$  is an optimal solution of problem (3).*

**Remark 2.** Before giving the proof, notice that the condition (ii) can be written as

$$\int_0^T H_s^{-1} U'(X_s^*) f(s) ds + (1 - F(T)) H_T^{-1} U'(X_T^*)$$

which is exactly the first-order Lagrange equation. It can be directly obtained using duality principle.

**Proof.**

- (a) Setting  $v_t = H_t^{-1} U'(H_t X_t^*)$ , we obtain  $X_t^* = I(v_t H_t)$ , and since  $X_t^*$  is a wealth process ( $H_t X_t^* = H_t I(v_t H_t); t \geq 0$ ) is a local martingale. Let  $X$  be another wealth process, i.e., a process such that  $(X_t H_t, t \geq 0)$  is a local martingale with initial value  $x$ . The function  $\Phi$  defined as

$$\Phi(\epsilon) = \mathbb{E} \left( \int_0^T U(\epsilon X_s^* + (1 - \epsilon) X_s) f(s) ds + (1 - F(T)) U'(\epsilon X_T^* + (1 - \epsilon) X_T) \right)$$

admits a maximum for  $\epsilon = 1$ . From

$$\begin{aligned} \Phi'(1) &= \mathbb{E} \left( \int_0^T f(s)U'(X_s^*)(X_s^* - X_s) ds + (1 - F(T))U'(X_T^*)(X_T^* - X_T) \right) \\ &= \mathbb{E} \left( \int_0^T f(s)U'(X_s^*)H_s^{-1}(H_s X_s^* - H_s X_s) ds + (1 - F(T))U'(X_T^*)H_T^{-1}(H_T X_T^* - H_T X_T) \right) \end{aligned}$$

and using the fact that  $(H_s X_s^* - H_s X_s, s \geq 0)$  is a martingale, we get

$$\Phi'(1) = \mathbb{E} \left( H_T(X_T^* - X_T) \left[ \int_0^T U'(X_s^*)H_s^{-1} f(s) ds + (1 - F(T))U'(X_T^*)H_T^{-1} \right] \right) = 0.$$

This equality holds for any  $X_T \in \mathcal{F}_T$  such that  $\mathbb{E}(X_T H_T) = x$ . Setting

$$Z = \int_0^T U'(X_s^*)H_s^{-1} f(s) ds + (1 - F(T))U'(X_T^*)H_T^{-1}$$

leads to  $\mathbb{E}(H_T Y Z) = 0$  for any  $Y \in \mathcal{F}_T$  such that  $\mathbb{E}(Y H_T) = 0$ , or  $\mathbb{E}(\tilde{Y} Z) = 0$  for any  $\tilde{Y} \in \mathcal{F}_T$  such that  $\mathbb{E}(\tilde{Y}) = 0$ . For  $\tilde{Y} = Z - \mathbb{E}(Z)$ , we get  $\mathbb{E}(Z^2) = \mathbb{E}^2(Z)$ , therefore  $Z$  is a constant.

It remains to note that the definition of  $\nu$  leads to

$$Z = \int_0^T U'(X_s^*)H_s^{-1} f(s) ds + (1 - F(T))U'(X_T^*)H_T^{-1} = \int_0^T \nu_s f(s) ds + (1 - F(T))U'(I(\nu_T H_T))H_T^{-1}.$$

(b) Let  $X$  be a wealth process, with the initial wealth  $x$  and  $X_t^* = I(\nu_t H_t)$ . Then

$$\begin{aligned} &\mathbb{E} \left( \int_0^T dt f(t)[U(X_t) - U(X_t^*)] + (1 - F(T))(U(X_T) - U(X_T^*)) \right) \\ &\leq \mathbb{E} \left( \int_0^T dt f(t)U'(X_t^*)(X_t - X_t^*) + (X_T - X_T^*)(1 - F(T))U'(X_T^*) \right) \\ &= \mathbb{E} \left( \int_0^T dt f(t)\nu_t H_t(X_t - X_t^*) + (X_T - X_T^*)(1 - F(T))U'(X_T^*) \right) \\ &= \mathbb{E} \left( H_T(X_T - X_T^*) \left[ \int_0^T dt f(t)\nu_t + \frac{(1 - F(T))U'(X_T^*)}{H_T} \right] \right) = 0. \quad \square \end{aligned}$$

In what follows, we provide an explicit solution to the optimal dynamic investment problem in the presence of an uncertain time-horizon for various preference specifications, including logarithmic and power utility functions.

### 3. Examples

For simplicity, we focus on the case of a one-dimensional Brownian motion and constant coefficients  $\mu, r$  and  $\sigma$ .

**Proposition 3.** *Let us assume that  $U(x) = \ln x$ . The value function and the optimal investment policy are respectively given by*

$$V(t, x) = p(t) + q(t) \ln x$$

where  $q(t) = 1 - F(t)$ ,  $p(t) = a[(T - t) - \int_t^T F(s) ds]$ ,  $a = r + (1/2)\theta^2$ , and

$$X_t^* = \frac{x}{H_t}, \quad \pi_t^* = \frac{\mu - r}{\sigma^2} X_t^*$$

**Proof.** Using Theorem 2 and Proposition 1, obviously  $v(t) = x$  is a solution to (6), hence the optimal wealth is  $X_t^* = x/H_t$  and the optimal portfolio is  $((\mu - r)/\sigma^2)X_t^*$ .

We are looking for a function  $V(t, x)$  that can be written as  $V(t, x) = p(t) + q(t) \ln x$ . Introducing the expression of the optimal portfolio given in (ii) in Eq. (5), we obtain the following equation for  $p$  and  $q$

$$0 = f(t) \ln x + p'(t) + q'(t) \ln x + q(t) \left[ r + \frac{1}{2}\theta^2 \right]$$

A sufficient and necessary condition for this equation to be satisfied is

$$q'(t) = -f(t), \quad p'(t) = -q(t) \left[ r + \frac{1}{2}\theta^2 \right]$$

which is equivalent to

$$q(t) = \lambda_1 - F(t), \quad p(t) = \lambda_2 - a \int_0^t du \{ \lambda_1 - F(u) \}$$

where  $\lambda_1$  and  $\lambda_2$  are two real numbers and  $a = r + (1/2)\theta^2$ . Using the boundary condition  $V(T, x) = \ln(x)(1 - F(T))$ , we get  $p(T) = 0$  and  $q(T) = 1 - F(T)$  from which we obtain  $\lambda_1 = \int_0^{+\infty} f(s) ds = 1$  and  $\lambda_2 = a \int_0^T du \{ 1 - F(u) \} = a(T - \int_0^T F(u) du)$ .  $\square$

We also obtain an explicit solution in the case of power utility.

**Proposition 4.** Let us assume that  $U(x) = (x^\alpha/\alpha)$  with  $0 < \alpha < 1$ . Then the value function is

$$V(t, x) = q(t) \frac{x^\alpha}{\alpha}$$

with  $q(t) = e^{-at} [e^{aT}(1 - F(T)) + \int_t^T e^{as} f(s) ds]$ ,  $a = r - (\theta^2/2)(\alpha - 1)$ .

The optimal wealth and portfolio are

$$X_t^* = (v(t)H_t)^\beta, \quad \pi_t^* = -\frac{\mu - r}{(\alpha - 1)\sigma^2} X_t^*$$

where  $v(t) = x \exp[-((\beta + 1)/\beta)(-r + (\beta/2)\theta^2)t]$  and  $\beta = 1/(\alpha - 1)$ .

**Proof.** The explicit form of  $v(t)$  follows from (6). The optimal wealth is  $X_t^* = (v(t)H_t)^\beta$  and the optimal portfolio is  $-((\mu - r)/(\alpha - 1)\sigma^2)X_t^*$ .

We are looking for a function  $V(t, x)$  that can be written as  $V(t, x) = q(t)U(x)$ . We thus obtain the differential equation which is satisfied by  $q$

$$0 = f(t) + q'(t) + q(t) \left[ r - \frac{\theta^2}{2(\alpha - 1)} \right]$$

Hence  $q(t) = e^{-at} [\gamma - \int_0^t e^{as} f(s) ds]$ . From the boundary condition  $q(T) = 1 - F(T)$ , we obtain  $q(t) = e^{-at} [e^{aT}(1 - F(T)) + \int_t^T e^{as} f(s) ds]$ .  $\square$

We find that the optimal strategies coincide to these obtained when the time-horizon is fixed, whatever the distribution  $F$  of the time-horizon. In the simple case of deterministic distribution of the time-horizon and coefficients, it is a striking result that the portfolio strategy expressed in term of optimal wealth is independent of the exit time distribution and identical to the one obtained in the case of a fixed time-horizon. This result is a confirmation and an extension of Merton (1973) and Richard (1975), who had focused on an exponentially distributed time-horizon.



This result intuitively comes from the fact that optimal portfolio strategies in these simple cases (CRRA utility functions and deterministic coefficients) do not depend on the time-horizon (“myopic strategies”). In other words, if the proportion of wealth optimally invested in the risky versus riskfree assets with a fixed time-horizon is independent of the maturity, then these proportions will be optimal in the case of a random time-horizon. On the other hand, the value function is affected by the presence of a random time of exit. We are able to obtain explicit solutions for the optimal strategy and the wealth process as a function of the distribution of the uncertain-time-horizon. We also notice that the solutions obtained above carry through to the case of non-constant parameters that are deterministic functions of time.

We now discuss how these results transport to the case of an economy with infinite time span.

#### 4. The case of an economy with infinite time span

One problem in the case of an infinite time span is that stronger restrictions than the one in Definition 1 have to be imposed on admissible strategies so as to avoid arbitrage. As in Huang and Pagès (1992), a strategy is now said to be admissible if it satisfies the following definition.

**Definition 2.** A portfolio  $(\pi_t, 0 \leq t)$  is said to be admissible if it is a weak admissible strategy, if the wealth process is positive and if

$$\int_0^{T_n} \left| \pi_s \sigma_s \exp \left( - \int_0^s r_u \, du \right) \right|^2 \, ds < \infty \quad \mathbb{P} \quad \text{a.s.}$$

for a sequence of stopping times  $T_n \uparrow \infty \quad \mathbb{P}$ -a.s.

**Remark 3.** Since  $\sigma$  is deterministic and bounded, if the process  $\pi$  is continuous, then the sequence  $T_n = n$  satisfies the previous condition.

As before, let  $P(t, x)$  denote the set of admissible portfolios starting from a wealth level  $x$  at date  $t$

$$P(t, x) = \{ \pi_u, t \leq u, X_u^{t, \pi, x} = x, \pi \text{ is admissible} \}$$

We define

$$V(t, x) = \sup_{\pi \in P(t, x)} \mathbb{E} \left[ \int_t^\infty f(u) U(X_u^{t, \pi, x}) \, du \right] \tag{7}$$

and obtain that  $V$  is a solution to the Hamilton–Jacobi–Bellman Eq. (5), except for the fact that the boundary condition has now changed (cf. Fleming and Soner (1993), p. 172, Theorem 5.1).

**Theorem 3.** If  $Y$  is the  $C^{1,2}$  solution to the following Hamilton–Jacobi–Bellman equation

$$0 = f(t)U(x) + \frac{\partial Y(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^d} A(t, x, \pi) \tag{8}$$

where  $A(t, x, \pi) = [x r_t + \pi(\mu_t - r_t)] Y_x(t, x) + (1/2) \pi^2 \sigma_t^2 Y_{xx}(t, x)$  and  $Y_x$  (resp.  $Y_{xx}$ ) denotes the first (resp. second) derivative with respect to the space variable, the boundary conditions being  $\lim_{t \rightarrow +\infty} Y(t, x) = 0$ , then  $V = Y$ .

As previously, we can determine the solution for some utility functions. These solutions are the same as before, except that we need to account for the stricter conditions for portfolios to be admissible.

#### Proposition 5.

(i) Let us assume that  $U(x) = \ln x$ . The value function and the optimal investment policy are respectively given by

$$V(t, x) = p(t) + q(t) \ln x, \quad \pi_t^{\log,*} = \frac{\mu - r}{\sigma^2} X_t^{\log,*}$$

where  $q(t) = 1 - \int_0^t f(s) \, ds$ ,  $p(t) = \lambda_2 - at + \int_0^t a \, ds \int_0^s f(u) \, du$ ,  $a = r + (1/2)\theta^2$ ,  $\lambda_2 = a \mathbb{E}(\tau)$ .

(ii) Let us assume that  $U(x) = (x^\alpha/\alpha)$  with  $0 < \alpha < 1$ ,  $(r - (\theta^2/2(\alpha - 1))) > 0$ , and  $\mathbb{E}[e^{(r - (\theta^2/2(\alpha - 1)))\tau}] < \infty$ . Then the value function is

$$V(t, x) = q(t) \frac{x^\alpha}{\alpha}$$

with  $q(t) = e^{-at} [\int_t^{+\infty} e^{as} f(s) ds]$ ,  $a = r - (\theta^2/2(\alpha - 1))$ .

The optimal portfolio strategy is

$$\pi_t^{(\alpha),*} = -\frac{\mu - r}{(\alpha - 1)\sigma^2} X_t^{(\alpha),*}$$

**Proof.** It is exactly the same that in Propositions 2 and 3, just changing the boundary condition. Using Remark (3) we notice that the portfolios are admissible.  $\square$

### 5. Stochastic probability of exiting

We extend the analysis to a framework with a stochastically time-varying density process. This generalization shall prove useful because the assumption of a deterministic density is neither realistic nor rich enough, as it implies independence between time-horizon uncertainty and stock return uncertainty. In particular, one would like to allow for a possible correlation between the instantaneous probability of exit from the market and risky asset returns, a feature of high potential practical relevance.

In this section, we enlarge the probability space and introduce a one-dimensional Brownian motion  $W^f$ .  $W^f$  and  $W^i$   $i = 1, \dots, n$ , are correlated Brownian motions, with correlation  $\sigma_{S,f} = (\sigma_{if})_{1 \leq i \leq n}$ . In what follows, we assume that  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t) = \int_0^t f_s ds$  with  $f$  a solution to the following stochastic differential equation

$$df_s = f_s(a(s) ds + b(s) dW_s^f), \quad f_0 = y, \quad 0 \leq s \leq T \tag{9}$$

where we assume that the coefficients  $a, b$  are deterministic and  $\int_0^\infty du \exp(\int_0^u a(s) ds) < \infty$ .

In an explicit form

$$f_s = y \exp\left(\int_0^s a(u) du\right) \xi_s$$

where  $\xi_s = \exp(\int_0^s b(u) dW_u^f - (1/2) \int_0^s b^2(u) du)$ .

As a word of caution, it should be noted at this stage that in principle nothing prevents  $F$  to take on values greater than 1 under such a general specification. A natural way to impose the desirable condition  $\int_0^\infty f_s ds \leq 1$  on the class of positive processes  $f$  would actually lead to consider a process with random coefficients for  $f$ , for which the problem is not solvable in closed-form. To see why imposing the condition  $F_t \leq 1$  is inconsistent with the assumption of deterministic coefficients for  $f$ , we let  $\lambda$  be a positive process such that  $d\lambda_t = \lambda_t(a_t dt + \sigma_t dW_t)$  and let  $f_t = \lambda_t \exp(-\int_0^t \lambda_s ds)$ , so that we do obtain the condition  $\int_0^\infty f_s ds \leq 1$ . Then, we have

$$df_t = f_t(a_t dt + \sigma_t dW_t - \lambda_t dt)$$

or

$$df_t = f_t(a_t dt + \sigma_t dW_t - \lambda_0 e^{A_t - \Sigma_t + Z_t} dt)$$

with  $A_t = \int_0^t a_s ds$ ,  $\Sigma_t = (1/2) \int_0^t \sigma_s^2 ds$ ,  $Z_t = \int_0^t \sigma_s dW_s$ . The problem here is then that the process  $f$  is a process with a stochastic drift even if  $a_t$  and  $\sigma_t$  are deterministic or even constant functions of time. In other words, it appears not to be possible to ensure the condition  $F_t \leq 1$  while staying in the class of process with deterministic coefficients for  $f$ . Since the assumption of deterministic parameters is critical for what follows, we need to revert to the rule of

<b>T=1 Year, <math>f_0=1/365</math></b>										
a \ b	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.001	0.50%	0.66%	0.85%	1.04%	1.23%	1.42%	1.59%	1.77%	1.95%	2.12%
0.002	0.54%	0.70%	0.88%	1.07%	1.25%	1.44%	1.62%	1.80%	1.97%	2.14%
0.003	0.58%	0.73%	0.91%	1.10%	1.28%	1.46%	1.64%	1.82%	1.99%	2.16%
0.004	0.62%	0.77%	0.94%	1.12%	1.31%	1.49%	1.67%	1.84%	2.02%	2.19%
0.005	0.66%	0.80%	0.97%	1.15%	1.34%	1.51%	1.69%	1.87%	2.04%	2.21%
0.006	0.71%	0.83%	1.00%	1.18%	1.36%	1.54%	1.72%	1.89%	2.06%	2.23%
0.007	0.75%	0.87%	1.03%	1.21%	1.39%	1.57%	1.74%	1.91%	2.09%	2.25%
0.008	0.80%	0.90%	1.06%	1.24%	1.41%	1.59%	1.76%	1.94%	2.11%	2.28%
0.009	0.84%	0.94%	1.10%	1.27%	1.44%	1.62%	1.79%	1.97%	2.13%	2.30%
0.010	0.89%	0.98%	1.13%	1.30%	1.47%	1.64%	1.82%	1.99%	2.16%	2.33%
<b>T=5 Years, <math>f_0=1/(5x365)</math></b>										
a \ b	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.001	0.67%	1.13%	1.58%	2.01%	2.42%	2.81%	3.19%	3.55%	3.90%	4.24%
0.002	0.83%	1.26%	1.70%	2.12%	2.53%	2.92%	3.30%	3.66%	4.00%	4.33%
0.003	1.00%	1.41%	1.83%	2.24%	2.65%	3.03%	3.40%	3.76%	4.10%	4.43%
0.004	1.19%	1.56%	1.97%	2.37%	2.76%	3.14%	3.51%	3.86%	4.20%	4.53%
0.005	1.39%	1.73%	2.11%	2.50%	2.89%	3.26%	3.62%	3.97%	4.31%	4.63%
0.006	1.60%	1.90%	2.26%	2.64%	3.01%	3.38%	3.74%	4.08%	4.41%	4.73%
0.007	1.82%	2.07%	2.42%	2.78%	3.14%	3.50%	3.85%	4.19%	4.52%	4.84%
0.008	2.04%	2.26%	2.58%	2.93%	3.28%	3.63%	3.97%	4.31%	4.63%	4.94%
0.009	2.27%	2.44%	2.74%	3.08%	3.42%	3.76%	4.09%	4.42%	4.74%	5.05%
0.010	2.50%	2.64%	2.91%	3.23%	3.56%	3.89%	4.22%	4.54%	4.85%	5.15%
<b>T=10 Years, <math>f_0=1/(10x365)</math></b>										
a \ b	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0.001	0.97%	1.58%	2.17%	2.71%	3.23%	3.71%	4.17%	4.59%	4.99%	5.36%
0.002	1.29%	1.85%	2.41%	2.95%	3.45%	3.93%	4.37%	4.79%	5.18%	5.55%
0.003	1.65%	2.14%	2.67%	3.18%	3.68%	4.14%	4.58%	4.99%	5.38%	5.74%
0.004	2.04%	2.45%	2.95%	3.44%	3.91%	4.37%	4.80%	5.20%	5.58%	5.93%
0.005	2.46%	2.78%	3.23%	3.70%	4.15%	4.59%	5.01%	5.41%	5.78%	6.13%
0.006	2.89%	3.13%	3.53%	3.97%	4.41%	4.83%	5.23%	5.62%	5.98%	6.32%
0.007	3.33%	3.50%	3.84%	4.25%	4.67%	5.07%	5.46%	5.83%	6.18%	6.52%
0.008	3.77%	3.87%	4.16%	4.54%	4.93%	5.32%	5.69%	6.05%	6.39%	6.72%
0.009	4.22%	4.26%	4.50%	4.83%	5.20%	5.57%	5.93%	6.27%	6.60%	6.92%
0.010	4.66%	4.66%	4.85%	5.14%	5.48%	5.83%	6.17%	6.50%	6.82%	7.13%

Fig. 1. Probability of obtaining a conditional distribution function for the uncertain time-horizon higher than one for various parameter values.

thumb stating that the problem may be neglected in practice as long as the probability of getting forbidden values remains sufficiently small for reasonable values of the parameters. This is somewhat similar to the Vasicek (1977) term structure model that allows negative values with positive probabilities. In other words, we offer a tractable stylized model that can be solved in closed form and argue that the insights obtained with it remain at least qualitatively valid as long as the probability of getting inconsistent values for  $F$  remains sufficiently low. To get a better sense of whether this is a serious problem, we run the following numerical experiment. For three different values of the time-horizon  $T$  (1 year, 5 years and 10 years), we generate 10,000 paths for  $f$  using Eq. (9) with constant parameters  $a$  and  $b$ , as well as corresponding paths for  $F$ . We then search for the space of parameter values such that the probability of having a path with a value greater than 1 for  $F_t$  remains low. The table below shows the probability of having a path with at least one negative value for  $F_t$  for various time-horizons (1 year, 5 years and 10 years), and various initial values of the  $f$  process (given the correspondence between time-horizon and the average levels obtained for  $f$ , we chose to take  $f_0$  to be inversely proportional to the time-horizon) (Fig. 1).

As we can see from this example, it is always possible to find a subset of the parameter space where the probability of values greater than one for the conditional distribution function of the uncertain exit-time can be maintained below a given threshold, e.g., 1% or 5%.

Under the assumption of constant coefficients  $r, \mu, \sigma, a, b$  and CRRA preferences, we then show that it is possible to obtain explicit solutions. We first note that the value function  $V$  now depends on  $t, x, y$ . At initial time 0, one has

$$V(0, x, y) = \sup_{\pi \in P(0,x)} \mathbb{E} \left[ \int_0^\infty f_u U(X_{u \wedge T}^x) du \right] = \sup_{\pi \in P(0,x)} \mathbb{E}_{\mathbb{Q}} \left[ y \int_0^\infty \exp(A(u)) U(X_{u \wedge T}^x) du \right]$$

where  $\mathbb{Q}$  the probability defined on  $\mathcal{F}_T$  by

$$d\mathbb{Q} = \xi_T d\mathbb{P},$$

and  $A$  is a primitive of  $a$ . It should be noted that the probability  $\mathbb{Q}$  is just a instrumental measure that should not be regarded as an equivalent martingale measure. When the random time-horizon is not a stopping time of the filtration generated by asset prices, uncertainty over the timing of the cash-flow induces some form of market incompleteness,

even when the conditional probability of exiting is deterministic, which actually implies that there is no unique equivalent martingale measure (EMM). The problem of providing an explicit characterization of the set of equivalent martingale measures in an economy with uncertain time-horizon is an interesting question that has been addressed by Blanchet-Scaillet et al. (2005) to which we also refer for more details on pricing cash-flows paid at a random date. In what follows, on the other hand, since we are not interested in a pricing problem, but instead in an optimal portfolio selection problem (which we solve through standard dynamic programming methods, as opposed to martingale methods), we do not need to use any particular equivalent martingale measure.

We thus obtain that:

$$V(0, x, y) = y \left[ \int_0^\infty du \exp(A(u)) \right] \sup_{\pi \in P(0,x)} \mathbb{E}_{\mathbb{Q}} \left[ \int_0^\infty \varphi(u) U(X_{u \wedge T}^x) du \right]$$

where  $\varphi(u)$  is the deterministic density  $\varphi(u) = (1 / \int_0^\infty \exp(A(s)) ds) \exp(A(u))$

$$V(0, x, y) = y \left[ \int_0^\infty du \exp(A(u)) \right] V^1(0, x)$$

where  $V^1$  is the solution associated with the density  $\varphi$  and a wealth process

$$dX_s^{t,\pi,x} = X_s^{t,\pi,x} r_s ds + \pi_s [(\mu_s + \sigma_s \sigma_{S,f} b(s) - r_s) ds + \sigma_s dW_s^{\mathbb{Q}}]. \tag{10}$$

where  $W_t^{\mathbb{Q}} = W_t + \int_0^t \sigma_{S,f} b(s) ds$ .

The following proposition provides an explicit solution for the optimal portfolio strategy.

**Proposition 6.**

(i) If  $U(x) = \log x$  and  $a < 0$ , then the value function  $V$  is given by

$$V(t, x, y) = -e^{at} \frac{y}{a} \left( \frac{C}{a} (e^{a(T-t)} - 1) + \log(x) \right)$$

where  $C = r + (1/2)((\mu - r + \sigma b \sigma_{S,f})^2 / \sigma^2)$ . The optimal portfolio strategy is

$$\pi_t^* = \left[ \frac{\mu - r + \sigma b \sigma_{S,f}}{\sigma^2} \right] X_t^* \tag{11}$$

(ii) If  $U(x) = (x^\alpha / \alpha)$  and  $a < 0$ , the value function and optimal strategy are

$$V(t, x, y) = -\frac{y x^\alpha}{a \alpha} e^{-Ct} \left[ \left( \frac{C}{C+a} \right) e^{(a+C)T} + \frac{a}{C+a} e^{(a+C)t} \right]$$

and

$$\pi_t^* = - \left[ \frac{\mu - r + \sigma b \sigma_{S,f}}{\sigma^2} \right] \frac{X_t^*}{\alpha - 1}$$

where  $C = (r - (1/2)((\mu - r + \sigma b \sigma_{S,f})^2 / \sigma^2 (\alpha - 1)))$ .

**Proof.** Using the result of Proposition 2 and Eq. (10).  $\square$

**Remark 4.** In the case of deterministic coefficients, using the dynamic programming approach, we obtain that  $V$  is solution of the following HJB equation

$$0 = yU(x) + \frac{\partial V(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^d} A(t, x, y, \pi) \tag{12}$$

with

$$\begin{aligned} A(t, x, y, \pi) = & [xr(t) + \pi(\mu(t) - r(t) + \sigma(t)b(t)\sigma_{S,f})]V_x(t, x, y) \\ & + \frac{1}{2}\pi^2\sigma^2(t)V_{xx}(t, x, y) \\ & + V_y(t, x, y)ya(t, y) + V_{xy}(t, x, y)yb(t, y)\pi\sigma(t)\sigma_{S,f} + \frac{1}{2}V_{yy}(t, x, y)y^2b^2(t, y) \end{aligned}$$

and

$$V(T, x, y) = U(x)\mathbb{E} \left( 1 - \frac{F_T}{f_T} = y \right) = yU(x) \int_T^{+\infty} \exp \int_T^u a(v) dv du \tag{13}$$

By taking  $b = 0$ , we recover that a random time-horizon with a deterministic conditional density leads to the same portfolio strategy as in the case of a fixed time-horizon. In general, however, agents optimally invest more or less in the risky assets than when time-horizon is fixed depending on the sign of the correlation. Let us consider for simplicity of exposure the case of a single risky asset. The result is as follows: if  $\sigma_{Sf} < 0$  (resp.  $\sigma_{Sf} = 0$ , resp.  $\sigma_{Sf} > 0$ ), i.e., if the probability of exiting the market is negatively correlated (resp. is not correlated, resp. positively correlated) with the return on the risky asset, then the fraction invested in the risky asset is lower than (resp. equal to, resp. greater than) what it is in the case of a certain time-horizon:  $((\mu - r + \sigma b \sigma_{Sf})/\sigma^2) < ((\mu - r)/\sigma^2)$  (resp.  $((\mu - r + \sigma b \sigma_{Sf})/\sigma^2) = ((\mu - r)/\sigma^2)$ , resp.  $((\mu - r + \sigma b \sigma_{Sf})/\sigma^2) > ((\mu - r)/\sigma^2)$ ).

One natural question is whether we have reasons to believe that  $\sigma_{Sf}$  should be different from zero, and if so, whether it should be positive or negative. This, of course, depends on the situation. There is for example abundant anecdotal evidence that agents may postpone their retirement decision in relation to stock market downturns that negatively affect the level of their expected pension benefits. This paper shows that such complex features can be accounted for in a tractable framework.

## 6. Conclusion

Uncertainty over exit time is an important practical issue facing most, if not all, investors. In order to address this question, we consider a suitable extension of the familiar optimal investment problem of Merton (1971), where we allow the conditional distribution function of an agent's time-horizon to be stochastic and correlated to returns on risky securities. In contrast to existing literature, which has focused on an independent time-horizon, we show that the portfolio decision is affected.

Given that economic models recognizing that agents do not necessarily exert the same level of rationality on various aspects of their decision-making process are not yet fully developed and well-understood, one might argue that our analysis raises a possible concern over an inconsistency in assuming simultaneously that a rational agent that dynamically chooses her portfolio allocation in an optimal way is not allowed to rationally choose the moment of liquidation. In fact, we view our model as a natural, even if highly stylized, attempt to analyze situations when the investor actually chooses the time of liquidation, but not solely on the basis of information contained in past asset prices. In practice, there are indeed various reasons why investors decide to exit a particular market. Some of these motives are related to past values of asset prices, which, if they were the only ones involved, would indeed command a rational exercise of the embedded put option that would then turn the random time into a stopping time of the asset price filtration. On the other hand some of these reasons are typically related to tax effects, as well as other non-modeled factors, which explain why investors' actual behaviors with respect to time-of-exit tend to be somewhat correlated, but not perfectly correlated, with asset returns. In other words, our framework is consistent either with assuming that the investor is rational with respect with her investment decision but not with

respect to her liquidation decision, or with assuming that the investor is rational with respect to both the asset allocation decision and the time-of-exit decision but that the latter depends on exogenous factors that are not easily modeled.

Our analysis can be extended in several directions. First, it would be interesting to try and apply the general framework we discuss in this paper to specific problems involving a random time-horizon. One example would relate to optimal investment decisions in mortgage-backed securities (MBS), which are subject to prepayment risk that implies a probability of a forced exit from the market that is negatively correlated to changes in interest rates (prepayment is more likely when interest rates are low). A similar example would involve life insurance contracts, where exit becomes more likely when interest rates increase, making alternative investments opportunities become more attractive. It would also be interesting to explore the equilibrium implications of an uncertain time-horizon, along the line of Huang (2003) analysis of the liquidity premium.

## Acknowledgements

It is a pleasure to thank Sanjiv Das, Nils Hakansson, Terry Marsh, Mark Rubinstein, Branko Urošević, Nick Wonder and Fernando Zapatero for very useful comments. All errors are, of course, the authors' sole responsibility.

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