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A DEA model for resource allocation

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Abstract

This paper concerns inverse DEA. The aim is to estimate input/output levels of a given Decision Making Unit (DMU) when some or all of its input/output levels are changed, under preserving the efficiency index. We show that in the case of estimating increased required input vector when the output vector is increased, the current method which uses weakly efficient solution of the relevant multiple objective optimization problem may fail. We propose some sufficient conditions for input estimation.

Keywords: Data Envelopment Analysis (DEA); Efficiency; Multiple Objective Linear Programming (MOLP)

1. Introduction

In the two recent decades, the data envelopment analysis (DEA) technique has allocated to itself a wide variety of research in operations research field, see, e.g., (Cooper et al., 1999). In fact, DEA has become increasingly popular for efficiency analysis in the practical projects of management, economy, education, sport, etc.

Recently Wei et al., (2000) proposed inverse DEA, to answer the following question: if among a group of DMUs, we increase certain inputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMUs, how much output could the unit produce, or if the outputs need to be increased to a certain level and the efficiency of the unit remains unchanged, how much input should be received by the unit? To estimate the output levels, Wei et al. proposed an MOLP when the DMU under evaluation is inefficient and a linear programming model when the DMU is weakly efficient. On the other hand, the treatment of inverse DEA for the case of tracing an increased input vector on possible output production maintaining the existing efficiency level is covered correctly by Wei et al., but we show that in the case of tracing an increased output vector on the increased required input consumption maintaining the existing efficiency level which is discussed in Wei et al. is not correct.

This paper proposes sufficient conditions for input estimation when output is increased. The established results are based on using any strongly efficient solution of a provided MOLP and using some certain weakly efficient solutions of

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that MOLP. In fact, the inverse DEA model is transformed into and solved as an MOLP. The studying of inverse DEA (resource reallocation) models has some practical advantages. Firstly, these models imply a new avenue for DEA applications, i.e., production analysis or production planning. At present, various DEA models are mainly used for relative technical efficiency measuring and analysis. The physical quantities of inputs and outputs associated with the concerned DMUs are considered fixed for parametric representation of the efficiency model. In the existing literature, the problem of resource reallocation is dealt with only for the purpose of guiding an inefficient unit shifting along the direction of a projected ray from its current position onto the frontier. The DEA model has not been used as a kind of conditioned production model for a DMU to deal with certain choices of inputs/ouputs, given its standing efficiency level (Wei et al., 2000). Secondly, the studied inverse DEA adds in a new and important class of application problems for research on the inverse optimization problem. Furthermore, some additional advantages of using the inverse DEA models for the tasks of production analysis or resource reallocation are as follows: it can be used naturally for multiple input/output production without preassigned weights; it can be used for production input/output estimation and planning without knowing the real form of the production function; and the inverse DEA model is related to multiple objective programming or single-objective linear programming, which are well structured and studied with well-developed theories and useful results (Yan et al., 2002).

The rest of this paper is organized as follows. In the following section we state the problem and then present a counter-example. In Section 3 we revise Wei et al.'s solution and Section 4 contains an extension. Section 3 proposes sufficient conditions for input estimation when output is increased. Although there are many practical DEA problems (projects) in the literature where there are no preferences between the units and input/output factors, another important advantage of using the inverse DEA models for production analysis or resource reallocation is that the decision makers' preferences can be incorporated into the production analysis. In Section 4, we extend the results of Section 3 to the case of generalized cone ratio DEA models. The preference cones used in these extended models are particularly important in short-term production planning or resource reallocation as it more closely reflects the management reality. Having said that, although incorporating the cone ratio structure into the inverse DEA models provides additional advantages in supporting resource reallocation and production planning decisions, it also involves additional complexity in model mathematics. Hence, for reducing the complexity, using ordinary inverse DEA models is sometimes unavoidable, especially when the difference between the importance of the considered factors is negligible. Finally, Section 5 provides an application for our study.

2. Statement of the problem

Suppose we have a set of n peer DMUs, $\{\mathrm{DMU}_j: j=1, 2,\ldots, n\}$, which produce multiple outputs y_{rj} $(r=1, 2,\ldots, s)$, by utilizing multiple inputs x_{ij} $(i=1, 2,\ldots, m)$. Let the inputs and outputs for DMU_j be $X_j=(x_{1j}, x_{2j},\ldots, x_{mj})^t$ and $Y_j=(y_{1j}, y_{2j}, \ldots, y_{sj})^t$, respectively. Also $X_j\in\mathbb{R}^m$, $Y_j\in\mathbb{R}^s$, $X_j > 0$, and $Y_j > 0$ for all $j=1, 2,\ldots, n$. When a DMU_o, $o\in\{1, 2, \ldots, n\}$, is under evaluation, we consider the following generalized DEA model:

$$
\begin{aligned}
(P_I)\ &\min \theta \\
&s.t. \sum_{j=1}^{n} X_j\lambda_j \le X_o\theta \\
&\quad\ \sum_{j=1}^{n} Y_j\lambda_j \ge Y_o \\
&\quad\ \lambda\in\Lambda,
\end{aligned}
$$

where

$$
\Lambda = \left\{\lambda|\lambda = (\lambda_1,\ldots,\lambda_n), \delta_1\left(\sum_{j=1}^{n}\lambda_j + \delta_2(-1)^{\delta_3}v\right) = \delta_1, v\ge 0, \lambda_j \ge 0, j=1,\ldots,n\right\}
$$

and δ_1, δ_2, δ_3 are parameters with 0–1 values. In fact, it is easy to see that

(i) if $\delta_1=0$, then (P_I) is the CCR model,
(ii) if $\delta_1=1$ and $\delta_2=0$, then (P_I) is the BCC model,

(iii) if $\delta_1=\delta_2=1$ and $\delta_3=0$, then (P_I) is the FG model,
(iv) if $\delta_1=\delta_2=\delta_3=1$, then (P_I) is the ST model.

The CCR model is the first DEA model, which has been provided by Charnes et al. (1978), and measures the efficiency under a constant returns to scale (RTS) assumption of technology. The BCC, FG, and ST models have been introduced by Banker et al. (1984), Färe and Grosskopf (1985), and Seiford and Thrall (1990), respectively. Also these three models evaluate the units under variable, nonincreasing, and nondecreasing RTS assumptions of technology, respectively. In all of the abovementioned models the vector variable λ exhibits the intensity vector variable. The components of this vector represent the contribution of efficient units to constructing the pattern (projection point) for inefficient units. Also variable v is an auxiliary variable for integrating four models into a general model, and this variable does not have any interpretation. It is clear that in all four models, the feasible region is non-empty and the optimal value θ_I must satisfy $\theta_I \leq 1$.

Definition 1. (Cooper et al., 1999) The optimal value θ_I of problem (P_I) is called the efficiency index of $\mathrm{DMU_o}$. If $\theta_I=1$, we say $\mathrm{DMU_o}$ is (at least) weakly efficient.

Now consider the following question: if the efficiency index θ_I remains unchanged, but the outputs increase, how much should the inputs of the DMU increase? To solve this problem, suppose the outputs of $\mathrm{DMU_o}$ are increased from Y_o to $\beta_\mathrm{o}=Y_\mathrm{o}+\Delta Y_\mathrm{o}$, where vector $\Delta Y_\mathrm{o}\geq 0$ and $\Delta Y_\mathrm{o}\neq 0$. We need to estimate the input vector α_o^* provided that the efficiency index of $\mathrm{DMU_o}$ is still θ_I. Here

$$\alpha_o^* = \left(\alpha_{1o}^*, \alpha_{2o}^*, \ldots, \alpha_{mo}^*\right)^t = X_o + \Delta X_o, \Delta X_o \geq 0.$$

For convenience, suppose DMU_{n+1} represents DMU_o after changing the inputs and outputs. Hence, to measure the efficiency of DMU_{n+1}, we use the following model:

$$\begin{aligned}(P_I^+)\ & \min \theta \\ s.t.\ & \sum_{j=1}^{n} X_j\lambda_j^I + \alpha_o^*\lambda_{n+1}^I \leq \alpha_o^*\theta \\ & \sum_{j=1}^{n} Y_j\lambda_j^I + \beta_o\lambda_{n+1}^I \geq \beta_o \\ & \lambda^I \in \Lambda^+,\end{aligned}$$

where

$$\Lambda^+ = \left\{\lambda^I | \lambda^I = \left(\lambda_1^I, \ldots, \lambda_{n+1}^I\right), \delta_1\left(\sum_{j=1}^{n+1}\lambda_j^I + \delta_2(-1)^{\delta_3}v\right) = \delta_1, v \geq 0, \lambda_j^I \geq 0, j = 1, \ldots, n+1\right\}.$$

Definition 2. If the optimal value of problem (P_I^+) is equal to the optimal value of problem (P_I), we say that the efficiency is unchanged and we write $eff(\alpha_o^*, \beta_o)=eff(X_o, Y_\mathrm{o})$.

Now consider the following MOLP model

$$\begin{aligned}(V_I)\ & \min\ (\alpha_{1o}, \alpha_{2o}, \ldots, \alpha_{mo}) \\ s.t.\ & \sum_{j=1}^{n} X_j\lambda_j^V \leq \theta_I\alpha_o \\ & \sum_{j=1}^{n} Y_j\lambda_j^V \geq \beta_o \\ & \alpha_o \geq X_o \\ & \lambda^V \in \Lambda,\end{aligned}$$

where β_o and Λ are defined as before and θ_I is the optimal value of problem (P_I).

Table 1
Data of Example 1

	DMU	A	B
Input	x_1	1	3
	x_2	1	5
Output	y	2	1

Definition 3. Let $(\alpha_o^*, \lambda^{V*})$ be a feasible solution of problem (V_I). If there is no feasible solution (α_o, λ^V) of (V_I) such that $\alpha_{io} \le \alpha_{io}^*$ for all $i=1, 2,\ldots, m$ and $\alpha_{io} < \alpha_{io}^*$ for at least one i, then we say $(\alpha_o^*, \lambda^{V*})$ is a strongly efficient solution of (V_I).

Definition 4. Suppose $(\alpha_o^*, \lambda^{V*})$ is a feasible solution of problem (V_I). If there is no feasible solution (α_o, λ^V) of (V_I) such that $\alpha_o < \alpha_o^*$, that is, $\alpha_{io} < \alpha_{io}^*$ for all $i=1, 2,\ldots, m$, then we say $(\alpha_o^*, \lambda^{V*})$ is a weakly efficient solution of problem (V_I).

Note that any strongly efficient solution is a weakly efficient solution but the converse is not necessarily true. According to Wei et al., 2000, if $\theta_I<1$ and the outputs are increased from Y_o to β_o, then the efficiency index θ_I remains unchanged if $(\alpha_o^*, \lambda^{V*})$ is a weakly efficient solution of (V_I). Here, using an example we show that this assertion is not always true.

Example 1. Consider Table 1.

Here we have two DMUs A and B with two inputs x_1 and x_2 and one output y. We take B into consideration in the BCC model, and solve problem (P_I)

$$
\begin{aligned}
&\min \theta \\
&s.t.\ \lambda_1 + 3\lambda_2 \le 3\theta \\
&\quad \lambda_1 + 5\lambda_2 \le 5\theta \\
&\quad 2\lambda_1 + \lambda_2 \ge 1 \\
&\quad \lambda_1 + \lambda_2 = 1 \\
&\quad \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

The optimal solution is $(\lambda_1, \lambda_2)=(1, 0)$ and $\theta_I = \frac{1}{3}$. Now suppose that the output is increased from 1 to $\frac{3}{2}$, and we would like to know how much more input the unit should receive? So, we solve problem (V_I)

$$
\begin{aligned}
&\min\,(\alpha_{1B}, \alpha_{2B}) \\
&s.t.\ \lambda_1^V + 3\lambda_2^V \le \frac{1}{3}\alpha_{1B} \\
&\quad \lambda_1^V + 5\lambda_2^V \le \frac{1}{3}\alpha_{2B} \\
&\quad 2\lambda_1^V + \lambda_2^V \ge \frac{3}{2} \\
&\quad \alpha_{1B} \ge 3 \\
&\quad \alpha_{2B} \ge 5 \\
&\quad \lambda_1^V + \lambda_2^V = 1 \\
&\quad \lambda_1^V, \lambda_2^V \ge 0.
\end{aligned}
$$

It can be seen that $(\alpha_{1B}^*, \alpha_{2B}^*, \lambda_1^{V*}, \lambda_2^{V*})=(5, 5, 1, 0)$ is a weakly efficient solution of the above problem. Now using $\alpha_B^*=(5, 5)$ we solve problem (P_I^+)

$$
\begin{aligned}
&\min \theta \\
&s.t.\ \lambda_1^I + 3\lambda_2^I + 5\lambda_3^I \le 5\theta \\
&\quad \lambda_1^I + 5\lambda_2^I + 5\lambda_3^I \le 5\theta \\
&\quad 2\lambda_1^I + \lambda_2^I + \frac{3}{2}\lambda_3^I \ge \frac{3}{2} \\
&\quad \lambda_1^I + \lambda_2^I + \lambda_3^I = 1 \\
&\quad \lambda_1^I, \lambda_2^I, \lambda_3^I \ge 0.
\end{aligned}
$$

The optimal solution is $(\theta_I^+, \lambda_1^I, \lambda_2^I, \lambda_3^I)=(\frac{1}{5}, 1, 0, 0)$ and we see that $\theta_I^+ = \frac{1}{5} \neq \frac{1}{3} = \theta_I$, hence $eff(5, 5, \frac{3}{2}) \neq eff(3, 5, 1)$. The above example shows that (Wei et al., 2000) must be revised. We provide a revision in the following section.

3. Revised solution

In this section we are going to determine the solutions of problem (V_I) that preserve the efficiency index. The following theorem shows that if we use strongly efficient solutions of problem (V_I), then $eff(\alpha_o^*, \beta_o)=eff(X_o, Y_o)$.

Theorem 1. *Suppose the optimal value of problem (P_I) is θ_I and $(\alpha_o^*, \lambda^{V*})$ is a strongly efficient solution of problem (V_I). Then, when the inputs of DMU_o are increased to α_o^*, the optimal value of problem (P_I^+) is (θ_I), that is, $eff(\alpha_o^*, \beta_o) = eff(X_o, Y_o)$.*

Proof. Suppose $(\alpha_o^*, \lambda^{V*})$ is a strongly efficient solution of problem (V_1) and $(\theta_I^+, \lambda^{I*})$ is the optimal solution of problem (P_I^+). We must show that $\theta_I^+ = \theta_I$. Since $(\alpha_o^*, \lambda^{V*})$ is a feasible solution of (V_I), it satisfies the following conditions

$$\sum_{j=1}^{n} X_j \lambda_j^{V*} \leq \theta_I \alpha_o^* \tag{1}$$

$$\sum_{j=1}^{n} Y_j \lambda_j^{V*} \geq \beta_o \tag{2}$$

$$\alpha_o^* \geq X_o \tag{3}$$

$$\lambda^{V*} \in \Lambda. \tag{4}$$

By Eqs. (1), (2) and (4) it is obvious that $(\theta_I, \bar{\lambda})$ is a feasible solution for (P_I^+), where $\bar{\lambda} = (\lambda^{V*}, 0) \in \mathbb{R}^{n+1}$, so $\theta_I^+ \leq \theta_I$. By Eq. (1) and since $\theta_I < 1$, we have

$$\sum_{j=1}^{n} X_j \lambda_j^{V*} \leq \alpha_o^*.$$

Using this in (P_I^+) yields

$$\theta_I^+ \alpha_o^* \geq \sum_{j=1}^{n} X_j \lambda_j^{I*} + \alpha_o^* \lambda_{n+1}^{I*} \geq \sum_{j=1}^{n} X_j \lambda_j^{I*} + \lambda_{n+1}^{I*} \sum_{j=1}^{n} X_j \lambda_j^{V*} = \sum_{j=1}^{n} X_j \left(\lambda_j^{I*} + \lambda_{n+1}^{I*} \lambda_j^{V*} \right). \tag{5}$$

Also using Eq. (2) and the restrictions in (P_I^+) we have

$$Y_o \leq \beta_o \leq \sum_{j=1}^{n} Y_j \lambda_j^{I*} + \beta_o \lambda_{n+1}^{I*} \leq \sum_{j=1}^{n} Y_j \lambda_j^{I*} + \lambda_{n+1}^{I*} \sum_{j=1}^{n} Y_j \lambda_j^{V*} = \sum_{j=1}^{n} Y_j \left(\lambda_j^{I*} + \lambda_{n+1}^{I*} \lambda_j^{V*} \right).$$

For each $j=1,\ldots, n$ set

$$\tilde{\lambda}_j = \lambda_j^{I*} + \lambda_{n+1}^{I*} \lambda_j^{V*}.$$

It is not difficult to see that $\tilde{\lambda}=(\tilde{\lambda}_1,\ldots, \tilde{\lambda}_n) \in \Lambda$. Now by contradiction assume that $\theta_I^+ < \theta_I$. There are two cases for constraint (3) at each strongly efficient solution of problem (V_I):

(i) $\alpha_o^* = X_o$. In this case, by Eq. (5) and since $\theta_I^+ < \theta_I$, we have

$$\sum_{j=1}^{n} X_j \tilde{\lambda}_j \leq X_o \theta_I^+ < X_o \theta_I.$$

That is, $(\theta_I^+, \tilde{\lambda})$ is a feasible solution to problem (P_I), but it is impossible because θ_I is the optimal value of (P_I).

(ii) $\alpha_o^* \gneqq X_o$. By Eq. (5) and the assumption that $\theta_I^+ < \theta_I$, we have

$$\sum_{j=1}^{n} X_j \tilde{\lambda}_j \leq \alpha_o^* \theta_I^+ < \alpha_o^* \theta_I.$$

On the other hand, in this case there exists at least one i, $1 \leq i \leq m$, such that $\alpha_{io}^* > x_{\text{io}}$. Therefore if we define

$$k = \min\left\{ \min_{1 \leq i \leq m} \left\{ \frac{\alpha_{io}^* \theta_I - \sum_{j=1}^{n} x_{ij} \tilde{\lambda}_j}{\theta_I} \right\}, \min_{1 \leq i \leq m} \left\{ \alpha_{io}^* - x_{io} : \alpha_{io}^* - x_{io} > 0 \right\} \right\},$$

then we have $k > 0$. Now defining

$$\overline{\alpha}_{io} = \begin{cases} \alpha_{\text{io}}^*, & \text{if } \alpha_{\text{io}}^* = x_{io}, \\ \alpha_{\text{io}}^* - k, & \text{if } \alpha_{\text{io}}^* > x_{io}, \end{cases}$$

we have

$$k \leq \frac{\alpha_{\text{io}}^* \theta_I - \sum_{j=1}^{n} x_{ij} \tilde{\lambda}_j}{\theta_I} \Rightarrow \sum_{j=1}^{n} x_{ij} \tilde{\lambda}_j \leq \theta_I \left(\alpha_{io}^* - k \right) \leq \theta_I \, \overline{\alpha}_{\text{io}}, \quad i = 1, 2, \ldots, m,$$

and

$$k \leq \alpha_{\text{io}}^* - x_{\text{io}} \Rightarrow \alpha_{\text{io}}^* - k \geq x_{\text{io}}^*, \quad \text{for } i : \alpha_{\text{io}}^* - x_{\text{io}} > 0,$$

which implies that $\overline{\alpha}_o \geq X_o$, because $\overline{\alpha}_{\text{io}} = x_{\text{io}}$ when $\alpha_{\text{io}}^* - x_{\text{io}} = 0$. Therefore $(\overline{\alpha}_o, \tilde{\lambda})$ is a feasible solution of problem (V_I), where $\overline{\alpha}_{\text{io}} \leq \alpha_{\text{io}}^*$ for all $i = 1, 2, \ldots, m$, and $\overline{\alpha}_{\text{io}} < \alpha_{\text{io}}^*$ for some $i = 1, 2, \ldots, m$. This contradicts the assumption that $(\alpha_o^*, \lambda^{V*})$ is a strongly efficient solution of problem (V_I).

Therefore in each case $\theta_I^+ \not< \theta_I$, and since $\theta_I^+ \leq \theta_I$, we have $\theta_I^+ = \theta_I$. □

Note that the converse of this theorem is not always true, that is, if (α_o, λ^V) is a feasible solution of problem (V_I) and $eff(\alpha_o, \beta_o) = eff(X_o, Y_o)$ then we cannot say that (α_o, λ^V) is a strongly efficient solution of (V_I). In fact, we can say that this solution is a weakly efficient solution of problem (V_I).

The next theorem shows that we can use some weakly efficient solutions of problem (V_I) for input estimation.

Theorem 2. *Suppose that $(\alpha_o^*, \lambda^{V*})$ denotes some weakly efficient solutions of problem (V_I) such that $\alpha_o^* > X_o$, then $eff(\alpha_o^*, \beta_o) = eff(X_o, Y_o)$.*

Proof. The proof is similar to case (ii) in the proof of Theorem 1; the only difference is that

$$k = \min\left\{ \min_{1 \leq i \leq m} \left\{ \frac{\alpha_{io}^* \theta_I - \sum_{j=1}^{n} x_{ij} \tilde{\lambda}_j}{\theta_I} \right\}, \min_{1 \leq i \leq m} \left\{ \alpha_{\text{io}}^* - x_{\text{io}} \right\} \right\},$$

$\overline{\alpha}_{io} = \alpha_{io}^* - k$ for $i = 1, 2, \ldots, m$, and hence $\overline{\alpha}_o < \alpha_o^*$. But this is a contradiction, because $(\alpha_o^*, \lambda^{V*})$ is a weakly efficient solution of (V_I). □

The following example demonstrates Theorems 1 and 2 for the BCC model. A similar example can be used for other models, too.

Example 2. Consider Table 2.

Here we have three DMUs A,B, and C with two inputs x_1 and x_2 and two outputs y_1 and y_2. We take C into consideration in the BCC model, and by solving the respective problem (P_I) we have $\theta_I = \frac{1}{4}$. Now suppose that the

Table 2
Data of Example 2

	DMU	A	B	C
Input	x_1	1	3	4
	x_2	1	1	4
Output	y_1	1	1	1
	y_2	2	4	1

output is increased from (1, 1) to (1, 3), and we would like to know how much more input the unit should receive? So we consider problem (V_I)

$$\begin{aligned}
&\min\ (\alpha_{1C},\alpha_{2C})\\
&s.t.\ \lambda_1^V+3\lambda_2^V+4\lambda_3^V \le \frac{1}{4}\alpha_{1C}\\
&\quad \lambda_1^V+\lambda_2^V+4\lambda_3^V \le \frac{1}{4}\alpha_{2C}\\
&\quad \lambda_1^V+\lambda_2^V+\lambda_3^V \ge 1\\
&\quad 2\lambda_1^V+4\lambda_2^V+\lambda_3^V \ge 3\\
&\quad \alpha_{1C} \ge 4\\
&\quad \alpha_{2C} \ge 4\\
&\quad \lambda_1^V+\lambda_2^V+\lambda_3^V = 1\\
&\quad \lambda_1^V,\lambda_2^V,\lambda_3^V \ge 0.
\end{aligned}$$

It can be seen that $(\alpha_{1C}^*, \alpha_{2C}^*, \lambda_1^{V*}, \lambda_2^{V*}, \lambda_3^{V*})=\left(8,4,\frac{1}{2},\frac{1}{2},0\right)$ and $(\alpha_{1C}^{**}, \alpha_{2C}^{**}, \lambda_1^{V**}, \lambda_2^{V**}, \lambda_3^{V*})=\left(8,5,\frac{1}{2},\frac{1}{2},0\right)$ are strongly and weak efficient solutions to the above problem, respectively, where $(\alpha_{1C}^{**}, \alpha_{2C}^{**}) > X_C$. Now using $\alpha_C^*=(8, 4)$ and $\alpha_C^{**}=(8, 5)$ we solve the following problems (P_I^+), respectively:

$$\begin{aligned}
&\min\ \theta\\
&s.t.\ \lambda_1^I+3\lambda_2^I+4\lambda_3^I+8\lambda_4^I \le 8\theta\\
&\quad \lambda_1^I+\lambda_2^I+4\lambda_3^I+4\lambda_4^I \le 4\theta\\
&\quad \lambda_1^I+\lambda_2^I+\lambda_3^I+\lambda_4^I \ge 1\\
&\quad 2\lambda_1^I+4\lambda_2^I+\lambda_3^I+3\lambda_4^I \ge 3\\
&\quad \lambda_1^I+\lambda_2^I+\lambda_3^I+\lambda_4^I = 1\\
&\quad \lambda_1^I,\lambda_2^I,\lambda_3^I,\lambda_4^I \ge 0
\end{aligned}$$

and

$$\begin{aligned}
&\min\ \theta\\
&s.t.\ \lambda_1^I+3\lambda_2^I+4\lambda_3^I+8\lambda_4^I \le 8\theta\\
&\quad \lambda_1^I+\lambda_2^I+4\lambda_3^I+5\lambda_4^I \le 5\theta\\
&\quad \lambda_1^I+\lambda_2^I+\lambda_3^I+\lambda_4^I \ge 1\\
&\quad 2\lambda_1^I+4\lambda_2^I+\lambda_3^I+3\lambda_4^I \ge 3\\
&\quad \lambda_1^I+\lambda_2^I+\lambda_3^I+\lambda_4^I = 1\\
&\quad \lambda_1^I,\lambda_2^I,\lambda_3^I,\lambda_4^I \ge 0.
\end{aligned}$$

Solving each of the above models gives $\theta_I^+ = \frac{1}{4}$ as the optimal value which is equal to θ_I as has been established in Theorems 1 and 2.

The following example, which uses the data of Example 1, clarifies the ability of Theorem 1. In fact, this example exhibits a case in which the strongly efficient solution preserves the efficiency level while the weakly efficient solution does not preserve the efficiency level.

Example 3. Consider the data of Example 1, listed in Table 1. We take B into consideration in the BCC model, and solve problem *(P_I)*

$$\begin{aligned}&\min \theta\\ &s.t.\ \lambda_1+3\lambda_2 \leq 3\theta\\ &\quad \lambda_1+5\lambda_2 \leq 5\theta\\ &\quad 2\lambda_1+\lambda_2 \geq 1\\ &\quad \lambda_1+\lambda_2 = 1\\ &\quad \lambda_1,\lambda_2 \geq 0.\end{aligned}$$

The optimal solution is $(\lambda_1, \lambda_2)=(1, 0)$ and $\theta_I=\frac{1}{3}$. Now suppose that the output is increased from 1 to $\frac{3}{2}$, and we would like to know how much more input the unit should receive? So we solve problem (V_I)

$$\begin{aligned}&\min\ (\alpha_{1B},\alpha_{2B})\\ &s.t.\ \lambda_1^V+3\lambda_2^V \leq \frac{1}{3}\alpha_{1B}\\ &\quad \lambda_1^V+5\lambda_2^V \leq \frac{1}{3}\alpha_{2B}\\ &\quad 2\lambda_1^V+\lambda_2^V \geq \frac{3}{2}\\ &\quad \alpha_{1B} \geq 3\\ &\quad \alpha_{2B} \geq 5\\ &\quad \lambda_1^V+\lambda_2^V = 1\\ &\quad \lambda_1^V,\lambda_2^V \geq 0.\end{aligned}$$

As it was seen in Example 1, $(\alpha_{1B}^*, \alpha_{2B}^*, \lambda_1^{V*}, \lambda_2^{V*})=(5, 5, 1, 0)$ is a weakly efficient solution of the above MOLP and $eff(5,5,\frac{3}{2})\neq eff(3, 5, 1)$. Now we consider a strongly efficient solution of the above MOLP. It can be seen that $(\alpha_{1B}^*, \alpha_{2B}^*, \lambda_1^{V*}, \lambda_2^{V*})=(3, 5, 1, 0)$ is a strongly efficient solution of the above MOLP. Now using $\alpha_B^*=(3, 5)$ we solve problem (P_I^+)

$$\begin{aligned}&\min \theta\\ &s.t.\ \lambda_1^I+3\lambda_2^I+3\lambda_3^I \leq 3\theta\\ &\quad \lambda_1^I+5\lambda_2^I+5\lambda_3^I \leq 5\theta\\ &\quad 2\lambda_1^I+\lambda_2^I+\frac{3}{2}\lambda_3^I \geq \frac{3}{2}\\ &\quad \lambda_1^I+\lambda_2^I+\lambda_3^I = 1\\ &\quad \lambda_1^I,\lambda_2^I,\lambda_3^I \geq 0.\end{aligned}$$

The optimal solution is $(\theta_I^+, \lambda_1^I, \lambda_2^I, \lambda_3^I)=(\frac{1}{3}, 1, 0, 0)$ and we see that $\theta_I^+=\frac{1}{3}=\theta_I$.

Regarding Theorems 1 and 2, by solving the provided MOLP (V_I) one can answer the question sketched in Section 1. To solve MOLP (V_I), we recommend using the method provided by Jahanshahloo and Foroughi (2004), see also (Steuer 1986).

The MOLP model (V_I) may obtain alternative optimal solutions. In this case we recommend using a manager-specified utility function to choose the most preferred solution. To this end, we must solve the optimization problem $\min\{f(\alpha_o^*)|(\alpha_o^*, \lambda^{V*})\in E\}$, in which f is the considered utility function and E is the set of all strongly efficient solutions of (V_I). This model can be solved using the technique provided by Tu (2000). Note that a convenient choosing of the related utility function can help the manager to select better efficient solution(s). For example, assume that $x_{1o}=x_{2o}$ and hence we would like to have this relation for α_o^*, as much as possible. In this case $f(\alpha_o^*)=\alpha_{1o}^*-\alpha_{2o}^*$ can be a convenient utility function.

4. Extension

In another paper in the inverse DEA filed, Yan et al. (2002) discussed the inverse DEA problem with preference cone constraints, to capture the top management's preferences on inputs and outputs, or on some DMUs. Denote by $V\subset\mathbb{R}_+^m$ the preference cone of the relative importance of m input entities; by $U\subset\mathbb{R}_+^s$ the preference cone of the relative importance of s output entities; and by $K \subset \mathbb{R}^n_+$ the preference cone which shows the predilection on some DMUs. In addition, denote by U^* and V^*, the negative polar cones of U and V, respectively, and by Int V^* and Int U^* the interior sets of V^* and U^*, respectively. We can assume that

$$x_j \in -\mathrm{Int}V^*, \quad y_j \in -\mathrm{Int}U^*, \quad j = 1, 2, \ldots, n.$$

Then the input-oriented generalized DEA model can be defined as follows:

$$\begin{aligned}
\left(\widehat{P}_I\right) \min\ & \theta \\
s.t\ & \sum_{j=1}^{n} X_j \lambda_j - \theta X_o \in V^* \\
& -\sum_{j=1}^{n} Y_j \lambda_j + Y_o \in U^* \\
& \lambda \in \widehat{\Lambda},
\end{aligned}$$

where

$$\widehat{\Lambda} = \left\{ \lambda \in -K^* | \lambda = (\lambda_1, \ldots, \lambda_n), \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} v \right) = \delta_1, v \geq 0 \right\},$$

and δ_1, δ_2, δ_3 are parameters with 0–1 values as presented for model (P_I). Suppose that the outputs of DMU$_o$ are increased from Y_o to $\widehat{\beta}_o$, such that $\widehat{\beta}_o - Y_o \in -U^*$. We need to estimate the input vector $\widehat{\alpha}^*{}_o$ provided that the efficiency index of DMU$_o$ is still $\widehat{\theta}_I$, the optimal value of ($\widehat{P}_I$). In this extended case, we consider the following MOLP:

$$\begin{aligned}
\left(\widehat{V}_I\right) \min & \left(\widehat{\alpha}_{1o}, \widehat{\alpha}_{2o}, \ldots, \widehat{\alpha}_{mo}\right) \\
s.t\ & \sum_{j=1}^{n} X_j \lambda_j^{\widehat{V}} - \widehat{\theta}_I \widehat{\alpha}_o \in V^* \\
& -\sum_{j=1}^{n} Y_j \lambda_j^{\widehat{V}} + \widehat{\beta}_o \in U^* \\
& \widehat{\alpha}_o - x_o \in -V^* \\
& \widehat{\lambda} \in \widehat{\Lambda}.
\end{aligned}$$

The following theorems show that if we use strongly efficient solutions or certain weakly efficient solutions of ($\widehat{V}_I$), then $eff\,(\widehat{\alpha}_o^*, \widehat{\beta}_o) = eff\,(X_o, Y_o)$. In these theorems the three considered cones are convex, and the proofs of these theorems are similar to those of Theorems 1 and 2, respectively, and are hence omitted.

Theorem 3. *Suppose the optimal value of problem ($\widehat{P}_I$) is $\widehat{\theta}_I$ and ($\widehat{\alpha}_o^*$, $\lambda^{\widehat{V}^*}$) is a strongly efficient solution of problem ($\widehat{V}_I$). Then, when the inputs of DMU$_o$ are increased to $\widehat{\alpha}_o^*$, $eff\,(\widehat{\alpha}_o^*, \widehat{\beta}_o) = eff\,(X_o, Y_o)$.*

Theorem 4. *Suppose ($\widehat{\alpha}_o^*$, $\lambda^{\widehat{V}^*}$) is a weakly efficient solution of problem ($\widehat{V}_I$) and $\widehat{\alpha}_o^* - X_o \in -Int\ V^*$, then $eff\,(\widehat{\alpha}_o^*, \widehat{\beta}_o) = eff\,(X_o, Y_o)$.*

5. An application

In this section we illustrate the application of the provided results by applying them to the real-world data of 17 university departments, denoted by *D1*, *D2*,…, *D17*. The data has been adopted from one of first-author's research projects in Azad University, Mobarakeh, Iran (Hadi-Vencheh, 2007). Each unit consumes two inputs to produce two outputs. The inputs are the number of bachelor students (X_1) and the number of (full time and part time) faculty members (X_2); and the outputs are the number of graduates (Y_1) and the number of research papers (Y_2). Data of the above factors for the 17 DMUs under consideration are reported in Table 3. Here, we have used the CCR model and produced input-oriented CCR efficiency scores have been listed in the last column of Table 3.

Table 3
Data related to the real application

DMU	X_1	X_2	Y_1	Y_2	θ
D1	26	7	12	3	0.86784
D2	29	6	10	7	1.00000
D3	40	8	20	6	0.99681
D4	42	7	12	6	0.81155
D5	45	9	18	6	0.82766
D6	92	12	40	2	0.63218
D7	83	11	58	3	1.00000
D8	87	14	52	7	0.96760
D9	149	16	61	4	0.75648
D10	177	17	54	12	0.94524
D11	191	19	61	11	0.87024
D12	185	14	73	4	0.99905
D13	186	20	85	10	0.98338
D14	74	12	36	5	0.79227
D15	164	22	69	8	0.72232
D16	225	20	80	5	0.78590
D17	108	10	27	3	0.61356

We take three units *D1*, *D5*, and *D11* into consideration in the CCR model and suppose that we would like to increase the output vectors of these units from (12, 3), (18, 6), and (61, 11) to (14, 5), (20, 8), and (63, 11), respectively, and we would like to know how much more input the unit should receive? In fact, the management of the university wants to improve the output levels of the above-mentioned departments as stated above. And we would like to help the management to find out how much to increase the level of inputs to obtain the desired level for the outputs, under the reference production technology and maintaining the efficiency level. To this end, we have considered the MOLP problem (V_I) corresponding to these units. For instance, this MOLP problem for unit D1 is as follows:

$$
\begin{aligned}
&\min\ (\alpha_{1C}, \alpha_{2C})\\
&s.t.\ 26\lambda_1^V + 29\lambda_2^V + 40\lambda_3^V + 42\lambda_4^V + 45\lambda_5^V + 92\lambda_6^V + 83\lambda_7^V + 87\lambda_8^V + 149\lambda_9^V + 177\lambda_{10}^V + 191\lambda_{11}^V + 185\lambda_{12}^V\\
&\qquad + 186\lambda_{13}^V + 74\lambda_{14}^V + 164\lambda_{15}^V + 225\lambda_{16}^V + 108\lambda_{17}^V \le 0.86784\alpha_{1,1}\\
&\quad 7\lambda_1^V + 6\lambda_2^V + 8\lambda_3^V + 7\lambda_4^V + 9\lambda_5^V + 12\lambda_6^V + 11\lambda_7^V + 14\lambda_8^V + 16\lambda_9^V + 17\lambda_{10}^V + 19\lambda_{11}^V + 14\lambda_{12}^V + 20\lambda_{13}^V\\
&\qquad + 12\lambda_{14}^V + 22\lambda_{15}^V + 20\lambda_{16}^V + 10\lambda_{17}^V \le 0.86784\alpha_{2,1}\\
&\quad 12\lambda_1^V + 10\lambda_2^V + 20\lambda_3^V + 12\lambda_4^V + 18\lambda_5^V + 40\lambda_6^V + 58\lambda_7^V + 52\lambda_8^V + 61\lambda_9^V + 54\lambda_{10}^V + 61\lambda_{11}^V + 73\lambda_{12}^V\\
&\qquad + 85\lambda_{13}^V + 36\lambda_{14}^V + 69\lambda_{15}^V + 80\lambda_{16}^V + 27\lambda_{17}^V \ge 14\\
&\quad 3\lambda_1^V + 7\lambda_2^V + 6\lambda_3^V + 6\lambda_4^V + 6\lambda_5^V + 2\lambda_6^V + 3\lambda_7^V + 7\lambda_8^V + 4\lambda_9^V + 12\lambda_{10}^V + 11\lambda_{11}^V + 4\lambda_{12}^V + 10\lambda_{13}^V + 5\lambda_{14}^V\\
&\qquad + 8\lambda_{15}^V + 5\lambda_{16}^V + 3\lambda_{17}^V \ge 5\\
&\quad \alpha_{1,1} \ge 26\\
&\quad \alpha_{2,1} \ge 7\\
&\quad \lambda_1^V, \lambda_2^V, \ldots, \lambda_{17}^V \ge 0.
\end{aligned}
$$

Considering MOLP problem (V1) for the three units mentioned, D1, D5 and D11, it can be seen that

$$
\begin{aligned}
\left(\lambda^{V*}, \alpha_1^*\right) &= (0, 0.06596, 0, 0, 0, 0, 0.1277, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 34.2499, 7),\\
\left(\lambda^{V**}, \alpha_5^*\right) &= (0, 1.0745, 0, 0, 0, 0, 0.1596, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 53.6504, 9.91),
\end{aligned}
$$

and

$$
\left(\lambda^{V***}, \alpha_{11}^*\right) = (0, 1.1941, 0, 0, 0, 0, 0.8803, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 191, 19.3606)
$$

are strongly efficient solutions for these three MOLP models. Then using α_1^* = (34.25, 7), α_5^* = (53.65, 9.91), and α_{11}^* = (191, 19.36) we have solved model (P_I^+), which led to the efficiency scores 0.86784, 0.82766, and 0.87024, respectively, which are the same as the efficiency scores obtained from the CCR model and listed in Table 3, as expected regarding the results established in Section 3 of the paper. For applied purposes, we can use the values $\alpha_1^* =$ (34, 7), $\alpha_5^* =$(54, 10), and $\alpha_{11}^* =$(191, 19), which have been rounded off.

As a supplementary study, owing to some educational reasons in the department numbered D5, the management expected the bachelor students (input-1) to be equal to four times as many as the faculty members (input-2). To meet this requirement, we have added a further constraint as

$$0 \leq \alpha_{1,5} - 4\alpha_{2,5} \leq 0$$

to MOLP model (V_I) corresponding to D5. After adding this constraint, it can be seen that

$$\left(\lambda^{VO}, \alpha_5^O\right) = (0, 1.0745, 0, 0, 0, 0, 0.1596, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 53.6504, 13.4126),$$

is a strongly efficient solution for this MOLP model corresponding to D5. Then using $\alpha_5^O =$(53.6504, 13.4126) led to the efficiency score 0.82766 which is the same as the efficiency score obtained from the CCR model. Here, also, we can use the numbers (54, 13), which have been rounded off.

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