Homotopy perturbation method for linear programming problems

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A R T I C L E   I N F O

A B S T R A C T

In this paper, He's homotopy perturbation method (HPM) is applied for solving linear programming (LP) problems. This paper shows that some recent findings about this topic cannot be applied for all cases. Furthermore, we provide the correct application of HPM for LP problems. The proposed method has a simple and graceful structure. Finally, a numerical example is displayed to illustrate the proposed method.

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1. Introduction

Experiential surveys show that linear programming (LP) is one of the most important techniques in applied mathematics. Many real-world problems can be transformed to linear programming model. Hence this model is an indispensable tool in today's applications such as energy, military, transportation, manufacturing, etc. There are many methods for solving LP (see [1] and references therein). Recently, Mehrabinezhad and Saberi-Nadjafi [2], presented a new method to solve LP based on the homotopy perturbation method (HPM). HPM was first proposed by He in 1998 [3] and was further developed and improved by him (see, [3–7] and the references therein). He, presented a homotopy perturbation technique based on the introduction of homotopy in topology, coupled with the traditional perturbation method for the solution of algebraic equations. This technique provides a summation of an infinite series with easily computable terms, which converges rapidly to the solution of the problem. In the literature, various authors have successfully applied HPM for many kinds of different problems such as nonlinear partial differential equations [8,9], nonlinear integral and integro-differential equations [10–12], fractional IVPs [13], nonlinear systems [14] and also linear systems [15,16]. For details of the number of publications on HPM according to web of science, we refer to [7, Fig. 1].

In this paper we study the homotopy perturbation method and point out some problems in the article [2]. Furthermore, we use HPM for solving linear programming (LP) problems under unrestricted variables. This paper is organized as follows. In Section 2, we present some notations and a brief review of the homotopy perturbation model for solving linear system. In Section 3, we point out the problems of the mentioned paper. In Section 4, we introduce application of HPM for LP under unrestricted variables with some examples.

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2. Homotopy perturbation method (HPM) for linear systems

Consider the following linear equation:

$$Ax = b,$$ (2.1)

where,

$$A = [a_{ij}], \quad b = [b_i] \quad x = [x_j], \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n.$$

Let $L(u) = Au - b$ and $F(u) = u - w_0$, where w_0 is a known vector.

Then we define *homotopy H* (u,p), as follows:

$$H(u,p) = (1-p)F(u) + \alpha p L(u) = 0,$$ (2.2)

where $p \in [0,1]$ is an embedding parameter and α is non-zero auxiliary parameter.

Obviously, we will have;

$$H(u,0) = F(u),$$ (2.3)
$$H(u,1) = L(u).$$ (2.4)

According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eq. (2.1) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots$$ (2.5)

and the exact solution is obtained as follows:

$$x = \lim_{p \to 1} u = \lim_{p \to 1}(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots) = \sum_{j=0}^{\infty} u_j.$$ (2.6)

Putting Eq. (2.5) into Eq. (2.2), and comparing the coefficients of identical degrees of p on both side, we find,

$$p^0 : u_0 = w_0$$
$$p^1 : (\alpha A - I)u_0 + u_1 - w_0 - \alpha b = 0, \quad u_1 = \alpha b - (\alpha A - I)u_0 + w_0,$$
$$p^2 : (\alpha A - I)u_1 + u_2 = 0, \quad u_2 = -(\alpha A - I)u_1,$$
$$\vdots$$

And in general,

$$u_{n+1} = -(\alpha A - I)u_n, \quad n = 1, 2, \ldots$$

Taking $u_0 = w_0 = 0$, yields:

$$u_1 = \alpha b$$
$$u_2 = -(\alpha A - I)u_1 = -(\alpha A - I)(\alpha b),$$
$$u_3 = -(\alpha A - I)u_2 = (\alpha A - I)^2 (\alpha b),$$
$$\vdots$$
$$u_{n+1} = -(\alpha A - I)u_n = (-1)^n (\alpha A - I)^n (\alpha b)$$

Therefore the solution can be of the form,

$$x = \sum_{i=0}^{\infty} (-1)^i (\alpha A - I)^i (\alpha b).$$ (2.7)

The convergence of the series (2.7), when $\alpha = 1$ is proved for diagonally dominant matrix A (see [2]).

3. Counter example

In this section, we point out an error in an algorithm that appeared in [2]. Consider the following linear programming with m constraints and n variables;

$$Min \quad z = C\mathbf{x},$$
$$s.t : \quad A\mathbf{x} = \mathbf{b},$$ (3.1)
$$\mathbf{x} \geqslant 0,$$

where $C = [c_j]$ is a n vector and $rank(A, b) = rank(A) = m$.

The corresponding dual of above LP is;

$$
\begin{aligned}
Max \quad & y = wb, \\
s.t: \quad & wA + v = C, \\
& v \geqslant 0.
\end{aligned}
\tag{3.2}
$$

Based on strong duality and Karush–Kuhn–Tucker (KKT) optimality conditions, Mehrabinezhad and Saberi-Nadjafi [2], transformed LP to following linear system and solve it, by HPM:

$$
BX = d,
\tag{3.3}
$$

where,

$$
B = \begin{bmatrix}
(A)_{m\times n} & (0)_{m\times m} & (0)_{m\times n} \\
(0)_{n\times n} & (A^t)_{n\times m} & (I)_{n\times n} \\
(C)_{1\times n} & (-b)_{1\times m} & (0)_{1\times n} \\
(0)_{n-1\times n} & (0)_{n-1\times m} & (0)_{n-1\times n}
\end{bmatrix}, \quad
X = \begin{bmatrix} x \\ w^t \\ v^t \end{bmatrix}, \quad
d = \begin{bmatrix} b \\ C^t \\ 0 \end{bmatrix}.
$$

This method is an interesting algorithm for solving linear programming. However, the main problem is, the solution of (3.3), does not guarantee that x, v are nonnegative. Therefore, the solution of mentioned method is not true for solving LP, necessarily. The following example is given to show this matter.

Example 3.1. Consider the following LP problem (P), and its corresponding dual problem (D),

$$
\begin{aligned}
(P)Min \quad & z = 2\mathbf{x}_1 + 3\mathbf{x}_2, \\
s.t: \quad & -2\mathbf{x}_1 + \mathbf{x}_2 = 1, \\
& \mathbf{x}_1, \mathbf{x}_2 \geqslant 0.
\end{aligned}
$$

$$
\begin{aligned}
(D)Max \quad & y = w, \\
s.t: \quad & -2w + v_1 = 2, \\
& w + v_2 = 3, \\
& v_1, v_2 \geqslant 0.
\end{aligned}
$$

These problems are solved by Lingo software and the following values are obtained:

$$
\begin{cases}
x^* = (x_1^*, x_2^*) = (0, 1), \\
(w^*, v^*) = (w^*, v_1^*, v_2^*) = (3, 8, 0), \\
z^* = y^* = 3.
\end{cases}
$$

But by the mentioned method [2], after some simple manipulations we have the following system of linear equations,

$$
\begin{bmatrix}
1 & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{2}{3} & 1 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ w \\ v_1 \\ v_2 \end{bmatrix}
=
\begin{bmatrix} -\frac{1}{2} \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},
$$

where, the second equation is the KKT condition. By considering ten terms of series (2.7), the solution of the above system will be obtained as $(x_1, x_2, w, v_1, v_2) = (-0.5, 0, -1, 0, 3.9220)$. Since the exact solution by Matlab software is $(x_1, x_2, w, v_1, v_2) = (-0.5, 0, -1, 0, 4)$, this is a good approximation for above linear system. However, x_1 is not nonnegative. Therefore, the method cannot obtain the optimal solution for above linear programming.

4. Correct application of HPM for solving LP problems

In this section we present the correct application of HPM for LP problems, i.e., the solution of LP problems under unrestricted variables.

Consider the following primal LP problem under unrestricted variables;

$$
\begin{aligned}
Min \quad & z = C\mathbf{x}, \\
s.t: \quad & A\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \ free,
\end{aligned}
\tag{4.1}
$$

where $C = [c_j]$ is a n vector and $rank(A, b) = rank(A) = m$.

The corresponding dual of above LP is;

$$Max \quad y = wb,$$
$$s.t: \quad wA = C,$$
$$\quad w \quad free.$$
(4.2)

Furthermore, based on strong duality, when y, x are optimal solutions of (4.1) and (4.2), we have

$$wb = Cx.$$

Therefore, we can transform linear programming (4.1) and (4.2) to the following systems:

$$\bar{H}\bar{Y} = \bar{g},$$
(4.3)

where,

$$\bar{H} = \begin{pmatrix} (A)_{m \times n} & (0)_{m \times m} \\ (0)_{n \times n} & (A^t)_{n \times m} \\ (C)_{1 \times n} & (-b)_{1 \times m} \end{pmatrix}, \quad \bar{Y} = \begin{pmatrix} x \\ w^t \end{pmatrix}, \quad \bar{g} = \begin{bmatrix} b \\ C^t \\ (0)_{1 \times 1} \end{bmatrix}.$$
(4.4)

Instead of the system (4.3), we solve the following column-padded $(m + n + 1) \times (m + n + 1)$ linear system:

$$HY = g,$$
(4.5)

where,

$$H = \begin{bmatrix} \bar{H} & (0)_{(m+n+1) \times 1} \end{bmatrix}, \quad Y = \begin{bmatrix} \bar{Y} \\ (0)_{1 \times 1} \end{bmatrix}, \quad g = \bar{g}.$$
(4.6)

Now, we can solve (4.5) based on HPM introduced in Section 2.

Example 4.1. Consider the following LP problem (P), and its corresponding dual problem (D),

$$(P)Min \quad z = 5\mathbf{x}_1 + \mathbf{x}_2$$
$$s.t: \quad 4\mathbf{x}_1 + 3\mathbf{x}_2 = 1,$$
$$\quad \mathbf{x}_1 + 2\mathbf{x}_2 = 4,$$
$$\quad \mathbf{x}_1, \mathbf{x}_2 \quad free.$$

$$(D)Max \quad y = w_1 + 4w_2$$
$$s.t: \quad 4w_1 + w_2 = 5,$$
$$\quad 3w_1 + 2w_2 = 1,$$
$$\quad w_1, w_2 \quad free.$$

These problems are solved by Lingo software and the following values are obtained:

$$\begin{cases} x^* = (x_1^*, x_2^*) = (-2, 3), \\ w^* = (w_1^*, w_2^*) = (1.8, -2.2), \\ z^* = y^* = -7. \end{cases}$$

Now we use HPM to solve the above example. To have a square matrix, based on (4.6), one zero column vectors is appended to the right side of the matrix $\bar{H}$.

After some simple manipulations we have the following system of linear equations,

$$\begin{bmatrix} 1 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 3/2 & 1 & 0 \\ 5 & 1 & -1 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 2 \\ 5/4 \\ 1/4 \\ 0 \end{bmatrix}.$$

By considering ten terms of series (2.7), the solution of above system will be obtained as $(x_1, x_2, w_1, w_2) = (-1.9852, 2.9778, 1.8859, -2.5807)$, which is a good approximation for optimal solutions.

5. Conclusions

In this paper, we have applied He's homotopy perturbation method (HPM) for linear programming (LP) problems. We have shown that the model of [2] cannot be applied for all cases. Finally, a numerical example is displayed to illustrate the proposed method.

Acknowledgment

References

[1] M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, Linear Programming and Network Flows, third ed., John Wiley & Sons, New York, 2004.
[2] M. Mehrabinezhad, J. Saberi-Nadjafi, Application of He's homotopy perturbation method to linear programming problems, Int. J. Comput. Math. 88 (2011) 341–347.
[3] J.-H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.
[4] J.-H. He, New interpretation of homotopy perturbation method, Int. J. Mod. Phys. B 20 (2006) 2561–2568.
[5] J.-H. He, Recent development of the homotopy perturbation method, Topol. Method Nonlinear Anal. 31 (2008) 205–209.
[6] J.-H. He, An elementary introduction to the homotopy perturbation method, Comput. Math. Appl. 57 (2009) 410–412.
[7] J.-H. He, Homotopy perturbation method with an auxiliary term, Abstr. Appl. Anal. (2012), http://dx.doi.org/10.1155/2012/857612.
[8] L. Cveticanin, Application of homotopy-perturbation to non-linear partial differential equations, Chaos Soliton Fract. 40 (2009) 221–228.
[9] M. Dehghan, F. Shakeri, The numerical solution of the second Painleve equation, Numer. Method Partial Differ. Equ. 25 (2009) 1238–1259.
[10] J. Saberi-Nadjafi, A. Ghorbani, He's homotopy perturbation method: an effective tool for solving nonlinear integral and integro-differential equations, Comput. Math. Appl. 58 (2009) 2379–2390.
[11] P. Roul, P. Meyer, Numerical solutions of systems of nonlinear integro-differential equations by homotopy-perturbation method, Appl. Math. Model. 35 (2011) 4234–4244.
[12] E. Hetmaniok, I. Nowak, D. Slota, R. Wituła, A study of the convergence of and error estimation for the homotopy perturbation method for the Volterra–Fredholm integral equations, Appl. Math. Lett. 26 (2013) 165–169.
[13] Q. Abdulaziz, I. Hashim, S. Momani, Application of homotopy perturbation method to fractional IVPs, J. Comput. Appl. Math. 216 (2008) 574–584.
[14] M.A. Noor, Some iterative methods for solving nonlinear equations using homotopy perturbation method, Int. J. Comput. Math. 87 (2010) 141–149.
[15] B. Keramati, An approach to the solution of linear system of equations by He's homotopy perturbation method, Chaos Soliton Fract. 41 (2009) 152–156.
[16] H.K. Liu, Application of homotopy perturbation methods for solving systems of linear equations, Appl. Math. Comput. 217 (2011) 5259–5264.